Regular multipartitions and representation theory

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80ème Séminaire Lotharingien de Combinatoire
Lyon
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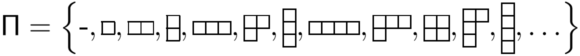
Partitions and multipartitions

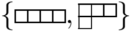
1. **Partitions.**
   Denote Π = set of integer partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$.



   For $n \in \mathbb{N}$, let $\Pi(n)$ = set of partitions of the integer n.

Definition

Let $p \in \mathbb{Z}_{\geq 1}$. An element $\lambda \in \Pi$ is called p-regular if no part is repeated more than $p - 1$ times.

Denote Π_p (resp. $\Pi_p(n)$) the set of p-regulars partitions (resp. of n).
Example: $\Pi_2(4) =$ .

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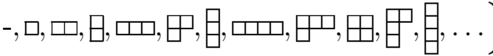
   $\Pi = \{$  $\}$

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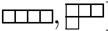
   Example: $\Pi_2(4) = \{$  $\}$.

2. **Multipartitions.** Fix $\ell \in \mathbb{Z}_{\geq 1}$.
   $\Pi^\ell := \Pi^{\times \ell}$ set of ℓ-partitions (or simply *multi*partitions).

Goal

Define *p*-regular ℓ-partitions.

Representations of the symmetric group

For $n \in \mathbb{N}$, let S_n be the symmetric group over $\{1, \dots, n\}$. Let K be a field. We are interested in the representation theory of S_n over K.

General problem

Understand irreducible representations $\rightsquigarrow \mathrm{Irr}_K(S_n)$, i.e. $\mathrm{Irr}(KS_n)$.

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- $K = \mathbb{C}$ (or $\mathrm{char}(K) = 0$):
  - $\mathrm{Irr}(\mathbb{C}S_n) \overset{\sim}{\longleftrightarrow} \Pi(n)$, $L_\lambda \longleftrightarrow \lambda$ (Specht module).
  - $\dim(L_\lambda) \rightsquigarrow$ hook-length formula.
  - Branching rule $\mathrm{Irr}(\mathbb{C}S_n) \xrightarrow{\mathrm{Ind}} \mathrm{Irr}(\mathbb{C}S_{n+1}) \rightsquigarrow$ Young lattice.

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  - Branching rule $\mathrm{Irr}(\mathbb{C}S_n) \xrightarrow{\mathrm{Ind}} \mathrm{Irr}(\mathbb{C}S_{n+1}) \rightsquigarrow$ Young lattice.
- $K = \mathbb{F}_p$ (p prime):
  - $\mathrm{Irr}(\mathbb{F}_p S_n) \overset{\sim}{\longleftrightarrow} \Pi_p(n)$, $D_\lambda \longleftrightarrow \lambda$.
  - $\dim(D_\lambda) =$?!?
  - "Modular branching rule" $\mathrm{Irr}(\mathbb{F}_p S_n) \xrightarrow{\mathrm{Ind}} \mathrm{Irr}(\mathbb{F}_p S_{n+1})$ (Kleshchev) $\rightsquigarrow$ LLT graph.

The group $W_{\ell,n}$ and its Hecke algebra

Let $W_{\ell,n} = S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n = G(\ell, 1, n)$.
Example: $W_{1,n} = S_n = \mathrm{Weyl}(A_{n-1})$ and $W_{2,n} = \mathrm{Weyl}(B_n)$.

$$\mathrm{Irr}(\mathbb{C}W_{\ell,n}) \stackrel{\sim}{\longleftrightarrow} \Pi^\ell(n)$$

Goal (reformulated)

Find $\Pi^\ell_p \subseteq \Pi^\ell$ such that $\mathrm{Irr}(\mathbb{F}_p W_{\ell,n}) \stackrel{\sim}{\longleftrightarrow} \Pi^\ell_p(n)$ and $\Pi^1_p(n) = \Pi_p(n)$.

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Idea: Use a deformation of the group algebra: the *Hecke algebra*.
For $\mathbf{v} = (v, v_1, \dots, v_\ell)$ multiparameter, $\mathbb{C}W_{\ell,n} \xrightarrow{\mathbf{v}\text{-deformation}} \mathcal{H}^{\mathbb{C}}_{\mathbf{v}}(W_{\ell,n})$.

Theorem [James, Dipper-Mathas]

Let $v = \sqrt[p]{1}$ and $v_i = v^{s_i}, s_i \in \mathbb{Z}$.

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p-regular multipartitions

Let $p \in \mathbb{Z}_{\geq 2}$, $0 \leq s_1 \leq s_2 \leq \cdots \leq s_\ell \leq s_1 + p$, $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \Pi^\ell$.

Definition [Foda-Leclerc-Okado-Thibon-Welsh]

We call $\boldsymbol{\lambda}$ p-regular and denote $\boldsymbol{\lambda} \in \Pi_p^\ell$ if

- $\forall 1 \leq k \leq \ell - 1$, $\lambda_i^{(k)} \geq \lambda_{i+s_{k+1}-s_k}^{(k+1)} \forall i \geq 1$ and $\lambda_i^{(\ell)} \geq \lambda_{i+p+s_1-s_\ell}^{(1)} \forall i \geq 1$, i.e. $\boldsymbol{\lambda}$ is *cylindric*.
- $\forall \alpha > 0, \left\{ (\lambda_i^{(k)} - i + s_k) \mod p \,|\, 1 \leq k \leq \ell - 1, \lambda_i^{(k)} = \alpha \right\} \neq \mathbb{Z}/p\mathbb{Z}$.

Examples: $\Pi_p^1 = \Pi_p$ for all p, $\mathbf{s}$. For $\mathbf{s} = (s_1, s_2, s_3) = (0, 0, 1)$, $\Pi_2^3(4) =$
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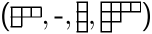
Example: $\ell = 4, p = 3$ and $\mathbf{s} = (0,1,1,2)$, $\boldsymbol{\lambda} =$ 

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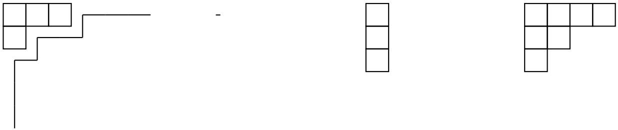
$\boldsymbol{\lambda} =$ 

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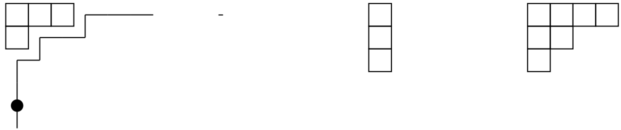
$\boldsymbol{\lambda} =$ 

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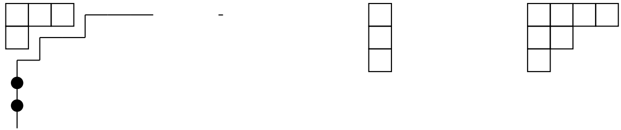
$\boldsymbol{\lambda} =$ 

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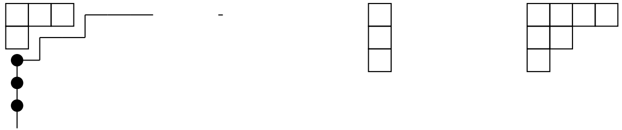
$\boldsymbol{\lambda} =$ 

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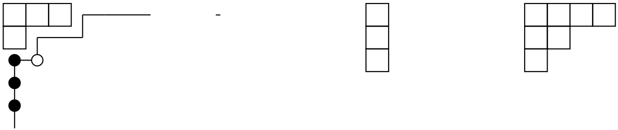
$\boldsymbol{\lambda} =$ 

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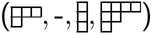
Example: $\ell = 4, p = 3$ and $\mathbf{s} = (0,1,1,2)$, $\boldsymbol{\lambda} =$ 

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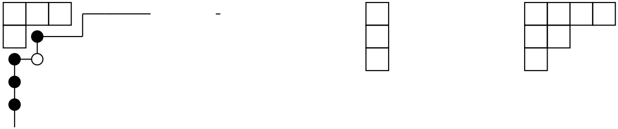

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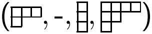
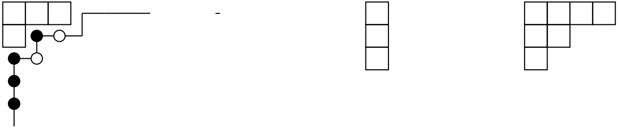
Example: $\ell = 4, p = 3$ and $\mathbf{s} = (0,1,1,2)$, $\boldsymbol{\lambda} = ($$)$

$\boldsymbol{\lambda} =$ 

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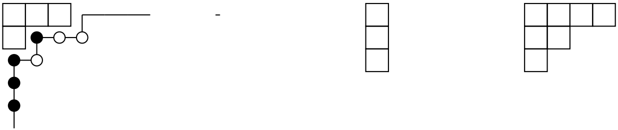
$\boldsymbol{\lambda} =$ 

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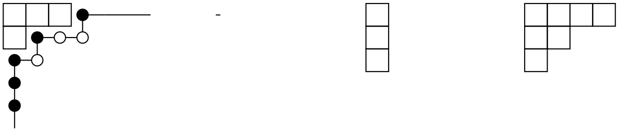
$\boldsymbol{\lambda} =$ 

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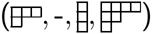
Example: $\ell = 4, p = 3$ and $\mathbf{s} = (0, 1, 1, 2)$, $\boldsymbol{\lambda} =$ 

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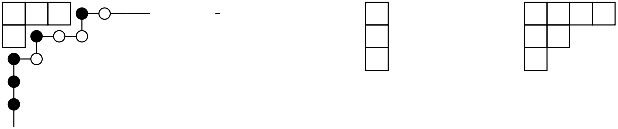

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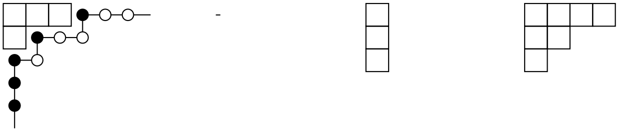
$\boldsymbol{\lambda} =$ 

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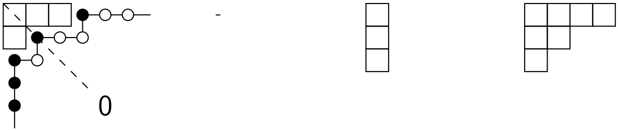
$\boldsymbol{\lambda} =$ 

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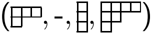
Example: $\ell = 4, p = 3$ and $\mathbf{s} = (0,1,1,2)$, $\boldsymbol{\lambda} = ($$)$

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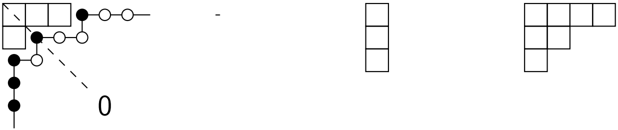

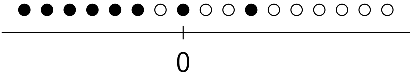

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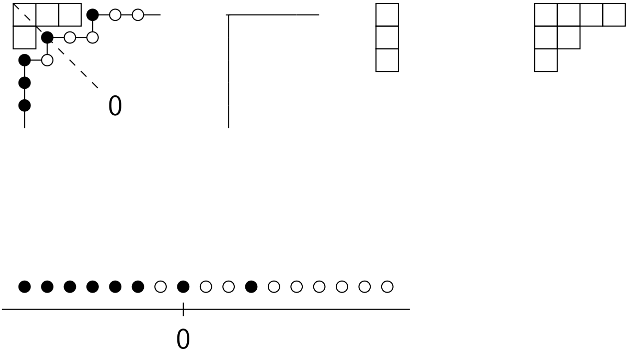

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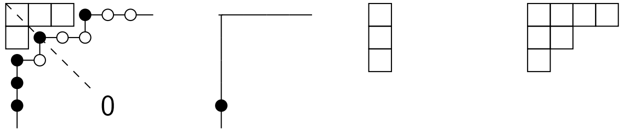
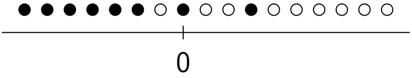
$\boldsymbol{\lambda} =$ 



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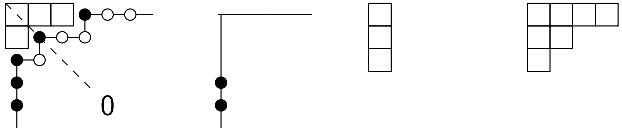
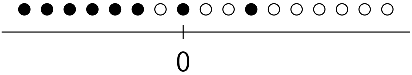
$\boldsymbol{\lambda} =$ 



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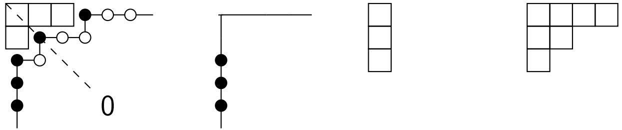

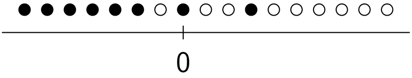

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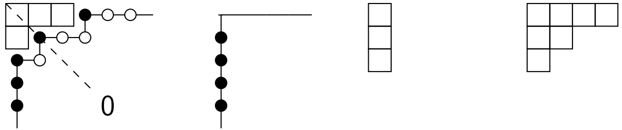

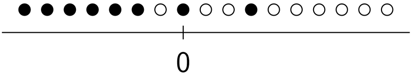

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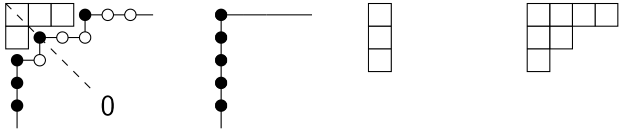

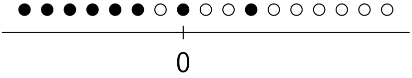

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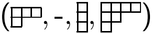
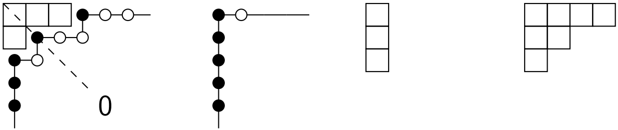
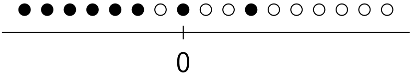
Example: $\ell = 4, p = 3$ and $\mathbf{s} = (0, 1, 1, 2)$, $\boldsymbol{\lambda} =$ 

$\boldsymbol{\lambda} =$ 



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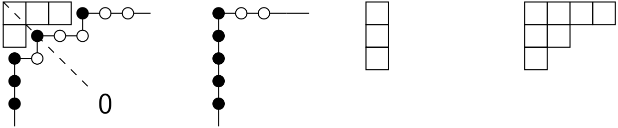
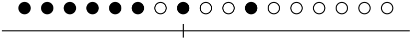
$\boldsymbol{\lambda} =$ 



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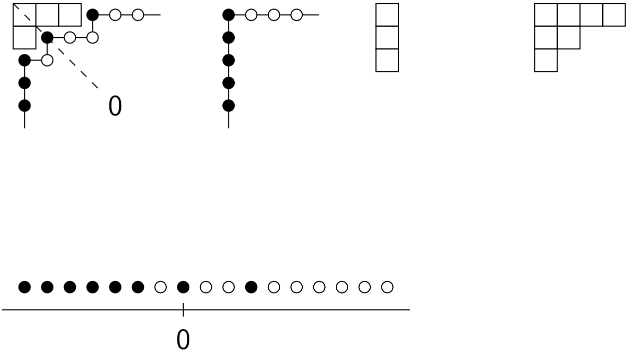

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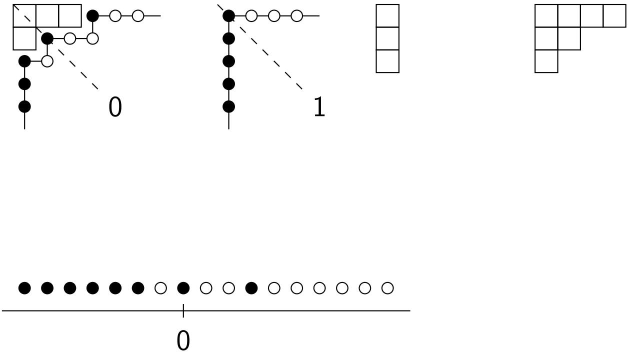

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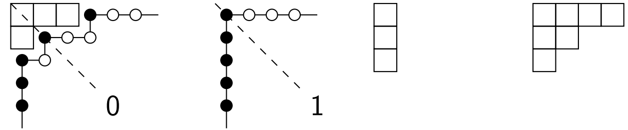
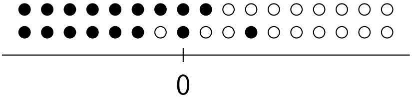
$\boldsymbol{\lambda} =$ 



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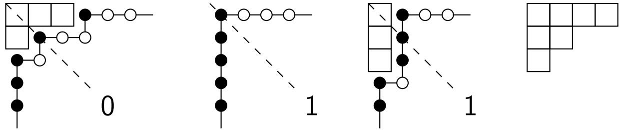
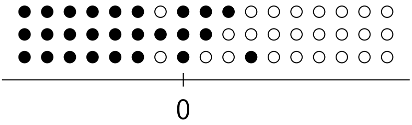
$\boldsymbol{\lambda} =$ 



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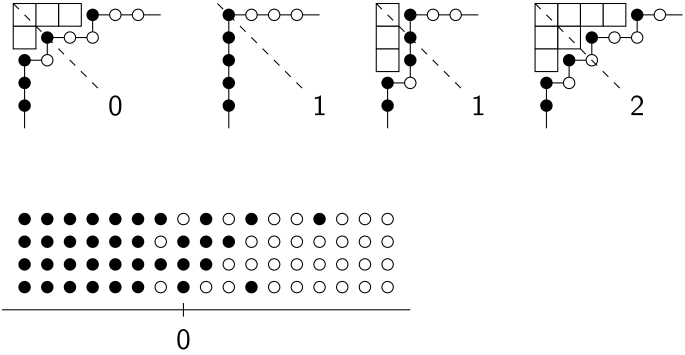

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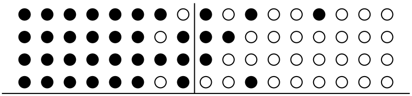

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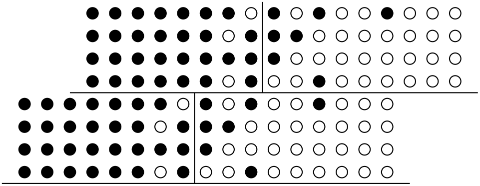

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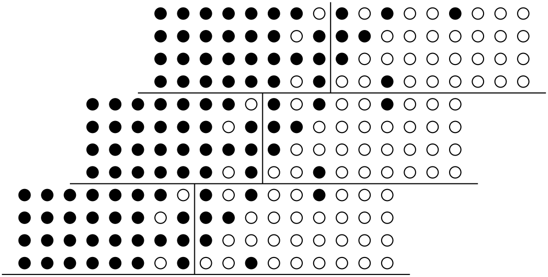

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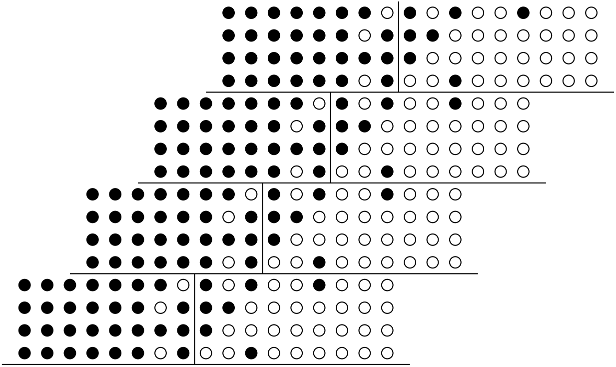

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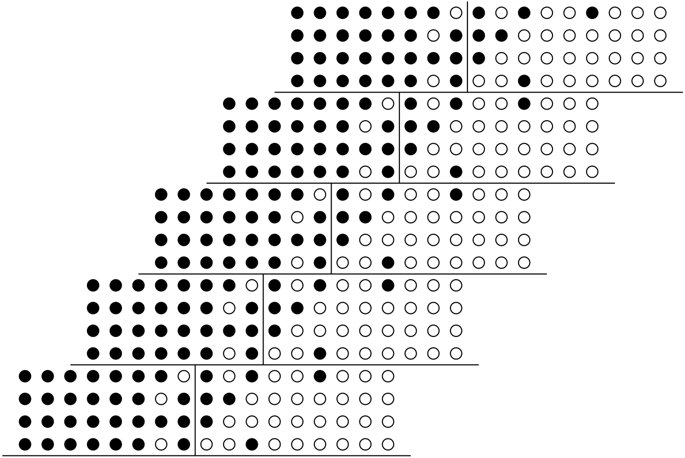

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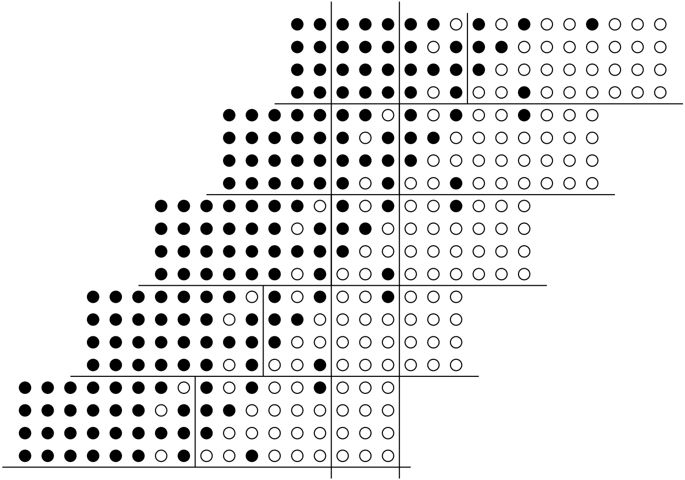

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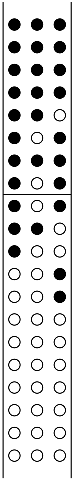

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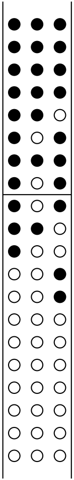

$$\dot{\mathbf{s}} = (-3, 1, -2)$$

$$\dot{\boldsymbol{\lambda}} = (\text{-}, (2,1,1), (6,2,2))$$

Crystal structures

Theorem [Jimbo-Misra-Miwa-Okado, Uglov, G.]

There are three crystal structures on $\Pi^\ell \times \mathbb{Z}^\ell$, arising from linear actions of

- the quantum group of type $\widehat{\mathfrak{sl}_{\mathfrak{p}}}$ (leaves $\Pi^\ell \times \{\mathbf{s}\}$ invariant),
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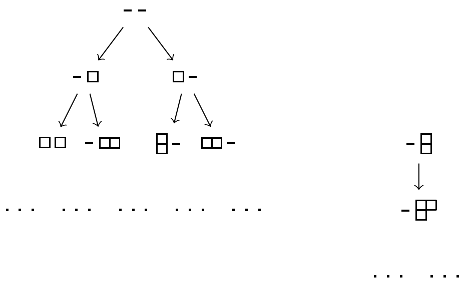
- the quantum Heisenberg algebra $\mathcal{H}$ (leaves both sets invariant).

Example: The beginning of the $\widehat{\mathfrak{sl}_\mathfrak{p}}$-crystal for $\ell = 2$, $\mathbf{s} = (0, 1)$, $p = 3$.



Regular multipartitions via crystals

Fix $\mathbf{s} \in \mathbb{Z}^\ell$ such that $0 \leq s_1 \leq s_2 \leq \cdots \leq s_\ell \leq s_1 + p$.

Theorem [Foda-Leclerc-Okado-Thibon-Welsh]

$\boldsymbol{\lambda} \in \Pi_p^\ell$ if and only if $\boldsymbol{\lambda}$ is in the connected component of the $\widehat{\mathfrak{sl}_\mathfrak{p}}$-crystal with source $(\text{-},\text{-},\dots,\text{-})$.

In fact, the $\widehat{\mathfrak{sl}_\mathfrak{p}}$-crystal gives the modular branching rule for $\mathcal{H}_{\mathbf{v}}^{\mathbb{C}}(W_{\ell,n})$...

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Theorem [G.]

1. $\boldsymbol{\lambda} \in \Pi_p^\ell$ if and only if $\dot{\boldsymbol{\lambda}}$ is a source in the $\widehat{\mathfrak{sl}_\ell}$-crystal and in the $\mathcal{H}$-crystal (associated to $\dot{\mathbf{s}}$).
2. $\boldsymbol{\lambda}$ is cylindric if and only if $\dot{\boldsymbol{\lambda}}$ is a source in the $\widehat{\mathfrak{sl}_\ell}$-crystal (associated to $\dot{\mathbf{s}}$).

In fact, the $\widehat{\mathfrak{sl}_\mathfrak{p}}$-crystal and the $\mathcal{H}$-crystal taken in conjunction give the branching rule for Cherednik algebras associated to $W_{\ell,n}$...