# Schützenberger's jeu de taquin

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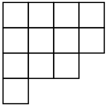
Conference dedicated to
the scientific legacy of Marcel-Paul Schützenberger
Bordeaux, March 22nd, 2016

# Outline

(Straight) Young diagram: set of boxes packed in the north-west corner.



(Straight) Young tableau: filling of a Young diagram

- weakly increasing along rows;
- strictly increasing along columns.

| | | | |
|---|---|---|---|
| 1 | 1 | 2 | 4 |
| 2 | 3 | 3 | 5 |
| 4 | 4 | 5 | |
| 6 | | | |

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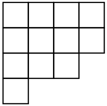

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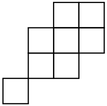
Skew diagram: difference set between two Young diagrams



Skew tableau: filling of a skew diagram with the same conditions

| | | | |
|---|---|---|---|
| | | 2 | 4 |
| | 3 | 3 | 5 |
| | 4 | 5 | |
| 2 | | | |

# Jeu de taquin

Start with a skew tableau and color in red one of its *inner corner*.

| | | | |
|---|---|---|---|
| | | 2 | 4 |
| | 3 | 3 | 5 |
| | 4 | 5 | |
| 2 | | | |

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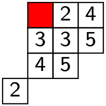
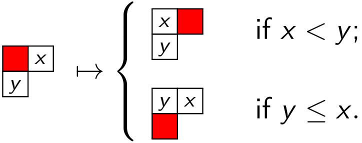



Move the red box according to the following rule:




until the red box is outside the diagram; then erase the red box.

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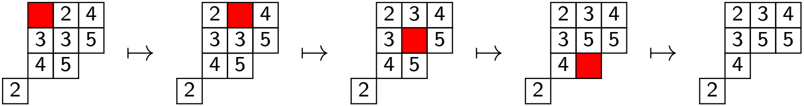


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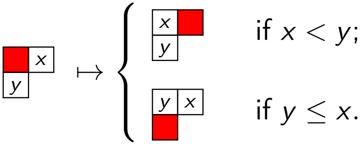


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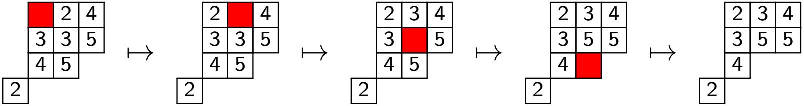


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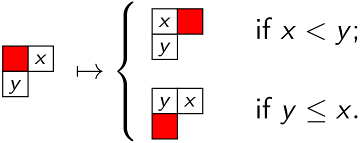
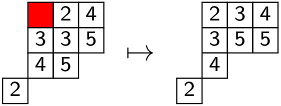



until the red box is outside the diagram; then erase the red box.

$\rightarrow$ The rule ensures that we keep a tableau.

# Rectification of a skew-tableau

Start with a skew tableau $T$. Successively choose corners and apply jeu de taquin until you get a tableau.



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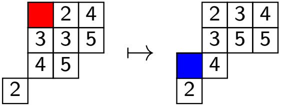

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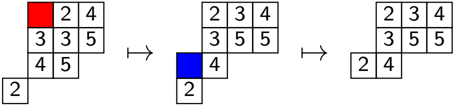

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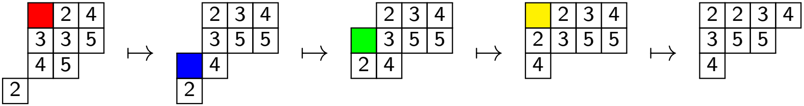

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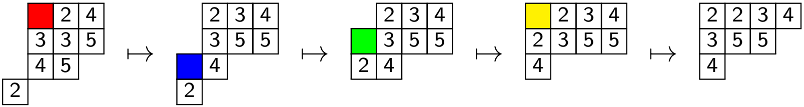

**Theorem**

*The final straight tableau does not depend on the order in which we choose corners.*

This final tableau is called the *rectification* of $T$ and denoted $\mathrm{Rect}(T)$.

Reading word of a skew tableau (from bottom to top, from left to right):

$$T = \begin{array}{cccc} & & \boxed{2} & \boxed{4} \\ & \boxed{3} & \boxed{3} & \boxed{5} \\ & \boxed{4} & \boxed{5} & \\ \boxed{2} & & & \end{array} \mapsto w(T) = 24533524.$$

Reading word of a skew tableau (from bottom to top, from left to right):

$$T = \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & 4 \\ \cline{2-4} \multicolumn{1}{c|}{} & 3 & 3 & 5 \\ \cline{2-4} \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \mapsto w(T) = 24533524.$$

## Definition

Two words are Knuth equivalent if one can be obtained from the other by *elementary Knuth transforms*

$$y\,z\,x \;\equiv\; y\,x\,z \text{ (if } x < y \leq z), \qquad x\,z\,y \;\equiv\; z\,x\,y \text{ (if } x \leq y < z).$$

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### Lemma

If one obtains $T'$ from $T$ by jeu-de-taquin, then $w(T) = w(T')$ are Knuth equivalent.

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The monoid of words up to Knuth equivalence is the *plactic monoid*, studied by Lascoux and Schützenberger.

# Uniqueness of the rectification

**Theorem**

*Each Knuth equivalence class contains the reading word of a unique straight tableau.*

**Corollary**

$Rect(T)$ *is the unique straight tableau, whose reading word is in the same class as* $w(T)$ *(and thus does not depend on the successive choices of corners).*

If $T$ is a tableau, write $x^T = x_1^{\#1 \text{ in } T} \cdot x_2^{\#2 \text{ in } T} \cdots$

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & 5 \\ \hline 4 & 5 & 5 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}, \quad x^T = x_1^2\, x_2^2\, x_3^3\, x_4\, x_5^3\, x_6.$$

Then if $\lambda$ is a diagram, the associated *Schur function* is

$$s_\lambda := \sum_T x^T,$$

where the sum is taken over all tableaux of shape $\lambda$.

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**Lemma**

$s_\lambda$ is symmetric and the family $(s_\lambda)_{\lambda \text{ straight}}$ is a basis of the symmetric function ring.

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### Problem

Describe *Littlewood-Richardson coefficients* $c_{\mu,\nu}^{\lambda}$ defined by

$$s_\mu s_\nu = \sum_\lambda c_{\mu,\nu}^{\lambda} s_\lambda.$$

$$\begin{tikzcd}
\text{Tableaux} \arrow[rr, "T \mapsto x^T"] \arrow[dr, "T \mapsto w(T)"'] & & \begin{array}{c}\text{Polynomials}\\ \text{Usual multiplication}\end{array} \\
& \begin{array}{c}\text{Plactic monoid}\\ \text{Concatenation}\end{array} \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c}\text{Tableaux}\\ \ast\ \text{product}\end{array} \arrow[rr, "T\mapsto x^T"] \arrow[dr, "T\mapsto w(T)"'] & & \begin{array}{c}\text{Polynomials}\\ \text{Usual multiplication}\end{array} \\
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\end{tikzcd}$$

**Definition**

$T * U := \mathrm{Rect}\left( \begin{smallmatrix} \square & U \\ T & \end{smallmatrix} \right)$. Example: $\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 4 & \\ \cline{1-2}\end{array} * \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} = \mathrm{Rect}\left( \begin{array}{ccccc} & & & \boxed{1} & \boxed{3} \\ & & & \boxed{2} & \\ \boxed{1} & \boxed{2} & \boxed{2} & & \\ \boxed{2} & \boxed{4} & & & \end{array} \right)$.

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### Proposition

*Denote* $S_\mu = \sum_{\substack{T \text{ of} \\ \text{shape } \mu}} T$*. In the algebra of tableaux*

$$S_\mu * S_\nu = \sum_\lambda c^\lambda_{\mu,\nu} S_\lambda.$$

$\rightarrow$ *enables to describe combinatorially products of Schur functions.*

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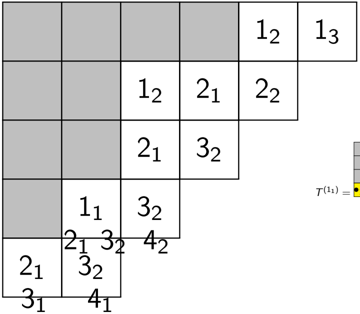
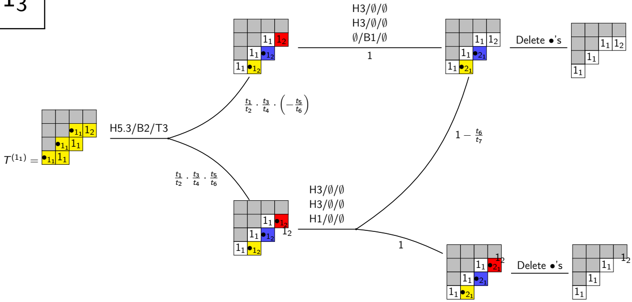
Remarkable fact: the algebra of the plactic monoid contains the symmetric function ring as a subalgebra.

# Pechenik-Yong's combinatorial rule

- Littlewood-Richardson coefficients also appears in the "cohomology ring" of the Grassmannian.
- Pechenik and Yong (2015) considers an analogue of Littlewood-Richardson coefficients in the "K-theory" of the Grassmannian.
- They find a combinatorial rule using enriched tableaux and adapting jeu-de-taquin to these tableaux.

# Pechenik-Yong's tableaux and jeu-de-taquin






©Pecheneik-Yong

→ jeu-de-taquin is a robust technique!

# Bijective proof of character identities (Fang)

Denote $f^\lambda$ the number of standard tableaux of shape $\lambda$.

**Theorem (Frobenius, Murnaghan-Nakayama)**

$$2\, f^\lambda \left( \sum_{(i,j)\in\lambda} (j-i) \right) = \binom{n}{2} \big( f^{\lambda/(2)} - f^\lambda(1,1) \big).$$

Both sides compute symmetric group characters on transpositions...

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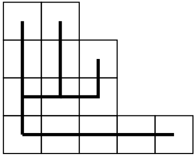
Both sides compute symmetric group characters on transpositions. . .

Fang (2015) gave a bijective proof using jeu-de-taquin

| | | | | |
|---|---|---|---|---|
| 6 | 12 | | | |
| 4 | 8 | 13 | | |
| 3 | 7 | 10 | | |
| 1 | 2 | 5 | 9 | 11 |



©Fang

# Jeu de taquin on random infinite tableaux (Romik, Śniady)

| | | ⋮ | | | ⋰ |
|---|---|---|---|---|---|
| 10 | 12 | 38 | 39 | 40 | |
| 7 | 11 | 23 | 36 | 37 | |
| 5 | 8 | 15 | 24 | 28 | ⋯ |
| 3 | 6 | 9 | 18 | 20 | |
| 1 | 2 | 4 | 13 | 16 | |

A random infinite tableau.

# Jeu de taquin on random infinite tableaux (Romik, Śniady)

$\vdots$ $\quad$ $\iddots$

| | | | | | |
|---|---|---|---|---|---|
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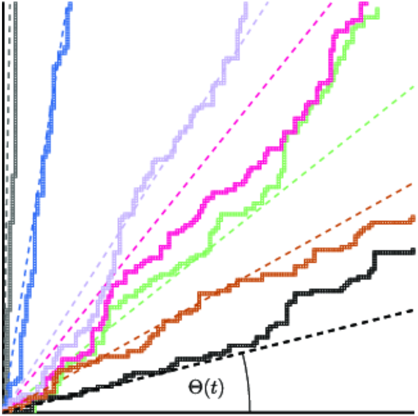
©Romik, Śniady



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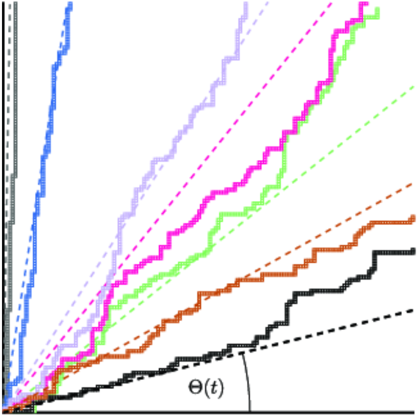


Romik and Śniady (2015) show that the jeu-de-taquin paths are

- almost surely asymptotically straight lines,
- with a random direction whose distribution is explicit.