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GENERALIZED SYLVESTER FORMULAS AND SKEW GIAMBELLI IDENTITIES

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Abstract. We obtain a common generalization of two types of Sylvester formulas for compound determinants and its Pfaffian analogue. As applications, we give generalizations of the Giambelli identity to skew Schur functions and the Schur identity to Schur's skew Q-functions.

1. Introduction

Determinant and Pfaffian formulas for general matrices are a powerful tool in proving relations among special functions and in evaluating specific determinants and Pfaffians. For example, in [7], we applied the Cauchy–Sylvester formula for compound determinants to give a transparent proof of the evaluation of the determinant involving BC_n-type Jackson integrals. Furthermore, in [5], we found a variant of the Cauchy–Sylvester formula and obtained product evaluations of determinants of classical group characters. The aims of this paper are to establish a common generalization of two types of Sylvester formulas for compound determinants and its Pfaffian analogue, and to obtain symmetric function identities by applying these generalized formulas.

Given a matrix $X = \left(x_{i,j}\right)$ and sequences $I = (i_1, \dots, i_p)$ of row indices and $I' = (i'_1, \dots, i'_q)$ of column indices, let $X \begin{pmatrix} I \\ I' \end{pmatrix}$ be the $p \times q$ matrix obtained from X by picking up the rows indexed by I and the columns indexed by I', i.e.,

$$X \begin{pmatrix} I \\ I' \end{pmatrix} = \left(x_{i_\alpha, i'_\beta} \right)_{1 \le \alpha \le p,\, 1 \le \beta \le q}.$$

For two sequences I and J, we denote by $I \sqcup J$ the concatenation of I and J, and, if J is a subsequence of I, then we denote by $I \setminus J$ the sequence obtained by removing the entries of J from I. For example, $(1,3,4) \sqcup (2,6) = (1,3,4,2,6)$ and $(1,3,4,2,6) \setminus (3,2) = (1,4,6)$. Then the Sylvester formula and its dual version (the Jacobi formula) are stated as follows.

Proposition 1.1. Let X be a matrix and let K and K' be sequences of row and column indices of the same length.

(1) (Sylvester [17]) For sequences $I = (i_1, \dots, i_p)$ of row indices and $I' = (i'_1, \dots, i'_p)$ of column indices, we have

$$\det \left(\det X \begin{pmatrix} (i_\alpha) \sqcup K \\ (i'_\beta) \sqcup K' \end{pmatrix} \right)_{1 \le \alpha, \beta \le p} = \det X \begin{pmatrix} I \sqcup K \\ I' \sqcup K' \end{pmatrix} \cdot \left(\det X \begin{pmatrix} K \\ K' \end{pmatrix} \right)^{p-1}. \tag{1.1}$$

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(2) (Jacobi [8]) For sequences $J = (j_1, \dots, j_q)$ of row indices and $J' = (j'_1, \dots, j'_q)$ of column indices, we have

$$\det \left(\det X \begin{pmatrix} (J \setminus (j_\alpha)) \sqcup K \\ (J' \setminus (j'_\beta)) \sqcup K' \end{pmatrix} \right)_{1 \le \alpha, \beta \le q} = \left(\det X \begin{pmatrix} J \sqcup K \\ J' \sqcup K' \end{pmatrix} \right)^{q-1} \cdot \det X \begin{pmatrix} K \\ K' \end{pmatrix}. \quad (1.2)$$

Let X, I, J, K, I', J', K' be as in Proposition 1.1, and let L and L' be sequences of row and column indices of the same length. If the submatrix $X \begin{pmatrix} I \sqcup K \sqcup L \\ I' \sqcup K' \sqcup L' \end{pmatrix}$ of X is non-singular with inverse matrix Y, then we can use Jacobi's complementary minor formula [8] to rewrite (1.1) in terms of Y as

$$\det \left(Y \begin{pmatrix} (I' \setminus (i'_\alpha)) \sqcup L' \\ (I \setminus (i_\beta)) \sqcup L \end{pmatrix} \right)_{1 \le \alpha, \beta \le p} = \left(\det Y \begin{pmatrix} I' \sqcup L' \\ I \sqcup L \end{pmatrix} \right)^{p-1} \cdot \det Y \begin{pmatrix} L' \\ L \end{pmatrix},$$

which has the same form as (1.2). Similarly, if the submatrix $X \begin{pmatrix} J \sqcup K \sqcup L \\ J' \sqcup K' \sqcup L' \end{pmatrix}$ is non-singular with inverse matrix Z, then (1.2) can be written as

$$\det \left(Z \begin{pmatrix} (j'_\alpha) \sqcup L' \\ (j_\beta) \sqcup L \end{pmatrix} \right)_{1 \le \alpha, \beta \le q} = \det Z \begin{pmatrix} J' \sqcup L' \\ J \sqcup L \end{pmatrix} \cdot \left(\det Z \begin{pmatrix} L' \\ L \end{pmatrix} \right)^{p-1}.$$

In this sense, Equations (1.1) and (1.2) can be regarded dual to each other.

One of the main results of this paper is the following theorem, which is a common generalization of (1.1) and (1.2). In fact, we can recover (1.1) (respectively (1.2)) by specializing $q = 0$ (respectively $p = 0$) in (1.3).

Theorem 1.2. *Let X be a matrix and I, J and K (respectively I', J' and K') sequences of row (respectively column) indices of length p, q and r. We define a $(p+q) \times (p+q)$ matrix $\widetilde{X} = \left(\widetilde{x}_{\alpha,\beta} \right)_{1 \le \alpha, \beta \le p+q}$ by*

$$\begin{cases} \widetilde{x}_{\alpha,\beta} = \det X \begin{pmatrix} (i_\alpha) \sqcup J \sqcup K \\ (i'_\beta) \sqcup J' \sqcup K' \end{pmatrix}, & \text{if } 1 \le \alpha, \beta \le p, \\ \widetilde{x}_{\alpha,p+\beta} = \det X \begin{pmatrix} (i_\alpha) \sqcup (J \setminus (j_\beta)) \sqcup K \\ J' \sqcup K' \end{pmatrix}, & \text{if } 1 \le \alpha \le p \text{ and } 1 \le \beta \le q, \\ \widetilde{x}_{p+\alpha,\beta} = \det X \begin{pmatrix} J \sqcup K \\ (i'_\beta) \sqcup (J' \setminus (j'_\alpha)) \sqcup K' \end{pmatrix}, & \text{if } 1 \le \alpha \le q \text{ and } 1 \le \beta \le p, \\ \widetilde{x}_{p+\alpha,p+\beta} = -\det X \begin{pmatrix} (J \setminus (j_\beta)) \sqcup K \\ (J' \setminus (j'_\alpha)) \sqcup K' \end{pmatrix}, & \text{if } 1 \le \alpha, \beta \le q. \end{cases}$$

Then we have

$$\det \widetilde{X} = (-1)^q \det X \begin{pmatrix} I \sqcup K \\ I' \sqcup K' \end{pmatrix} \cdot \left(\det X \begin{pmatrix} J \sqcup K \\ J' \sqcup K' \end{pmatrix} \right)^{p+q-1}. \quad (1.3)$$

For example, if $I = I' = (1,2)$, $J = J' = (3,4)$, $K = K' = (5)$, then Equation (1.3) reads

$$\det \begin{pmatrix} \begin{vmatrix} 1,3,4,5 \\ 1,3,4,5 \end{vmatrix} & \begin{vmatrix} 1,3,4,5 \\ 2,3,4,5 \end{vmatrix} & \begin{vmatrix} 1,4,5 \\ 3,4,5 \end{vmatrix} & \begin{vmatrix} 1,3,5 \\ 3,4,5 \end{vmatrix} \\ \begin{vmatrix} 2,3,4,5 \\ 1,3,4,5 \end{vmatrix} & \begin{vmatrix} 2,3,4,5 \\ 2,3,4,5 \end{vmatrix} & \begin{vmatrix} 2,4,5 \\ 3,4,5 \end{vmatrix} & \begin{vmatrix} 2,3,5 \\ 3,4,5 \end{vmatrix} \\ \begin{vmatrix} 3,4,5 \\ 1,4,5 \end{vmatrix} & \begin{vmatrix} 3,4,5 \\ 2,4,5 \end{vmatrix} & -\begin{vmatrix} 4,5 \\ 4,5 \end{vmatrix} & -\begin{vmatrix} 3,5 \\ 4,5 \end{vmatrix} \\ \begin{vmatrix} 3,4,5 \\ 1,3,5 \end{vmatrix} & \begin{vmatrix} 3,4,5 \\ 2,3,5 \end{vmatrix} & -\begin{vmatrix} 4,5 \\ 3,5 \end{vmatrix} & -\begin{vmatrix} 3,5 \\ 3,5 \end{vmatrix} \end{pmatrix} = (-1)^2 \begin{vmatrix} 1,2,5 \\ 1,2,5 \end{vmatrix} \cdot \begin{vmatrix} 3,4,5 \\ 3,4,5 \end{vmatrix}^{2+2-1},$$

where $\begin{vmatrix} i_1, \dots, i_p \\ i'_1, \dots, i'_p \end{vmatrix}$ stands for $\det X \begin{pmatrix} i_1, \dots, i_p \\ i'_1, \dots, i'_p \end{pmatrix}$.

Knuth [10] gave Pfaffian analogues of the Sylvester and Jacobi formulas in Proposition 1.1. For a skew-symmetric matrix Y and a sequence $I = (i_1, \dots, i_l)$ of row/column indices, we write $Y(I)$ for $Y \begin{pmatrix} I \\ I \end{pmatrix}$.

Proposition 1.3. Let Y be a skew-symmetric matrix and K be a sequence of row/column indices of even length.

(1) (Knuth [10, 2.5]) For a sequence $I = (i_1, \dots, i_l)$ of row/column indices of even length l, we have

$$\operatorname{Pf}\Big(\operatorname{Pf} Y((i_\alpha, i_\beta) \sqcup K)\Big)_{1 \le \alpha < \beta \le l} = \operatorname{Pf} Y(I \sqcup K) \cdot \big(\operatorname{Pf} Y(K)\big)^{l/2-1}. \tag{1.4}$$

(2) (Knuth [10, (2.7)]) For a sequence $J = (j_1, \dots, j_m)$ of row/column indices of even length m, we have

$$\operatorname{Pf}\Big(\operatorname{Pf} Y((J \setminus (j_\alpha, j_\beta)) \sqcup K)\Big)_{1 \le \alpha < \beta \le m} = \big(\operatorname{Pf} Y(J \sqcup K)\big)^{m/2-1} \cdot \operatorname{Pf} Y(K). \tag{1.5}$$

Another main result is the following common generalization of (1.4) and (1.5). If $m = 0$ (respectively $l = 0$), then Equation (1.6) reduces to (1.4) (respectively (1.5)). Note that Hirota [4] gave the special case $l = m$.

Theorem 1.4. Let Y be a skew-symmetric matrix. Let l, m and n be nonnegative integers with the same parity, and let I, J and K be sequences of row/column indices of Y of length l, m and n respectively. We define an $(l+m) \times (l+m)$ skew-symmetric matrix $\widetilde{Y} = \big(\widetilde{y}_{\alpha,\beta}\big)_{1 \le \alpha,\beta \le l+m}$ by

$$\begin{cases} \widetilde{y}_{\alpha,\beta} = \operatorname{Pf} Y((i_\alpha, i_\beta) \sqcup J \sqcup K), & \text{if } 1 \le \alpha < \beta \le l, \\ \widetilde{y}_{\alpha,l+\beta} = \operatorname{Pf} Y((i_\alpha) \sqcup (J \setminus (j_\beta)) \sqcup K), & \text{if } 1 \le \alpha \le l \text{ and } 1 \le \beta \le m, \\ \widetilde{y}_{l+\alpha,l+\beta} = \operatorname{Pf} Y((J \setminus (j_\alpha, j_\beta)) \sqcup K), & \text{if } 1 \le \alpha < \beta \le m. \end{cases}$$

Then we have

$$\operatorname{Pf} \widetilde{Y} = \operatorname{Pf} Y(I \sqcup K) \cdot \big(\operatorname{Pf} Y(J \sqcup K)\big)^{(l+m)/2-1}. \tag{1.6}$$

As an application of our generalized Sylvester formula (Theorem 1.2), we give an extension of the Giambelli identity to skew Schur functions, which expresses a skew Schur function $s_{\lambda/\mu}$ in terms of a determinant of skew Schur functions of the form $\pm s_{\nu/\mu}$. Similarly, we use the generalized Knuth formula (Theorem 1.4) to obtain a generalization of the Schur identity to Schur's skew Q-functions, which expresses a skew Q-function $Q_{\lambda/\mu}$ in terms of a Pfaffian of skew Q-functions of the form $\pm Q_{\nu/\mu}$.

This article is organized as follows. In Section 2, we prove Theorem 1.4 by using a Pfaffian analogue of the Desnanot–Jacobi identity. In Section 3, we derive Theorem 1.2 from Theorem 1.4 by using a relation between determinants and Pfaffians. Applications to symmetric function identities are provided in Sections 4 and 5.

2. Proof of Theorem 1.4

Recall a definition and some properties of Pfaffians (see [6] for details). Given an $2m \times 2m$ skew-symmetric matrix $A = \left(a_{ij}\right)_{1 \le i,j \le 2m}$, the Pfaffian of A, denoted by $\operatorname{Pf} A$, is defined by

$$\operatorname{Pf} A = \sum_{\pi \in F_{2m}} \operatorname{sgn}(\pi) a_{\pi(1),\pi(2)} a_{\pi(3),\pi(4)} \cdots a_{\pi(2m-1),\pi(2m)},$$

where F_{2m} is the subset of the symmetric group S_{2m} given by

$$F_{2m} = \{\pi \in S_{2m} : \pi(1) < \pi(3) < \cdots < \pi(2m-1),\ \pi(2i-1) < \pi(2i)\,(1 \le i \le m)\}.$$

For example, if $2m = 4$, then we have

$$\operatorname{Pf} A = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

For a $2m \times 2m$ skew-symmetric matrix A and a $2m \times 2m$ matrix T, we have

$$\operatorname{Pf}\left({}^{t}TAT\right) = \det T \cdot \operatorname{Pf} A.$$

Hence the Pfaffian is alternating in the sense that $\operatorname{Pf}\left(a_{\sigma(i),\sigma(j)}\right) = \operatorname{sgn}(\sigma) \operatorname{Pf}\left(a_{i,j}\right)$ for any permutation $\sigma \in S_{2m}$.

One key ingredient of the proof of Theorem 1.4 is the Pfaffian analogue of the Desnanot–Jacobi formula, which is a special case of the basic identity in [10].

Lemma 2.1. (See e.g. [10, (1.1)].) Let A be a skew-symmetric matrix with rows/columns indexed by $(1, 2, \dots, 2m)$. For row/column indices $i < j < k < l$, we have

$$\operatorname{Pf} A^{i,j} \cdot \operatorname{Pf} A^{k,l} - \operatorname{Pf} A^{i,k} \cdot \operatorname{Pf} A^{j,l} + \operatorname{Pf} A^{i,l} \cdot \operatorname{Pf} A^{j,k} = \operatorname{Pf} A \cdot \operatorname{Pf} A^{i,j,k,l}, \tag{2.1}$$

where $A^{i_1,\dots,i_p}$ denotes the skew-symmetric matrix obtained from A by removing rows and columns with indices $i_1, \dots, i_p$.

First we show the following identities, which are special cases of Theorem 1.4.

Lemma 2.2. Let Y be a skew-symmetric matrix. Let (a, b, c, d) and K be sequences of row/column indices.

(1) If the length of K is even, then we have

$$\begin{aligned}\operatorname{Pf} Y((a,b,c,d) \sqcup K) \cdot \operatorname{Pf}(K) - \operatorname{Pf} Y((a,d) \sqcup K) \cdot \operatorname{Pf} Y((b,c) \sqcup K) \\ + \operatorname{Pf} Y((a,c) \sqcup K) \cdot \operatorname{Pf} Y((b,d) \sqcup K) = \operatorname{Pf} Y((a,b) \sqcup K) \cdot \operatorname{Pf} Y((c,d) \sqcup K).\end{aligned} \tag{2.2}$$

(2) If the length of K is odd, then we have

$$\begin{aligned}\operatorname{Pf} Y((a,c,d) \sqcup K) \cdot \operatorname{Pf} Y((b) \sqcup K) - \operatorname{Pf} Y((a,b,d) \sqcup K) \cdot \operatorname{Pf} Y((c) \sqcup K) \\ + \operatorname{Pf} Y((a,b,c) \sqcup K) \cdot \operatorname{Pf} Y((d) \sqcup K) = \operatorname{Pf} Y((b,c,d) \sqcup K) \cdot \operatorname{Pf} Y((a) \sqcup K).\end{aligned} \tag{2.3}$$

Proof. The first identity (2.2) is a restatement of Lemma 2.1. We shall prove the second identity (2.3). Let e be the first entry of K and let $K' = K \setminus (e)$ be the remaining sequence. If we write $[i_1, \dots, i_r] = \operatorname{Pf} Y((i_1, \dots, i_r) \sqcup K')$, then the left hand side of (2.3) is written as

$$[a, c, d, e] \cdot [b, e] - [a, b, d, e] \cdot [c, e] + [a, b, c, e] \cdot [d, e].$$

By using Lemma 2.1, we have

$$\begin{aligned} [a, c, d, e] \cdot [\emptyset] &= [a, c] \cdot [d, e] - [a, d] \cdot [c, e] + [a, e] \cdot [c, d], \\ [a, b, d, e] \cdot [\emptyset] &= [a, b] \cdot [d, e] - [a, d] \cdot [b, e] + [a, e] \cdot [b, d], \\ [a, b, c, e] \cdot [\emptyset] &= [a, b] \cdot [c, e] - [a, c] \cdot [b, e] + [a, e] \cdot [b, c], \end{aligned}$$

where $[\emptyset] = \operatorname{Pf} Y(K')$. Hence the left hand side of (2.3) is equal to

$$\frac{1}{[\emptyset]} \cdot [a, e] \left([b, c] \cdot [d, e] - [b, d] \cdot [c, e] + [b, e] \cdot [c, d] \right),$$

which turns out to be equal to $[a, e] \cdot [b, c, d, e] = \operatorname{Pf} Y((a) \sqcup K) \cdot \operatorname{Pf} Y((b, c, d) \sqcup K)$ again by using Lemma 2.1. $\square$

Proof of Theorem 1.4. For three sequences I, J and K, we denote by $Z(I, J, K)$ the $(l + m) \times (l + m)$ skew-symmetric matrix given by

$$Z(I, J, K) = \begin{pmatrix} \left(\operatorname{Pf} Y((i_\alpha, i_\beta) \sqcup J \sqcup K) \right)_{1 \le \alpha < \beta \le l} & \left(\operatorname{Pf} Y((i_\alpha) \sqcup (J \setminus (j_\beta) \sqcup K) \right)_{\substack{1 \le \alpha \le l \\ 1 \le \beta \le m}} \\ & \left(\operatorname{Pf} Y((J \setminus (j_\alpha, j_\beta)) \sqcup K) \right)_{1 \le \alpha < \beta \le m} \end{pmatrix},$$

where l and m are the lengths of I and J respectively. We proceed by induction on l and m.

First we consider the case where $l \ge 4$. In this case, we apply Lemma 2.1 to the matrix $A = \widetilde{Y} = Z(I, J, K)$ with $(i, j, k, l) = (1, 2, 3, 4)$. By using the induction hypothesis on l, we have

$$\begin{aligned} \widetilde{Y}^{\alpha, \beta} &= \operatorname{Pf} Z(I \setminus (i_\alpha, i_\beta), J, K) \\ &= \operatorname{Pf} Y((I \setminus (i_\alpha, i_\beta)) \sqcup K) \cdot \left(\operatorname{Pf} Y(J \sqcup K) \right)^{((l-2)+m)/2-1} \quad (1 \le \alpha < \beta \le 4), \end{aligned}$$

and

$$\begin{aligned} \widetilde{Y}^{1,2,3,4} &= \operatorname{Pf} Z(I \setminus (i_1, i_2, i_3, i_4), J, K) \\ &= \operatorname{Pf} Y((I \setminus (i_1, i_2, i_3, i_4)) \sqcup K) \cdot \left(\operatorname{Pf} Y(J \sqcup K) \right)^{((l-4)+m)/2-1}. \end{aligned}$$

Hence we see that

$$\begin{aligned} &\operatorname{Pf} Z(I, J, K) \\ &= \frac{1}{\operatorname{Pf} Y((I \setminus (i_1, i_2, i_3, i_4)) \sqcup K)} \left\{ \begin{array}{l} \operatorname{Pf} Y((I \setminus (i_1, i_2)) \sqcup K) \cdot \operatorname{Pf} Y((I \setminus (i_3, i_4)) \sqcup K) \\ - \operatorname{Pf} Y((I \setminus (i_1, i_3)) \sqcup K) \cdot \operatorname{Pf} Y((I \setminus (i_2, i_4)) \sqcup K) \\ + \operatorname{Pf} Y((I \setminus (i_1, i_4)) \sqcup K) \cdot \operatorname{Pf} Y((I \setminus (i_2, i_3)) \sqcup K) \end{array} \right\} \\ &\quad \times \left(\operatorname{Pf} Y(J \sqcup K) \right)^{(l+m)/2-1}. \end{aligned}$$

Again by applying Lemma 2.1 to $A = Y(I \sqcup K)$, we obtain the desired formula (1.6) for $Z(I, J, K)$.

Next we consider the case where $m \geq 4$. In this case we apply Lemma 2.1 to the matrix $A = \widetilde{Y} = Z(I, J, K)$ with $(i, j, k, l) = (l+1, l+2, l+3, l+4)$. We see that

$$\begin{aligned} \widetilde{Y}^{l+\alpha,l+\beta} &= Z(I, J \setminus (j_\alpha, j_\beta), (j_\alpha, j_\beta) \sqcup K) \quad \text{if } (\alpha, \beta) = (1,2),\ (2,3) \text{ or } (3,4), \\ \widetilde{Y}^{l+1,l+2,l+3,l+4} &= Z(I, J \setminus (j_1, j_2, j_3, j_4), (j_1, j_2, j_3, j_4) \sqcup K), \end{aligned}$$

and

- $\widetilde{Y}^{l+1,l+3}$ is equal to the matrix obtained from $-Z(I, J \setminus (j_1, j_3), (j_1, j_3) \sqcup K)$ by multiplying the $(l+1)$st row/column (corresponding to j_2) by -1,
- $\widetilde{Y}^{l+2,l+4}$ is equal to the matrix obtained from $-Z(I, J \setminus (j_2, j_4), (j_2, j_4) \sqcup K)$ by multiplying the $(l+2)$nd row/column (corresponding to j_3) by -1,
- $\widetilde{Y}^{l+1,l+4}$ is equal to the matrix obtained from $Z(I, J \setminus (j_1, j_4), (j_1, j_4) \sqcup K)$ by multiplying the $(l+1)$st and $(l+2)$nd rows/columns (corresponding to j_2 and j_3) by -1.

Hence, by using the induction hypothesis on m, we have

$$\begin{aligned} &Z(I, J, K) \\ &= \frac{1}{\operatorname{Pf} Y(I \sqcup (j_1, j_2, j_3, j_4) \sqcup K)} \left\{ \begin{array}{l} \operatorname{Pf} Y(I \sqcup (j_1, j_2) \sqcup K) \cdot \operatorname{Pf} Y(I \sqcup (j_3, j_4) \sqcup K) \\ - \operatorname{Pf} Y(I \sqcup (j_1, j_3) \sqcup K) \cdot \operatorname{Pf} Y(I \sqcup (j_2, j_4) \sqcup K) \\ + \operatorname{Pf} Y(I \sqcup (j_1, j_4) \sqcup K) \cdot \operatorname{Pf} Y(I \sqcup (j_2, j_3) \sqcup K) \end{array} \right\} \\ &\quad \times \left(\operatorname{Pf} Y(J \sqcup K) \right)^{(l+m)/2-1}. \end{aligned}$$

Again by using Lemma 2.1 to $A = Y(I \sqcup (j_1, j_2, j_3, j_4) \sqcup K)$ with $(i, j, k, l) = (l+1, l+2, l+3, l+4)$, we obtain the desired formula (1.6) for $Z(I, J, K)$.

Now it remains to show the cases where

$$(l, m) = (1,1), \quad (2,2), \quad (1,3), \quad (3,1) \quad \text{and} \quad (3,3).$$

The case $(l, m) = (1,1)$ is trivial. The case $(l, m) = (2,2)$ follows from (2.2) and the cases $(l, m) = (1,3)$ and $(3,1)$ follow from (2.3) in Lemma 2.2. Hence it is enough to consider the case $(l, m) = (3,3)$.

Let $l = m = 3$, $I = (a, b, c)$, $J = (d, e, f)$, and write $\langle i_1, \dots, i_m \rangle = \operatorname{Pf} Y((i_1, \dots, i_m) \sqcup K)$. Then we have

$$Z(I, J, K) = \begin{pmatrix} 0 & \langle a,b,d,e,f \rangle & \langle a,c,d,e,f \rangle & \langle a,e,f \rangle & \langle a,d,f \rangle & \langle a,d,e \rangle \\ & 0 & \langle b,c,d,e,f \rangle & \langle b,e,f \rangle & \langle b,d,f \rangle & \langle b,d,e \rangle \\ & & 0 & \langle c,e,f \rangle & \langle c,d,f \rangle & \langle c,d,e \rangle \\ & & & 0 & \langle f \rangle & \langle e \rangle \\ & & & & 0 & \langle d \rangle \\ & & & & & 0 \end{pmatrix}.$$

We apply Lemma 2.1 to this matrix, say W. It follows from (2.3) that

$$\operatorname{Pf} W^{1,2} = \langle c \rangle \cdot \langle d,e,f \rangle, \quad \operatorname{Pf} W^{1,3} = \langle b \rangle \cdot \langle d,e,f \rangle, \quad \operatorname{Pf} W^{2,3} = \langle a \rangle \cdot \langle d,e,f \rangle.$$

Moreover, it follows from (2.2) that

$$\operatorname{Pf} W^{3,4} = \langle a,b,d \rangle \cdot \langle d,e,f \rangle, \quad \operatorname{Pf} W^{2,4} = \langle a,c,d \rangle \cdot \langle d,e,f \rangle, \quad \operatorname{Pf} W^{1,4} = \langle b,c,d \rangle \cdot \langle d,e,f \rangle.$$

Hence, again by using (2.3), we see that

$$\operatorname{Pf} W^{1,2} \cdot \operatorname{Pf} W^{3,4} - \operatorname{Pf} W^{1,3} \cdot \operatorname{Pf} W^{2,4} + \operatorname{Pf} W^{1,4} \cdot \operatorname{Pf} W^{2,3}$$

$$= \langle d,e,f\rangle^2 \cdot (\langle a\rangle \cdot \langle b,c,d\rangle - \langle b\rangle \cdot \langle a,c,d\rangle + \langle c\rangle \cdot \langle a,b,d\rangle)$$
$$= \langle d,e,f\rangle^2 \cdot \langle a,b,c\rangle \cdot \langle d\rangle.$$

Since $\operatorname{Pf} W^{1,2,3,4} = \langle d\rangle$, we conclude that

$$\operatorname{Pf} W = \langle a,b,c\rangle \cdot \langle d,e,f\rangle^2 = \operatorname{Pf} Y((a,b,c) \sqcup K) \cdot \big(\operatorname{Pf} Y((d,e,f) \sqcup K)\big)^2.$$

This completes the proof of Theorem 1.4. □

3. Proof of Theorem 1.2

In this section we derive Theorem 1.2 from Theorem 1.4. Some determinant formulas can be deduced from Pfaffian formulas by using the following fundamental relation between determinants and Pfaffians.

Lemma 3.1. (See e.g. [6].) Suppose that $m+n$ is even. If M is an $m \times n$ matrix, then we have

$$\operatorname{Pf}\begin{pmatrix} O & M \\ -{}^tM & O \end{pmatrix} = \begin{cases} (-1)^{\binom{m}{2}} \det M, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases} \tag{3.1}$$

More generally, if S is an $m \times m$ skew-symmetric matrix and M is an $m \times n$ matrix, then we have

$$\operatorname{Pf}\begin{pmatrix} S & M \\ -{}^tM & O \end{pmatrix} = \begin{cases} (-1)^{\binom{m}{2}} \det M, & \text{if } m = n, \\ 0, & \text{if } m < n. \end{cases} \tag{3.2}$$

Proof of Theorem 1.2. We put

$$A = X\begin{pmatrix} I \\ I' \end{pmatrix}, \quad B = X\begin{pmatrix} I \\ J' \end{pmatrix}, \quad P = X\begin{pmatrix} I \\ K' \end{pmatrix},$$
$$C = X\begin{pmatrix} J \\ I' \end{pmatrix}, \quad D = X\begin{pmatrix} J \\ J' \end{pmatrix}, \quad Q = X\begin{pmatrix} J \\ K' \end{pmatrix},$$
$$R = X\begin{pmatrix} K \\ I' \end{pmatrix}, \quad S = X\begin{pmatrix} K \\ J' \end{pmatrix}, \quad T = X\begin{pmatrix} K \\ K' \end{pmatrix},$$

that is,

$$X = \begin{array}{c} \\ I \\ J \\ K \end{array} \begin{array}{c} \begin{array}{ccc} I' & J' & K' \end{array} \\ \begin{pmatrix} A & B & P \\ C & D & Q \\ R & S & T \end{pmatrix} \end{array},$$

and consider the following $2(p+q+r) \times 2(p+q+r)$ skew-symmetric matrix Y with rows/columns indexed by $I \sqcup I' \sqcup J \sqcup J' \sqcup K \sqcup K'$:

$$Y = \begin{array}{c} \\ I \\ I' \\ J \\ J' \\ K \\ K' \end{array} \begin{array}{c} \begin{array}{cccccc} I & I' & J & J' & K & K' \end{array} \\ \begin{pmatrix} O & A & O & B & O & P \\ -{}^tA & O & -{}^tC & O & -{}^tR & O \\ O & C & O & D & O & Q \\ -{}^tB & O & -{}^tD & O & -{}^tS & O \\ 0 & R & O & S & O & T \\ -{}^tP & O & -{}^tQ & 0 & -{}^tT & O \end{pmatrix} \end{array}.$$

We apply Theorem 1.4 to the sequences $\widetilde{I} = I \sqcup I'$, $\widetilde{J} = J \sqcup J'$ and $\widetilde{K} = K \sqcup K'$.

We compute the entries of the matrix $\widetilde{Y}$ given in Theorem 1.4. By permuting row/columns and then by using (3.1), we see that, for $1 \le \alpha, \beta \le p$, we have

$$\begin{aligned}
&\operatorname{Pf} Y((i_\alpha, i_\beta) \sqcup \widetilde{J} \sqcup \widetilde{K}) \\
&= (-1)^{qr} \operatorname{Pf} \begin{pmatrix} O_{q+r+2} & X \begin{pmatrix} (i_\alpha, i_\beta) \sqcup J \sqcup K \\ J' \sqcup K' \end{pmatrix} \\ -{}^t\left(X \begin{pmatrix} (i_\alpha, i_\beta) \sqcup J \sqcup K \\ J' \sqcup K' \end{pmatrix} \right) & O_{q+r} \end{pmatrix} \\
&= 0, \\
&\operatorname{Pf} Y((i_\alpha, i'_\beta) \sqcup \widetilde{J} \sqcup \widetilde{K}) \\
&= (-1)^{qr+(q+r)} \operatorname{Pf} \begin{pmatrix} O_{q+r+1} & X \begin{pmatrix} (i_\alpha) \sqcup J \sqcup K \\ (i'_\beta) \sqcup J' \sqcup K' \end{pmatrix} \\ -{}^t\left(X \begin{pmatrix} (i_\alpha) \sqcup J \sqcup K \\ (i'_\beta) \sqcup J' \sqcup K' \end{pmatrix} \right) & O_{q+r+1} \end{pmatrix} \\
&= (-1)^{qr+(q+r)} \cdot (-1)^{\binom{q+r+1}{2}} \det X \begin{pmatrix} (i_\alpha) \sqcup J \sqcup K \\ (i'_\beta) \sqcup J' \sqcup K' \end{pmatrix}, \\
&\operatorname{Pf} Y((i'_\alpha, i'_\beta) \sqcup \widetilde{J} \sqcup \widetilde{K}) \\
&= (-1)^{r^2+q(q+r)} \operatorname{Pf} \begin{pmatrix} O_{q+r+2} & -{}^t\left(X \begin{pmatrix} J \sqcup K \\ (i'_\alpha, i'_\beta) \sqcup J' \sqcup K' \end{pmatrix} \right) \\ X \begin{pmatrix} J \sqcup K \\ (i'_\alpha, i'_\beta) \sqcup J' \sqcup K' \end{pmatrix} & O_{q+r} \end{pmatrix} \\
&= 0,
\end{aligned}$$

where O_n denotes the $n \times n$ zero matrix. Similarly, for $1 \le \alpha \le p$ and $1 \le \beta \le q$, we have

$$\begin{aligned}
&\operatorname{Pf} Y((i_\alpha) \sqcup (\widetilde{J} \setminus (j_\beta)) \sqcup \widetilde{K}) = (-1)^{qr+\binom{q+r}{2}} \det X \begin{pmatrix} (i_\alpha) \sqcup (J \setminus (j_\beta)) \sqcup K \\ J' \sqcup K' \end{pmatrix}, \\
&\operatorname{Pf} Y((i_\alpha) \sqcup (\widetilde{J} \setminus (j'_\beta)) \sqcup \widetilde{K}) = 0, \\
&\operatorname{Pf} Y((i'_\alpha) \sqcup (\widetilde{J} \setminus (j_\beta)) \sqcup \widetilde{K}) = 0, \\
&\operatorname{Pf} Y((i'_\alpha) \sqcup (\widetilde{J} \setminus (j'_\beta)) \sqcup \widetilde{K}) = (-1)^{q+qr} \cdot (-1)^{\binom{q+r}{2}} \det X \begin{pmatrix} J \sqcup K \\ (i'_\alpha) \sqcup (J' \setminus (j'_\beta)) \sqcup K' \end{pmatrix},
\end{aligned}$$

and, for $1 \le \alpha, \beta \le q$,

$$\begin{aligned}
&\operatorname{Pf} Y((\widetilde{J} \setminus (j_\alpha, j_\beta)) \sqcup \widetilde{K}) = 0, \\
&\operatorname{Pf} Y((\widetilde{J} \setminus (j_\alpha, j'_\beta)) \sqcup \widetilde{K}) = (-1)^{(q-1)r} \cdot (-1)^{\binom{q+r-1}{2}} \det X \begin{pmatrix} (J \setminus (j_\alpha)) \sqcup K \\ (J' \setminus (j'_\beta)) \sqcup K' \end{pmatrix}, \\
&\operatorname{Pf} Y((\widetilde{J} \setminus (j'_\alpha, j'_\beta)) \sqcup \widetilde{K}) = 0.
\end{aligned}$$

If we put

$$\begin{aligned}
U &= \left(\det X \begin{pmatrix} (i_\alpha) \sqcup J \sqcup K \\ (i'_\beta) \sqcup J' \sqcup K' \end{pmatrix} \right)_{1 \le \alpha, \beta \le p}, \\
V &= \left(\det X \begin{pmatrix} (i_\alpha) \sqcup (J \setminus (j_\beta)) \sqcup K \\ J' \sqcup K' \end{pmatrix} \right)_{1 \le \alpha \le p, 1 \le \beta \le q},
\end{aligned}$$

$$V' = \left(\det X \begin{pmatrix} J \sqcup K \\ (i'_\alpha) \sqcup (J' \setminus (j'_\beta)) \sqcup K' \end{pmatrix} \right)_{1 \le \alpha \le p, 1 \le \beta \le q},$$
$$W = \left(\det X \begin{pmatrix} (J \setminus (j_\alpha)) \sqcup K \\ (J' \setminus (j'_\beta)) \sqcup K' \end{pmatrix} \right)_{1 \le \alpha, \beta \le q},$$

then the matrix $\widetilde{Y}$ in Theorem 1.4 is given by

$$\widetilde{Y} = \begin{pmatrix} O & \varepsilon U & \varepsilon V & O \\ -\varepsilon {}^t U & O & O & (-1)^q \varepsilon V' \\ -\varepsilon {}^t V & O & O & (-1)^{q-1} \varepsilon W \\ O & -(-1)^q \varepsilon {}^t V' & -(-1)^{q-1} \varepsilon {}^t W & O \end{pmatrix},$$

where $\varepsilon = (-1)^{qr + \binom{q+r}{2}}$. By permuting rows/columns and using Lemma 3.1, we have

$$\begin{aligned} \operatorname{Pf} \widetilde{Y} &= (-1)^{q(p+q)} \cdot (-1)^{\binom{p+q}{2}} \det \begin{pmatrix} \varepsilon U & \varepsilon V \\ -(-1)^q \varepsilon {}^t V' & -(-1)^{q-1} \varepsilon {}^t W \end{pmatrix} \\ &= (-1)^{q(p+q) + \binom{p+q}{2} + (p+q)(qr + \binom{q+r}{2})} \cdot (-1)^{q(q+1)} \det \begin{pmatrix} U & V \\ {}^t V' & -{}^t W \end{pmatrix}. \end{aligned}$$

Since we have

$$\operatorname{Pf} Y(\widetilde{I} \sqcup \widetilde{K}) = (-1)^{pr + \binom{p+r}{2}} \det X \begin{pmatrix} I \sqcup K \\ I' \sqcup K' \end{pmatrix},$$
$$\operatorname{Pf} Y(\widetilde{J} \sqcup \widetilde{K}) = (-1)^{qr + \binom{q+r}{2}} \det X \begin{pmatrix} J \sqcup K \\ J' \sqcup K' \end{pmatrix},$$

we obtain

$$\begin{aligned} &(-1)^{q(p+q) + \binom{p+q}{2} + (p+q)(qr + \binom{q+r}{2}) + q(q+1)} \det \widetilde{X} \\ &\quad = (-1)^{pr + \binom{p+r}{2} + (p+q-1)(qr + \binom{q+r}{2})} \det X \begin{pmatrix} I \sqcup K \\ I' \sqcup K' \end{pmatrix} \cdot \left(\det X \begin{pmatrix} J \sqcup K \\ J' \sqcup K' \end{pmatrix} \right)^{p+q-1}. \end{aligned}$$

Now, by noticing the congruence

$$pq + \binom{p+q}{2} + qr + \binom{q+r}{2} + rp + \binom{r+p}{2} \equiv 0 \mod 2,$$

we complete the proof of Theorem 1.2. □

From Theorem 1.2 we can derive the following formula due to Bazin [1].

Corollary 3.2 (Bazin [1]). Let n and p be positive integers such that $p \le n$. If Z is a matrix with columns indexed by $(1, 2, \dots, n)$, and $I = (i_1, \dots, i_p)$, $J = (j_1, \dots, j_p)$, and K are sequences of row indices of length p, p, and $n - p$ respectively, then we have

$$\begin{aligned} &\det \left(\det Z \begin{pmatrix} (i_\alpha) \sqcup (J \setminus (j_\beta)) \sqcup K \\ (1, 2, \dots, n) \end{pmatrix} \right)_{1 \le \alpha, \beta \le p} \\ &\quad = (-1)^{p(p-1)/2} \det Z \begin{pmatrix} I \sqcup K \\ (1, 2, \dots, n) \end{pmatrix} \cdot \left(\det Z \begin{pmatrix} J \sqcup K \\ (1, 2, \dots, n) \end{pmatrix} \right)^{p-1}. \qquad (3.3) \end{aligned}$$

Proof. We apply Theorem 1.2 to the matrix $X = (x_{i,j})$ with rows indexed by $I \sqcup J \sqcup K$ and columns indexed by $(\overline{1}, \dots, \overline{p}, 1, \dots, p, p+1, \dots, n)$, whose entries are given by

$$\begin{cases} x_{i,\overline{j}} = z_{i,j}, & \text{if } 1 \le j \le p, \\ x_{i,j} = z_{i,j}, & \text{if } 1 \le j \le n. \end{cases}$$

Let $I' = (\overline{1}, \dots, \overline{p})$, $J' = (1, \dots, p)$, $K' = (p+1, \dots, n)$. Then we have

$$\begin{aligned}
\det X \begin{pmatrix} (i_\alpha) \sqcup J \sqcup K \\ (\overline{i}) \sqcup J' \sqcup K' \end{pmatrix} &= 0 \quad (1 \le \alpha \le p, 1 \le i \le p), \\
\det X \begin{pmatrix} (i_\alpha) \sqcup (J \setminus (j_\beta)) \sqcup K \\ J' \sqcup K' \end{pmatrix} &= \det Z \begin{pmatrix} (i_\alpha) \sqcup (J \setminus (j_\beta)) \sqcup K \\ (1, 2, \dots, n) \end{pmatrix} \quad (1 \le \alpha, \beta \le p), \\
\det X \begin{pmatrix} J \sqcup K \\ (\overline{i}) \sqcup (J' \setminus (j)) \sqcup K' \end{pmatrix} &= \delta_{i,j} (-1)^{i-1} \det Z \begin{pmatrix} J \sqcup K \\ (1, 2, \dots, n) \end{pmatrix} \quad (1 \le i, j \le p), \\
\det X \begin{pmatrix} I \sqcup K \\ I' \sqcup K' \end{pmatrix} &= \det Z \begin{pmatrix} I \sqcup K \\ (1, 2, \dots, n) \end{pmatrix}, \\
\det X \begin{pmatrix} J \sqcup K \\ J' \sqcup K' \end{pmatrix} &= \det Z \begin{pmatrix} J \sqcup K \\ (1, 2, \dots, n) \end{pmatrix}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\det \begin{pmatrix} O_p & \left(\det Z \begin{pmatrix} (i_\alpha) \sqcup (J \setminus (j_\beta)) \sqcup K \\ (1, 2, \dots, n) \end{pmatrix} \right)_{1 \le \alpha, \beta \le p} \\ \left(\delta_{i,j} (-1)^{i-1} \det Z \begin{pmatrix} J \sqcup K \\ (1, 2, \dots, n) \end{pmatrix} \right)_{1 \le i, j \le p} & * \end{pmatrix} \\
&\qquad = (-1)^p \det Z \begin{pmatrix} I \sqcup K \\ (1, 2, \dots, n) \end{pmatrix} \cdot \left(\det Z \begin{pmatrix} J \sqcup K \\ (1, 2, \dots, n) \end{pmatrix} \right)^{2p-1}.
\end{aligned}$$

By permuting columns in the determinant of the left hand side, we see that the left hand side is equal to

$$\begin{aligned}
&(-1)^{p^2} \det \left(\det Z \begin{pmatrix} (i_\alpha) \sqcup (J \setminus (j_\beta)) \sqcup K \\ (1, 2, \dots, n) \end{pmatrix} \right)_{1 \le \alpha, \beta \le p} \\
&\qquad \times (-1)^{\sum_{i=1}^p (i-1)} \left(\det Z \begin{pmatrix} J \sqcup K \\ (1, 2, \dots, n) \end{pmatrix} \right)^p.
\end{aligned}$$

By cancelling the common factor, we obtain the Bazin formula (3.3). □

Remark 3.3. As is shown in [12], many determinant identities (including (1.1), (1.2) and (3.3)) are easily derived from Turnbull's identity. The generalized Sylvester formula (1.3) too is a special case of Turnbull's identity. For example, if X is a 7×7 matrix and $I = I' = (1, 2, 3)$, $J = J' = (4, 5)$, $K = K' = (6, 7)$, then we can use the tableau notation of [12] to write (1.3) as

$$\begin{array}{|ccccccc|}\hline \boxed{x_1} & 2 & 3 & x_4 & x_5 & x_6 & x_7 \\ \hline 1 & \boxed{x_2} & 3 & x_4 & x_5 & x_6 & x_7 \\ \hline 1 & 2 & \boxed{x_3} & x_4 & x_5 & x_6 & x_7 \\ \hline 1 & 2 & 3 & \boxed{4} & x_5 & x_6 & x_7 \\ \hline 1 & 2 & 3 & x_4 & \boxed{5} & x_6 & x_7 \\ \hline \end{array} = \begin{array}{|ccccccc|}\hline x_1 & x_2 & x_3 & 4 & 5 & x_6 & x_7 \\ \hline 1 & 2 & 3 & x_4 & x_5 & x_6 & x_7 \\ \hline 1 & 2 & 3 & x_4 & x_5 & x_6 & x_7 \\ \hline 1 & 2 & 3 & x_4 & x_5 & x_6 & x_7 \\ \hline 1 & 2 & 3 & x_4 & x_5 & x_6 & x_7 \\ \hline \end{array}$$

where $x_1, \dots, x_7$ are the column vectors of X and $1, \dots, 7$ are the column vectors of the 7×7 identity matrix.

4. Skew generalizations of the Giambelli identity

In this section, we use the generalized Sylvester formula (Theorem 1.2) to obtain a skew generalization of the Giambelli identity to skew Schur functions.

A partition of a nonnegative integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers such that $|\lambda| = \sum_{i \ge 1} \lambda_i = n$. The length, denoted by $l(\lambda)$, of a partition λ is the number of nonzero entries of λ. We sometimes write $\lambda = (\lambda_1, \dots, \lambda_{l(\lambda)})$ by omitting the 0s at the end. We identify a partition λ of n with its Young diagram, which is a left-justified array of n cells with λ_i cells in the ith row. We denote by $\emptyset$ the empty partition $(0, 0, \dots)$ of 0. Given a partition λ, we put

$$p(\lambda) = \#\{i : \lambda_i \ge i\}, \quad \alpha_i = \lambda_i - i, \quad \beta_i = \lambda'_i - i \quad (1 \le i \le p(\lambda)),$$

where λ'_i is the number of cells in the ith column of the Young diagram of λ. Then we write $\lambda = (\alpha_1, \dots, \alpha_{p(\lambda)} | \beta_1, \dots, \beta_{p(\lambda)})$ and call it the Frobenius notation of λ.

Let s_λ be the Schur function corresponding to a partition λ, and $s_{\lambda/\mu}$ the skew Schur function associated to a pair of partitions λ and μ. Note that $s_\lambda = s_{\lambda/\emptyset}$ and $s_{\mu/\mu} = 1$ and that $s_{\lambda/\mu} = 0$ unless $\lambda_i \ge \mu_i$ for $i \ge 1$. (Refer to [13, Chapter I] for details on Schur functions.) Giambelli [2] gave a formula which expresses any Schur function as a determinant of Schur functions of hook shapes $(a|b)$, and Lascoux and Pragacz [11] gave a generalization of the Giambelli identity to skew Schur functions.

Proposition 4.1. (1) (Giambelli [2]) For a partition $\lambda = (\alpha_1, \dots, \alpha_p | \beta_1, \dots, \beta_p)$, we have

$$s_\lambda = \det \left(s_{(\alpha_i | \beta_j)} \right)_{1 \le i,j \le p}. \tag{4.1}$$

(2) (Lascoux, Pragacz [11]) If $\lambda = (\alpha_1, \dots, \alpha_p | \beta_1, \dots, \beta_p)$ and $\mu = (\gamma_1, \dots, \gamma_q | \delta_1, \dots, \delta_q)$ in Frobenius notation, then we have

$$s_{\lambda/\mu} = (-1)^q \det \begin{pmatrix} \left(s_{(\alpha_i|\beta_j)} \right)_{1 \le i,j \le p} & \left(h_{\alpha_i - \gamma_j} \right)_{\substack{1 \le i \le p \\ 1 \le j \le q}} \\ \left(e_{\beta_j - \delta_i} \right)_{\substack{1 \le i \le q \\ 1 \le j \le p}} & O \end{pmatrix}, \tag{4.2}$$

where h_k (respectively e_k) is the kth complete (respectively elementary) symmetric function and $h_k = e_k = 0$ for $k < 0$.

We apply Theorem 1.2 to give another generalization of the Giambelli identity to skew Schur functions. In order to state the identity, we extend the definition of skew Schur functions as follows.

Definition 4.2. Given two nonnegative integer sequences $\alpha = (\alpha_1, \dots, \alpha_p)$ and $\beta = (\beta_1, \dots, \beta_p)$ of the same length p and a partition μ, we define $s_{(\alpha|\beta)/\mu}$ by putting

$$s_{(\alpha|\beta)/\mu} = (-1)^q \det \begin{pmatrix} \left(s_{(\alpha_i|\beta_j)} \right)_{1 \le i,j \le p} & \left(h_{\alpha_i - \gamma_j} \right)_{\substack{1 \le i \le p \\ 1 \le j \le q}} \\ \left(e_{\beta_j - \delta_i} \right)_{\substack{1 \le i \le q \\ 1 \le j \le p}} & O \end{pmatrix},$$

where $\mu = (\gamma_1, \dots, \gamma_q | \delta_1, \dots, \delta_q)$ in Frobenius notation.

If the entries of α (or β) are not distinct, then $s_{(\alpha|\beta)/\mu} = 0$. Otherwise, if σ and $\tau \in S_p$ are permutations such that $\alpha_{\sigma(1)} > \cdots > \alpha_{\sigma(p)}$ and $\beta_{\tau(1)} > \cdots > \beta_{\tau(p)}$, then by (4.2) we have $s_{(\alpha|\beta)/\mu} = \operatorname{sgn}(\sigma\tau) s_{\lambda/\mu}$, where λ is a partition given by the Frobenius notation $\lambda = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)} | \beta_{\tau(1)}, \dots, \beta_{\tau(p)})$. Then we have the following skew generalization of the Giambelli identity.

Theorem 4.3. If two partitions λ and μ are represented as $\lambda = (\alpha_1, \dots, \alpha_p | \beta_1, \dots, \beta_p)$ and $\mu = (\gamma_1, \dots, \gamma_q | \delta_1, \dots, \delta_q)$ in Frobenius notation, then we have

$$\begin{aligned} &s_{\lambda/\mu} \\ &= (-1)^q \det \begin{pmatrix} \left(s_{(\alpha_i, \gamma_1, \dots, \gamma_q | \beta_j, \delta_1, \dots, \delta_q)/\mu} \right)_{1 \le i,j \le p} & \left(s_{(\alpha_i, \gamma_1, \dots, \widehat{\gamma_j}, \dots, \gamma_q | \delta_1, \dots, \delta_q)/\mu} \right)_{\substack{1 \le i \le p \\ 1 \le j \le q}} \\ \left(s_{(\gamma_1, \dots, \gamma_q | \beta_j, \delta_1, \dots, \widehat{\delta_i}, \dots, \delta_q)/\mu} \right)_{\substack{1 \le i \le q \\ 1 \le j \le p}} & O \end{pmatrix}, \end{aligned} \tag{4.3}$$

where the symbol $\widehat{a}$ means that a is removed from the sequence.

If $\mu = \emptyset$, then (4.3) reduces to the Giambelli identity (4.1). Note that the nonzero entries of the determinant (4.3) are of the form $\pm s_{\nu/\mu}$ with ν/μ a border strip, i.e., a connected skew Young diagram containing no 2×2 block of cells. Since these entries are skew Schur functions skewed by a fixed partition μ, our determinant formula (4.3) is not a case of the border strip determinants given by Hamel–Goulden [3].

Proof. Apply Theorem 1.2 to the matrix

$$X = \begin{pmatrix} \left(s_{(\alpha_i|\beta_j)} \right)_{1 \le i,j \le p} & \left(s_{(\alpha_i|\delta_j)} \right)_{\substack{1 \le i \le p \\ 1 \le j \le q}} & \left(h_{\alpha_i - \gamma_j} \right)_{\substack{1 \le i \le p \\ 1 \le j \le q}} \\ \left(s_{(\gamma_i|\beta_j)} \right)_{\substack{1 \le i \le q \\ 1 \le j \le p}} & \left(s_{(\gamma_i|\delta_j)} \right)_{1 \le i,j \le q} & \left(h_{\gamma_i - \gamma_j} \right)_{1 \le i,j \le q} \\ \left(e_{\beta_j - \delta_i} \right)_{\substack{1 \le i \le q \\ 1 \le j \le p}} & \left(e_{\delta_j - \delta_i} \right)_{1 \le i,j \le q} & O \end{pmatrix}$$

with $I = I' = (1, \dots, p)$, $J = J' = (p+1, \dots, p+q)$ and $K = K' = (p+q+1, \dots, p+2q)$. Then by definition we have

$$\det X \begin{pmatrix} (i) \sqcup J \sqcup K \\ (j) \sqcup J \sqcup K \end{pmatrix} = (-1)^q s_{(\alpha_i, \gamma_1, \dots, \gamma_q | \beta_j, \delta_1, \dots, \delta_q)/\mu},$$

$$\det X \begin{pmatrix} (i) \sqcup (J \setminus (p+j)) \sqcup K \\ J \sqcup K \end{pmatrix} = (-1)^q s_{(\alpha_i, \gamma_1, \dots, \widehat{\gamma_j}, \dots, \gamma_q | \delta_1, \dots, \delta_q)/\mu},$$

$$\det X \begin{pmatrix} J \sqcup K \\ (i) \sqcup (J \setminus (p+j)) \sqcup K \end{pmatrix} = (-1)^q s_{(\gamma_1, \dots, \gamma_q | \beta_i, \delta_1, \dots, \widehat{\delta_j}, \dots, \delta_q)/\mu},$$

$$\det X \begin{pmatrix} I \sqcup K \\ I \sqcup K \end{pmatrix} = (-1)^q s_{(\alpha|\beta)/\mu} = (-1)^q s_{\lambda/\mu},$$
$$\det X \begin{pmatrix} J \sqcup K \\ J \sqcup K \end{pmatrix} = (-1)^q s_{(\gamma|\delta)/\mu} = (-1)^q s_{\mu/\mu} = (-1)^q.$$

Since $X \begin{pmatrix} (J \setminus (p+i)) \sqcup K \\ (J \setminus (p+j)) \sqcup K \end{pmatrix}$ is a $(2s-1) \times (2s-1)$ matrix whose bottom-right block is the $s \times s$ zero matrix, we see that

$$\det X \begin{pmatrix} (J \setminus (p+i)) \sqcup K \\ (J \setminus (p+j)) \sqcup K \end{pmatrix} = 0.$$

Hence, by applying Theorem 1.2, we obtain the desired identity. □

Remark 4.4. As is shown in [14], Theorem 4.3 is obtained by using the Giambelli-type determinant formula for the expansion coefficients of the τ-function $\tau(x)$ of the KP hierarchy ([14, Theorem 1.1]) and the fact that $\tau(x) = \sum_\lambda s_{\lambda/\mu}(y) s_\lambda(x)$ is a solution of the KP hierarchy.

5. Skew generalizations of the Schur identity

In this section we use Theorem 1.4 to obtain a skew generalization of the Schur Pfaffian identity for Schur Q-functions.

A partition λ is called strict if $\lambda_1 > \lambda_2 > \cdots > \lambda_{l(\lambda)}$. Let Q_λ be the Schur Q-function corresponding to a strict partition λ, and $Q_{\lambda/\mu}$ the skew Q-function associated with a pair of strict partitions λ and μ. Note that $Q_{\lambda/\emptyset} = Q_\lambda$, $Q_{\mu/\mu} = 1$ and $Q_{\lambda/\mu} = 0$ unless $\lambda_i \geq \mu_i$ for all i. See [13, Chapter III, Section 8] for details on Schur Q-functions.

Schur [15] defined the Schur Q-function corresponding to any strict partition as a Pfaffian of Schur Q-functions corresponding to strict partitions with at most two rows, and Józefiak and Pragacz [9] gave a generalization of Schur's identity to skew Q-functions.

Proposition 5.1. For sequences $\alpha = (\alpha_1, \dots, \alpha_l)$ and $\beta = (\beta_1, \dots, \beta_m)$ of nonnegative integers, let S^α_β and T^α_β be $l \times m$ matrices defined by

$$S^\alpha_\beta = \left(Q_{(\alpha_i, \beta_j)} \right)_{1 \leq i \leq l, 1 \leq j \leq m}, \quad T^\alpha_\beta = \left(Q_{(\alpha_i - \beta_{m+1-j})} \right)_{1 \leq i \leq l, 1 \leq j \leq m},$$

where we use the conventions

$$Q_{(a,b)} = -Q_{(b,a)}, \quad Q_{(a,0)} = -Q_{(0,a)} = Q_{(a)}, \quad Q_{(0,0)} = 0$$

for positive integers a and b, and $Q_{(a)} = 0$ for $a < 0$. Then we have the following Pfaffian identities:

(1) (Schur [15]) If λ is a strict partition, then we have

$$Q_\lambda = \begin{cases} \operatorname{Pf} S^\lambda_\lambda, & \text{if } l(\lambda) \text{ is even}, \\ \operatorname{Pf} S^{\lambda^0}_{\lambda^0}, & \text{if } l(\lambda) \text{ is odd}, \end{cases} \tag{5.1}$$

where $\lambda = (\lambda_1, \dots, \lambda_{l(\lambda)})$ and $\lambda^0 = (\lambda_1, \dots, \lambda_{l(\lambda)}, 0)$.

(2) (Józefiak, Pragacz [9]) For two strict partitions λ and μ, we have

$$Q_{\lambda/\mu} = \begin{cases} \operatorname{Pf} \begin{pmatrix} S^\lambda_\lambda & T^\lambda_\mu \\ -{}^tT^\lambda_\mu & O \end{pmatrix}, & \text{if } l(\lambda) + l(\mu) \text{ is even,} \\ \operatorname{Pf} \begin{pmatrix} S^\lambda_\lambda & T^\lambda_{\mu^0} \\ -{}^tT^\lambda_{\mu^0} & O \end{pmatrix}, & \text{if } l(\lambda) + l(\mu) \text{ is odd.} \end{cases} \tag{5.2}$$

We find another generalization of the Schur identity (5.1) to skew Q-functions. To state the identity, we extend the definition of skew Q-functions as follows.

Definition 5.2. Given a nonnegative integer sequence α of length l and a strict partition μ, we define $Q_{\alpha/\mu}$ by putting

$$Q_{\alpha/\mu} = \begin{cases} \operatorname{Pf} \begin{pmatrix} S^\alpha_\alpha & T^\alpha_\mu \\ -{}^tT^\alpha_\mu & O \end{pmatrix}, & \text{if } l + l(\mu) \text{ is even,} \\ \operatorname{Pf} \begin{pmatrix} S^\alpha_\alpha & T^\alpha_{\mu^0} \\ -{}^tT^\alpha_{\mu^0} & O \end{pmatrix}, & \text{if } l + l(\mu) \text{ is odd.} \end{cases}$$

We note that, if $l + l(\mu)$ is odd, then we have

$$\begin{pmatrix} S^\alpha_\alpha & T^\alpha_{\mu^0} \\ -{}^tT^\alpha_{\mu^0} & O \end{pmatrix} = \begin{pmatrix} S^{\alpha^0}_{\alpha^0} & T^{\alpha^0}_\mu \\ -{}^tT^{\alpha^0}_\mu & O \end{pmatrix},$$

where $\alpha^0 = (\alpha_1, \dots, \alpha_l, 0)$. If the entries of α are not distinct, then $Q_{\alpha/\mu} = 0$. Otherwise, if $\sigma \in S_l$ is a permutation such that $\alpha_{\sigma(1)} > \cdots > \alpha_{\sigma(l)}$, then $Q_{\alpha/\mu} = \operatorname{sgn}(\sigma) Q_{\lambda/\mu}$ with $\lambda = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(l)})$. Then we have the following skew-generalization of Schur identity (5.1).

Theorem 5.3. For strict partitions λ and μ, we have

$$Q_{\lambda/\mu} = \operatorname{Pf} \begin{pmatrix} \left(Q_{(\lambda_i, \lambda_j, \mu_1, \dots, \mu_m)/\mu} \right)_{1 \le i,j \le l} & \left(Q_{(\lambda_i, \mu_1, \dots, \widehat{\mu_j}, \dots, \mu_s)/\mu} \right)_{\substack{1 \le i \le l \\ 1 \le j \le s}} \\ -{}^t\left(Q_{(\lambda_i, \mu_1, \dots, \widehat{\mu_j}, \dots, \mu_s)/\mu} \right)_{\substack{1 \le i \le l \\ 1 \le j \le s}} & O \end{pmatrix}, \tag{5.3}$$

where $l = l(\lambda)$, $m = l(\mu)$ and $s = m$ or $m + 1$ according to whether $l + m$ is even or odd.

If $\mu = \emptyset$, then (5.3) reduces to the Schur identity (5.1).

Proof. We put $l = l(\lambda)$ and $m = l(\mu)$.

First we consider the case where $l \equiv m \bmod 2$. In this case we apply Theorem 1.4 to the $(l + 2m) \times (l + 2m)$ skew-symmetric matrix

$$Y = \begin{pmatrix} S^\lambda_\lambda & S^\lambda_\mu & T^\lambda_\mu \\ -{}^tS^\lambda_\mu & S^\mu_\mu & T^\mu_\mu \\ -{}^tT^\lambda_\mu & -{}^tT^\mu_\mu & O \end{pmatrix},$$

with $I = (1, \dots, l)$, $J = (l + 1, \dots, l + m)$ and $K = (l + m + 1, \dots, l + 2m)$. Then by definition we have

$$\operatorname{Pf} Y((i,j) \sqcup J \sqcup K) = Q_{(\lambda_i, \lambda_j, \mu_1, \dots, \mu_m)/\mu},$$
$$\operatorname{Pf} Y((i) \sqcup (J \setminus (l+j)) \sqcup K) = Q_{(\lambda_i, \mu_1, \dots, \widehat{\mu_j}, \dots, \mu_m)/\mu}.$$

Since $Y((J \setminus (l+i, l+j)) \sqcup K)$ is a $(2m-2) \times (2m-2)$ skew-symmetric matrix whose bottom-right block is the $m \times m$ zero matrix, we have $\operatorname{Pf} Y((J \setminus (l+i, l+j)) \sqcup K) = 0$ by (3.2). Also we have

$$\operatorname{Pf} Y(I \sqcup K) = Q_{\lambda/\mu}, \quad \operatorname{Pf} Y(J \sqcup K) = Q_{\mu/\mu} = 1.$$

Hence, by applying (1.6), we obtain (5.3).

Next we consider the case where $l \not\equiv m \bmod 2$. In this case, we apply Theorem 1.4 to the $(l+1+2m) \times (l+1+2m)$ skew-symmetric matrix

$$Y = \begin{pmatrix} S^{\lambda^0}_{\lambda^0} & S^{\lambda^0}_{\mu} & T^{\lambda^0}_{\mu} \\ -{}^tS^{\lambda^0}_{\mu} & S^{\mu}_{\mu} & T^{\mu}_{\mu} \\ -{}^tT^{\lambda^0}_{\mu} & -{}^tT^{\mu}_{\mu} & O \end{pmatrix},$$

with $I = (1, \dots, l+1)$, $J = (l+2, \dots, l+m+1)$ and $K = (l+m+2, \dots, l+2m+1)$ and $\lambda_{l+1} = 0$. Then by definition we have

$$\operatorname{Pf} Y((i,j) \sqcup J \sqcup K) = Q_{(\lambda_i, \lambda_j, \mu_1, \dots, \mu_m)/\mu} \quad \text{if } 1 \le i < j \le l,$$
$$\operatorname{Pf} Y((i) \sqcup (J \setminus (l+1+j)) \sqcup K) = Q_{(\lambda_i, \mu_1, \dots, \widehat{\mu_j}, \dots, \mu_m)/\mu} \quad \text{if } 1 \le i \le l \text{ and } 1 \le j \le m.$$

If $1 \le i \le l$, then by moving the 2nd column/row to the $(m+1)$st column/row we see that

$$\begin{aligned}\operatorname{Pf} Y((i, l+1) \sqcup J \sqcup K) &= (-1)^m \operatorname{Pf} \begin{pmatrix} S^{(\lambda_i, \mu_1, \dots, \mu_m)}_{(\lambda_i, \mu_1, \dots, \mu_m)} & T^{(\lambda_i, \mu_1, \dots, \mu_m)}_{(\mu_1, \dots, \mu_m, 0)} \\ -{}^tT^{(\lambda_i, \mu_1, \dots, \mu_m)}_{(\mu_1, \dots \mu_m, 0)} & O \end{pmatrix} \\ &= (-1)^m Q_{(\lambda_i, \mu_1, \dots, \mu_m)/\mu}.\end{aligned}$$

If $1 \le j \le m$, then by moving the 1st row/column to the mth row/column and by noting $Q_{(\lambda_{l+1} - \mu_k)} = 0$ for $1 \le k \le m$, we have

$$\operatorname{Pf} Y((l+1) \sqcup (J \setminus (l+1+j)) \sqcup K) = (-1)^{m-1} \operatorname{Pf} \begin{pmatrix} S^{(\mu_1, \dots, \widehat{\mu_j}, \dots, \mu_m)}_{(\mu_1, \dots, \widehat{\mu_j}, \dots, \mu_m)} & T^{(\mu_1, \dots, \widehat{\mu_j}, \dots, \mu_m)}_{(\mu_1, \dots, \mu_m, 0)} \\ -{}^tT^{(\mu_1, \dots, \widehat{\mu_j}, \dots, \mu_m)}_{(\mu_1, \dots, \mu_m, 0)} & O_{m+1} \end{pmatrix}.$$

Hence by using (3.2), we have $\operatorname{Pf} Y((l+1) \sqcup (J \setminus (l+1+j)) \sqcup K) = 0$. Also we have

$$\operatorname{Pf} Y(I \sqcup K) = Q_{\lambda/\mu}, \quad \operatorname{Pf} Y(J \sqcup K) = Q_{\mu/\mu} = 1.$$

Therefore by applying (1.6), we obtain

$$Q_{\lambda/\mu} = \operatorname{Pf} \widetilde{Y},$$

where the entries of the skew-symmetric matrix $\widetilde{Y} = \left(\widetilde{y}_{i,j}\right)_{1 \le i,j \le l+m+1}$ are given by

$$\begin{cases} \widetilde{y}_{i,j} = Q_{(\lambda_i, \lambda_j, \mu_1, \dots, \mu_m)/\mu}, & \text{if } 1 \le i, j \le l, \\ \widetilde{y}_{i,l+1} = (-1)^m Q_{(\lambda_i, \mu_1, \dots, \mu_m)/\mu}, & \text{if } 1 \le i \le l, \\ \widetilde{y}_{i,l+1+j} = Q_{(\lambda_i, \mu_1, \dots, \widehat{\mu_j}, \dots, \mu_m)/\mu}, & \text{if } 1 \le i \le l \text{ and } 1 \le j \le m, \\ \widetilde{y}_{i,j} = 0, & \text{if } l+1 \le i, j \le l+m+1. \end{cases}$$

By pulling out the common factor $(-1)^m$ from the $(l+1)$st row/column and moving the $(l+1)$st row/column to the last row/column, we see that

$$\operatorname{Pf} \widetilde{Y} = (-1)^m \cdot (-1)^m$$

$$\times \operatorname{Pf} \begin{pmatrix} \left(Q_{(\lambda_i,\lambda_j,\mu_1,\dots,\mu_m)/\mu} \right)_{1 \le i,j \le l} & \left(Q_{(\lambda_i,\mu_1,\dots,\widehat{\mu_j},\dots,\mu_{m+1})/\mu} \right)_{\substack{1 \le i \le l \\ 1 \le j \le m+1}} \\ -{}^t\!\left(Q_{(\lambda_i,\mu_1,\dots,\widehat{\mu_j},\dots,\mu_{m+1})/\mu} \right)_{\substack{1 \le i \le l \\ 1 \le j \le m+1}} & O \end{pmatrix}.$$

This completes the proof of Theorem 5.3. □

Remark 5.4. (This remark is due to A. Nakayashiki.) Theorem 5.3 can be also obtained from the theory of the BKP hierarchy by using the same idea as in [14] (see Remark 4.4). We fix a strict partition μ and consider a formal power series of the form

$$\tau(x) = Q_\mu \left(\frac{x}{2} \right) + \sum_\lambda \xi_\lambda Q_\lambda \left(\frac{x}{2} \right),$$

where λ runs over strict partitions with $|\lambda| > |\mu|$ and $x = (x_1, x_3, x_5, \dots)$ is the so-called Sato variables, i.e., $x_i = p_i/i$ in the symmetric function language. Then Shigyo [16, Theorem 3] proves that $\tau(x)$ is a solution of the BKP hierarchy if and only if the coefficients ξ_λ satisfy the following Pfaffian formulas:

$$\xi_\lambda = \operatorname{Pf} \begin{pmatrix} \left(\xi_{(\lambda_i,\lambda_j,\mu_1,\dots,\mu_m)} \right)_{1 \le i,j \le l} & \left(\xi_{(\lambda_i,\mu_1,\dots,\widehat{\mu_j},\dots,\mu_s)} \right)_{\substack{1 \le i \le l \\ 1 \le j \le s}} \\ -{}^t\!\left(\xi_{(\lambda_i,\mu_1,\dots,\widehat{\mu_j},\dots,\mu_s)} \right)_{\substack{1 \le i \le l \\ 1 \le j \le s}} & O \end{pmatrix},$$

where $l = l(\lambda)$, $m = l(\mu)$, and $s = m$ or $m+1$ according to whether $l+m$ is even or odd, and we use the convention that $\xi_{(\alpha_1,\dots,\alpha_p)}$ is alternating in $\alpha_1, \dots, \alpha_p$. By using the fact that $Q_\mu(x/2)$ is a solution of the BKP hierarchy [18], we can show that

$$\tau(x) = \sum_\lambda Q_{\lambda/\mu}(y) Q_\lambda \left(\frac{x}{2} \right) = Q_\mu \left(\frac{x}{2} \right) \exp \left(\sum_n n x_n y_n \right),$$

where n runs over all positive odd integers and $y = (y_1, y_3, y_5, \dots)$ is another set of variables, is a solution of the BKP hierarchy. Applying Shigyo's formula to this special solution $\tau(x)$, we obtain the generalized Schur identity (5.3).

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