# Character restrictions and hook removal operators on the odd Young graph

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March 26, 2018

# Irreducible complex characters of $\mathfrak{S}_n$

*Partitions* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of $n$ (written as $\lambda \vdash n$),

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ are integers with $\sum_{i=1}^{\ell} \lambda_i = n$.

**Theorem (Frobenius)**

*The irreducible complex characters of $\mathfrak{S}_n$ are naturally labelled by partitions of $n$,*

$$\mathrm{Irr}_{\mathbb{C}}(\mathfrak{S}_n) = \{\chi^\lambda \mid \lambda \vdash n\}.$$

**Remark** Connection to symmetric functions:
Via the Frobenius characteristic, there is a correspondence

$$\chi^\lambda \longleftrightarrow s_\lambda \quad \text{(Schur function)}.$$

# Character restriction from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$

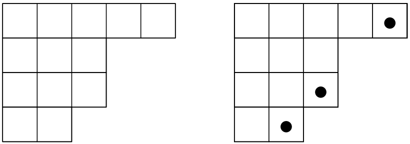
**Theorem (Branching formula)**

*For $\lambda \vdash n$,*

$$\chi^\lambda \downarrow_{\mathfrak{S}_{n-1}} = \sum_A \chi^{\lambda \setminus A}$$

*where the sum runs over the removable corners $A$ of the Young diagram of $\lambda$.*

$\lambda = (5, 3, 3, 2)$



$$\chi^{(5,3,3,2)} \downarrow_{\mathfrak{S}_{12}} = \chi^{(4,3,3,2)} + \chi^{(5,3,2,2)} + \chi^{(5,3,3,1)}$$

# Odd degree characters

Let $\lambda \vdash n$.

$$\chi^\lambda(\mathrm{id}) = f^\lambda = \#\{\text{standard Young tableaux of shape } \lambda\}.$$

If $f^\lambda$ is odd, we call $\lambda$ an *odd partition*, for short: $\lambda \vdash_o n$.
What is the number of odd partitions? I.e., we consider

$$\mathcal{O}(n) = \{\lambda \vdash n \mid f^\lambda \text{ odd } \}$$

or equivalently, $\mathrm{Irr}_{2'}(\mathfrak{S}_n) = \{\chi^\lambda \in \mathrm{Irr}(\mathfrak{S}_n) \mid 2 \nmid f^\lambda\}$

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**Theorem (Macdonald 1971, McKay 1972)**

*Let $n = 2^{a_1} + \ldots + 2^{a_r}$ with $a_1 < \ldots < a_r$. Then*

$$\#\{\lambda \vdash n \mid f^\lambda \text{ odd }\} = 2^{a_1+\ldots+a_r}.$$

# Odd degree characters and hooks

Denote by $\chi^\lambda(\mu)$ the value of $\chi^\lambda$ on elements of cycle type $\mu$. Observe that for $n = 2^{a_1} + \ldots + 2^{a_r}$ as above,

$$\chi^\lambda(1^n) \equiv \chi^\lambda(2^{a_1}, \ldots, 2^{a_r}) \mod 2.$$

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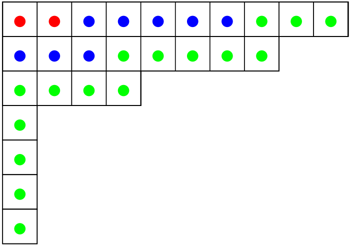
For example, for $n = 26 = 2 + 8 + 16$:



$$\mu = (10, 8, 4, 1^4) \in \mathcal{O}(26)$$

# McKay's Conjecture

Let $G$ be a finite group, $p$ a prime; set

$$k_{p'}(G) = \#\{\chi \in \mathrm{Irr}(G) \mid p \nmid \chi(1)\}.$$

**Conjecture (McKay)**

*Let $P$ be a Sylow $p$-subgroup of $G$. Then*

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**Remark** For $G = \mathfrak{S}_n$, $P_n \in \mathrm{Syl}_2(\mathfrak{S}_n)$, a canonical bijection

$$\mathrm{Irr}_{2'}(\mathfrak{S}_n) \to \mathrm{Irr}_{2'}(N_{\mathfrak{S}_n}(P_n)) = \mathrm{Lin}(P_n)$$

was constructed by [Giannelli, Kleshchev, Navarro, Tiep 2016].

# Branching revisited

For $\lambda \in \mathcal{O}(n)$, what can we say about constituents of $\chi^\lambda \downarrow_{\mathfrak{S}_{n-1}}$ and of $\chi^\lambda \uparrow^{\mathfrak{S}_{n+1}}$ of odd degree?

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Let $\nu_2(n)$ be the exponent of the highest power of 2 dividing $n$.

**Theorem (Ayyer, Prasad, Spallone, SLC 2015)**

*Let $\lambda \in \mathcal{O}(n)$. Let $m = \nu_2(n+1)$.*

1. $\chi^\lambda \downarrow_{\mathfrak{S}_{n-1}}$ *has exactly one constituent of odd degree.*
2. $\chi^\lambda \uparrow^{\mathfrak{S}_{n+1}}$ *has exactly two constituents of odd degree if the $2^m$-core of $\lambda$ is a hook, and no such constituent otherwise.*

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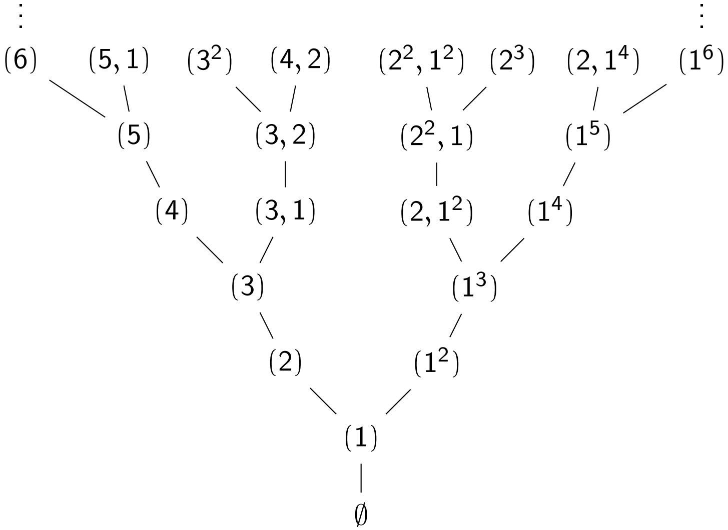
**Example** $\lambda = (5,3,3,2)$.

| | $\chi^{(5,3,3,2)} \downarrow_{\mathfrak{S}_{12}}$ | $=$ | $\chi^{(4,3,3,2)}$ | $+$ | $\chi^{(5,3,2,2)}$ | $+$ | $\chi^{(5,3,3,1)}$ |
|---|---|---|---|---|---|---|---|
| degree: | 11583 | | 2970 | | 4455 | | 4158 |

# The odd Young graph



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*Let $k \in \mathbb{N}$ such that $2^k < n$, and let $\chi \in \mathrm{Irr}_{2'}(\mathfrak{S}_n)$.*
*Then there exists a unique odd-degree irreducible constituent $f_k^n(\chi)$ of $\chi_{\mathfrak{S}_{n-2^k}}$ appearing with* **odd multiplicity**.

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*Then there exists a unique odd-degree irreducible constituent $f_k^n(\chi)$ of $\chi_{\mathfrak{S}_{n-2^k}}$ appearing with* **odd multiplicity**.

Thus, for $2^k < n$ we have a naturally defined map

$$f_k^n\colon \ \mathrm{Irr}_{2'}(\mathfrak{S}_n) \longrightarrow \mathrm{Irr}_{2'}(\mathfrak{S}_{n-2^k}).$$

# More general restrictions - combinatorial

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Let $\lambda \in \mathcal{O}(n)$, $\mu \in \mathcal{O}(n-2^k)$. Then

$$f_k^n(\chi^\lambda) = \chi^\mu \text{ if and only if } \mu = \lambda \setminus H \text{ for a } 2^k\text{-hook } H.$$

In fact, there is a *unique* $2^k$-hook $H$ of $\lambda$ such that $\lambda \setminus H$ is odd. Correspondingly, we write

$$f_k^n \colon \mathcal{O}(n) \longrightarrow \mathcal{O}(n-2^k),\ \lambda \mapsto \mu = \lambda \setminus H.$$

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**Example**

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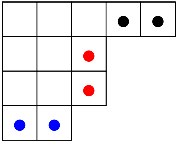
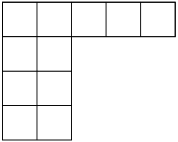
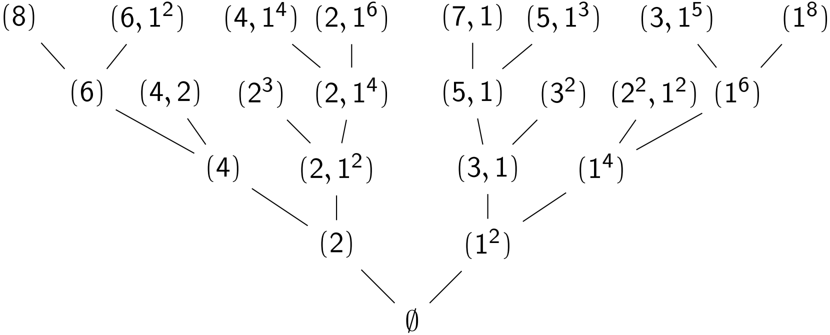




$f_1^{13}(\lambda) = (5,2,2,2)$

# $f_1$ on the odd Young graph




**Remark** $f_1 \neq f_0 f_0$!
For example,

$$f_1((31)) = (1^2) \neq (2) = f_0 f_0((3,1)).$$

# Hook removal on the odd Young graph

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**Question (INOT 2017)**

*Let $k \leq \ell$ be such that $2^k + 2^\ell \leq n$.*
*When do the hook removal operators $f_k$ and $f_\ell$ commute?*
*More specifically: when is $f_k^{n-2^\ell} f_\ell^n = f_\ell^{n-2^k} f_k^n$?*

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**Remark** Not always! But, for example, if $2^\ell$ is the largest binary digit of $n$.

# 2-cores and 2-quotients: 2-data

Let $\lambda \vdash n$. Construct the 2-*quotient tower* $\mathcal{Q}_2(\lambda)$:

Row 0: $\mathcal{Q}_2^{(0)}(\lambda) = (\lambda_1^{(0)}) = (\lambda)$.

Row $k$: $\mathcal{Q}_2^{(k)}(\lambda) = (\lambda_1^{(k)}, \ldots, \lambda_{2^k}^{(1)})$ contains the partitions in the 2-quotients of the partitions in $\mathcal{Q}_2^{(k-1)}(\lambda)$ (the same as those in the $2^k$-quotient $Q_{2^k}(\lambda)$ of $\lambda$).

2-*core tower* $\mathcal{C}_2(\lambda)$:

$j \geq 0$: $\mathcal{C}_2^{(j)}(\lambda) = (C_2(\lambda_1^{(j)}), \ldots, C_2(\lambda_{2^k}^{(j)}))$, the 2-cores of the $\lambda_i^{(j)}$.

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$k$-*data* $\mathcal{D}_2^{(k)}(\lambda)$: row $j < k$ equals $\mathcal{C}_2^{(j)}(\lambda)$, row $k$ equals $\mathcal{Q}_2^{(k)}(\lambda)$.

**Theorem (BGO 2017)**

*Fix $k \geq 0$. Then $\lambda$ is odd if and only if for $j < k$ the sum over all partitions in $\mathcal{C}_2^{(j)}(\lambda)$ is $\leq 1$, and the partitions in $\mathcal{Q}_2^{(k)}$ are odd and their sizes are pairwise 2-disjoint (i.e., in binary expansion).*

# Example

$\lambda = (5, 4, 2^2, 1^2),\ k = 2$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\mathcal{Q}_2^{(0)}(\lambda)$: | | | $\lambda$ | | | $\mathcal{C}_2^{(0)}(\lambda)$: | $(1)$ |
| $\mathcal{Q}_2^{(1)}(\lambda)$: | $(2^2, 1^2)$ | | | $(1)$ | | $\mathcal{C}_2^{(1)}(\lambda)$: | $(0)$ $(1)$ |
| $\mathcal{Q}_2^{(2)}(\lambda)$: | $(1^2)$ | $(1)$ | $(0)$ | $(0)$ | | | |

$\mathcal{D}_2^{(2)}(\lambda)$:

| | | | | |
|---|---|---|---|---|
| $\mathcal{C}_2^{(0)}(\lambda)$: | | $(1)$ | | |
| $\mathcal{C}_2^{(1)}(\lambda)$: | $(0)$ | | $(1)$ | |
| $\mathcal{Q}_2^{(2)}(\lambda)$: | $(1^2)$ | $(1)$ | $(0)$ | $(0)$ |

**Remark** Removal of a $2^k$-hook from $\lambda$ is equivalent to removing a box from one of the partitions in row $k$ of $\mathcal{D}_2^{(k)}(\lambda)$.

# Walking around the odd Young graph

For an odd partition $\mu$, we define the set of odd extensions:

$$\mathcal{E}(\mu, 2^k) = \{\lambda \vdash_o n \mid f_k^n(\lambda) = \mu\}, \ e(\mu, 2^k) = \#\mathcal{E}(\mu, 2^k).$$

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**Theorem (B., Giannelli, Olsson 2017)**

*Assume $2^k < n$. Let $d = d(n,k) = \nu_2\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right)$. Let $\mu \vdash_o n - 2^k$. Then $e(\mu, 2^k) \neq 0$ if and only if there is a partition $\mu_i^{(k)}$ in $\mathcal{Q}_2^{(k)}(\mu)$ such that*

- $\left|\mu_i^{(k)}\right| \equiv 2^d - 1 \mod 2^{d+1}$, *and*
- $C_{2^d}\left(\mu_i^{(k)}\right)$ *is a hook partition.*

*In this case, $e(\mu, 2^k) = 2^k$ if $d = 0$, and $e(\mu, 2^k) = 2$ if $d > 0$.*

# Properties of the hook removal operators

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1. *If $k = 0$ then $f_k^n$ is surjective if and only $d(n,k) \leq 2$.*
   *If $k > 0$ then $f_k^n$ is surjective if and only $d(n,k) \leq 1$.*
2. *The map $f_k^n$ is injective if and only if $k = 0$ and $n$ is odd.*
   *In this case, the map $f_k^n$ is bijective.*

# Commutativity of hook removal operators

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**Theorem (B., Giannelli, Olsson 2017)**

*Let $n = 2^t + m$ where $0 \leq m < 2^t$.*
*Assume $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$.*
*Then,* with the exception of the case $n = 6$, $k = 0$, $\ell = 1$,

$$f_k f_\ell = f_\ell f_k \text{ if and only if } 2^k > m \text{ or } \ell = t.$$

**Remark** Explicit description of the set

$$T_{k,\ell}(n) = \{\lambda \vdash_o n \mid f_k f_\ell(\lambda) = f_\ell f_k(\lambda)\},$$

and counting formula for $\# T_{k,\ell}(n)$ [BGO 2017].

# Compatibility of hook removal operators

Let $k \in \mathbb{N}_0$ be such that $2^{k+1} \leq n$.
We define

$$\mathcal{G}_k(n) = \{\lambda \vdash_o n \mid f_k f_k(\lambda) = f_{k+1}(\lambda)\}\,, G_k(n) = |\mathcal{G}_k(n)|.$$

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Write $n = 2^{a_1} + \ldots + 2^{a_r}$, where $a_1 > a_2 > \ldots > a_r$.
Define $p, q \leq r$ to be maximal with $a_p \geq k+1$ and $a_q \geq k$, resp.
For $J \subseteq I = \{1, \ldots, q\}$ define its *G-weight*

$$w_G(J) = (\prod_{i \in I \setminus J} 2^{a_i - k}) \cdot (2^k - 1)^{|I \setminus J|} \cdot G_0(\sum_{j \in J} 2^{a_j - k}).$$

# Counting formulae for $G_0$

$G_0(n)$ and hence the $G$-weights can be computed explicitly:

**Theorem (BGO 2017)**

1. $G_0(1) = 1,\ G_0(2) = 2 = G_0(3)$.
2. *For* $t > 1$, $G_0(2^t) = 2^{t-1}$ *and* $G_0(2^t + 1) = 4$.
3. *Let* $n = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_r} + \varepsilon \geq 2$, *where* $a_1 > \cdots > a_r > 0$ *and* $\varepsilon \in \{0, 1\}$. *Then*

$$G_0(n) = G_0(2^{a_r} + \varepsilon) \cdot \prod_{j=1}^{r-1} (2^{a_j} - 2).$$

# Counting formulae for $G_k$

As before, $n = 2^{a_1} + \ldots + 2^{a_r}$, where $a_1 > a_2 > \ldots > a_r$, and $p, q \leq r$ are maximal with $a_p \geq k+1$ and $a_q \geq k$, respectively.

**Theorem (BGO 2017)**

*For $k > 0$,*

$$G_k(n) = \Big(\prod_{j=q+1}^{r} 2^{a_j}\Big) \cdot 2^k \cdot \sum_{\{p,q\}\subseteq J\subseteq I} w_G(J)\,.$$

**Corollary**

*Assume $2^{k+1} \leq n$.*
*For $k > 0$, $\mathcal{G}_k(n) = \mathcal{O}(n)$ if and only if $\lfloor n/2^k \rfloor = 2$.*
*For $k = 0$, $\mathcal{G}_0(n) = \mathcal{O}(n)$ only holds for $n \in \{2, 3, 5\}$.*

**Remark** Explicit description of $\mathcal{G}_k(n)$ [BGO 2017].

# The end

Thank you!