Skew Hook Formula for d-Complete Posets

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Young Diagrams and Standard Tableaux

For a partition λ of n, we define its diagram by

$$D(\lambda) = \{(i,j) \in \mathbb{Z}^2 : 1 \leq j \leq \lambda_i\}.$$

Let λ and μ be partitions such that $\lambda \supset \mu$ (i.e., $D(\lambda) \supset D(\mu)$). A standard tableau of skew shape λ/μ is a filling T of the cells of $D(\lambda)$ with numbers $1, 2, \ldots, n = |\lambda| - |\mu|$ satisfying

- each integer appears exactly once,
- the entries in each row and each column are increasing.

Example

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 8 \\ \cline{1-3} 7 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline & & 2 & 3 \\ \hline 1 & 5 & 6 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$$

are standard tableaux of shape $(4,3,1)$ and skew shape $(4,3,1)/(2)$ respectively.

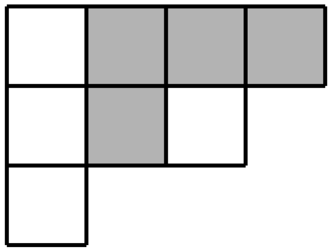
Frame–Robinson–Thrall's Hook Formulas for Young Diagrams

Theorem (Frame–Robinson–Thrall) The number f^λ of standard tableaux of shape λ is given by

$$f^\lambda = \frac{n!}{\prod_{v \in D(\lambda)} h_\lambda(v)}, \quad n = |\lambda|,$$

where $h_\lambda(i,j) = \lambda_i + \lambda'_j - i - j + 1$ is the hook length of (i,j) in $D(\lambda)$.

Example The hook of $(1,2)$ in $D(4,3,1)$ and the hook lengths are given by



$$\begin{array}{|c|c|c|c|} \hline 6 & 4 & 3 & 1 \\ \hline 4 & 2 & 1 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array}$$

Hence we have

$$f^{(4,3,1)} = \frac{8!}{6 \cdot 4 \cdot 3 \cdot 1 \cdot 4 \cdot 2 \cdot 1 \cdot 1} = 70.$$

Naruse's Hook Formulas for skew Young Diagrams

Theorem (Naruse) The number $f^{\lambda/\mu}$ of standard tableaux of skew shape λ/μ is given by

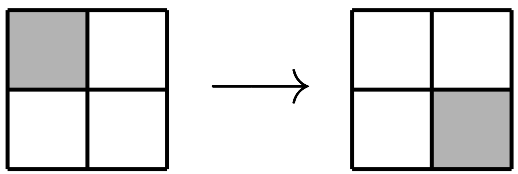
$$f^{\lambda/\mu} = n! \sum_{D} \frac{1}{\prod_{v \in D(\lambda) \setminus D} h_\lambda(v)}, \quad n = |\lambda| - |\mu|,$$

where D runs over all excited diagrams of $D(\mu)$ in $D(\lambda)$.

- If a subset $D \subset D(\lambda)$ and $u = (i, j)$ satisfy $(i, j+1)$, $(i+1, j)$, $(i+1, j+1) \in D(\lambda) \setminus D$, then we define

$$\alpha_u(D) = D \setminus \{(i, j)\} \cup \{(i+1, j+1)\}.$$

- We say that D is an excited diagram of $D(\mu)$ in $D(\lambda)$ if D is obtained from $D(\mu)$ after a sequence of elementary excitations $D \to \alpha_u(D)$.



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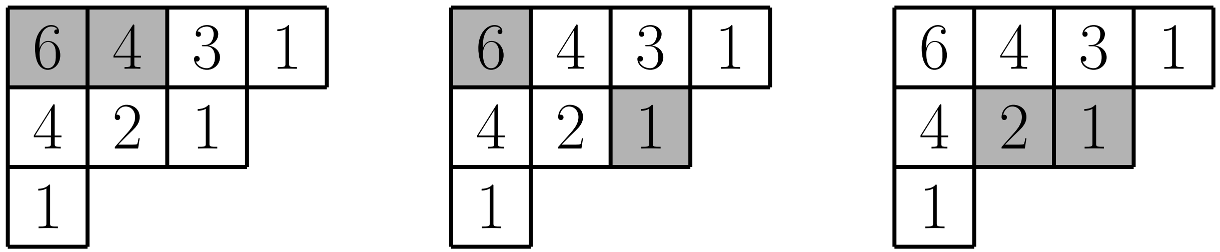
Example If $\lambda = (4, 3, 1)$ and $\mu = (2)$, then there are three excited diagrams of $D(\mu)$ in $D(\lambda)$:



and we have

$$f^{(4,3,1)/(2)} = 6! \left(\frac{1}{3 \cdot 1 \cdot 4 \cdot 2 \cdot 1 \cdot 1} + \frac{1}{4 \cdot 3 \cdot 1 \cdot 4 \cdot 2 \cdot 1} + \frac{1}{6 \cdot 4 \cdot 3 \cdot 1 \cdot 4 \cdot 1} \right) = 40.$$

Reverse Plane Partitions

For a poset P, a P-partition is a map $\pi : P \to \mathbb{N}$ satisfying

$$x \leq y \text{ in } P \quad \Longrightarrow \quad \pi(x) \geq \pi(y) \text{ in } \mathbb{N}.$$

Let $\mathcal{A}(P)$ be the set of P-partitions, and write $|\pi| = \sum_{x \in P} \pi(x)$ for $\pi \in \mathcal{A}(P)$.

The Young diagrams can be regarded as posets by defining

$$(i, j) \geq (i', j') \Longleftrightarrow i \leq i', \; j \leq j'.$$

If $P = D(\lambda) \setminus D(\mu)$, then P-partitions are called reverse plane partitions of shape λ/μ.

Example

$$\pi = \begin{array}{|c|c|c|c|} \hline & & 3 & 3 \\ \hline 0 & 1 & 3 & \\ \cline{1-3} 2 & & & \\ \cline{1-1} \end{array}$$

is a reverse plane partition of shape $(4, 3, 1)/(2)$.

Univariate Generating Functions of Reverse Plane Partitions

Theorem (Stanley) For a partition λ, the generating function of reverse plane partitions of shape λ is given by

$$\sum_{\pi \in \mathcal{A}(D(\lambda))} q^{|\pi|} = \frac{1}{\prod_{v \in P} (1 - q^{h_\lambda(v)})}.$$

Theorem (Morales–Pak–Panova) For partitions $\lambda \supset \mu$, the generating function of reverse plane partition of skew shape λ/μ is given by

$$\sum_{\pi \in \mathcal{A}(D(\lambda) \setminus D(\mu))} q^{|\pi|} = \sum_{D} \frac{\prod_{v \in B(D)} q^{h_\lambda(v)}}{\prod_{v \in D(\lambda) \setminus D} (1 - q^{h_\lambda(v)})},$$

where D runs over all excited diagrams of $D(\mu)$ in $D(\lambda)$, and $B(D)$ is the set of excited peaks of D.

Generalization of Hook Formulas

The Frame–Robinson–Thrall-type hook formula holds for shifted Young diagrams and rooted trees. Proctor introduced a wide class of posets, called d-complete posets.

Theorem (Peterson–Proctor) Let P be a d-complete poset. Then the univariate generating function of P-partitions is given by

$$\sum_{\pi \in \mathcal{A}(P)} q^{|\pi|} = \frac{1}{\prod_{v \in P} (1 - q^{h_P(v)})}.$$

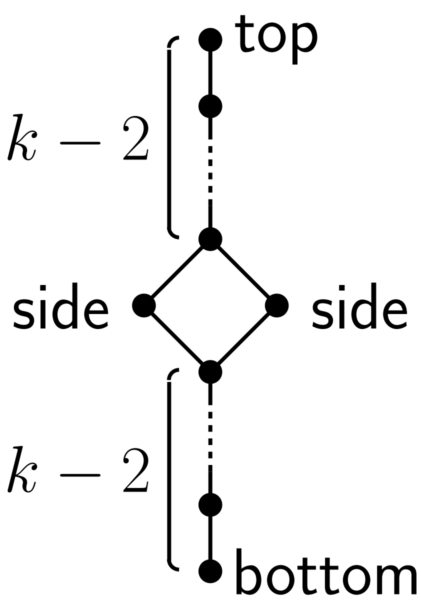
More generally, the multivariate generating function of P-partitions is given by

$$\sum_{\pi \in \mathcal{A}(P)} \boldsymbol{z}^{\pi} = \frac{1}{\prod_{v \in P} (1 - \boldsymbol{z}[H_P(v)])}.$$

Goal Generalize Naruse's and Morales–Pak–Panova's skew hook formulas to d-complete posets (in other words, generalize Peterson–Proctor's hook formula to skew setting).

Double-tailed Diamond

- The double-tailed diamond poset $d_k(1)$ ($k \geq 3$) is the poset depicted below:



- A d_k-interval is an interval isomorphic to $d_k(1)$.
- A d_k^--convex set is a convex subset isomorphic to $d_k(1) - \{\text{top}\}$.

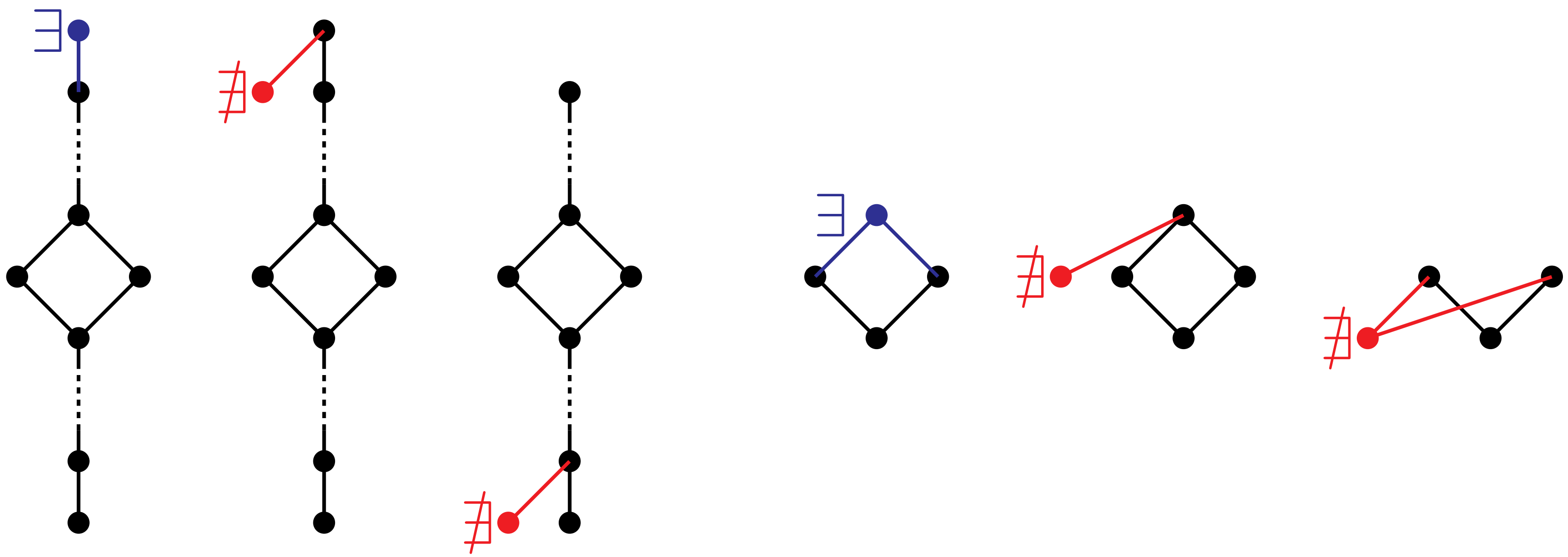
d-Complete Posets

Definition A finite poset P is d-complete if it satisfies the following three conditions for every $k \geq 3$:

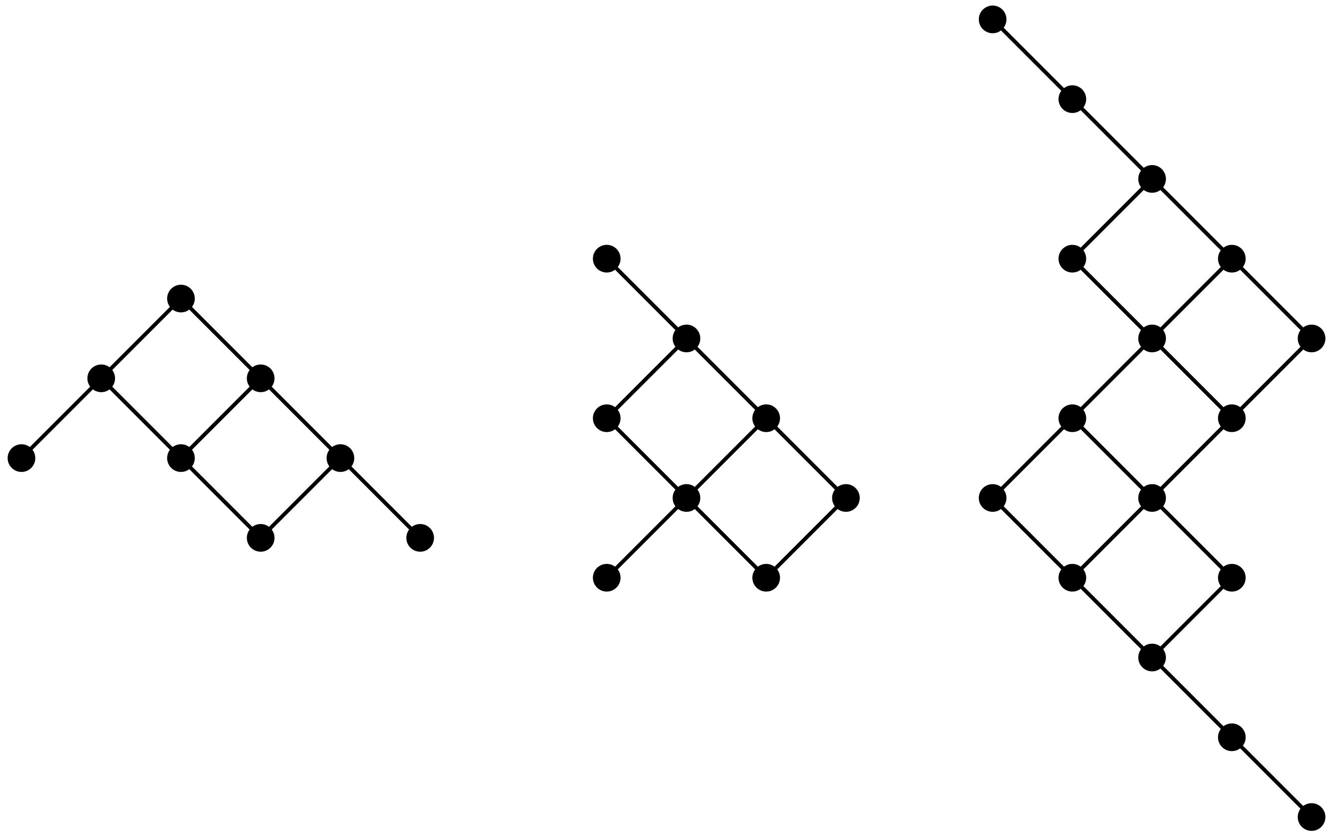
(D1) If I is a d_k^--convex set, then there exists an element v such that v covers the maximal elements of I and $I \cup \{u\}$ is a d_k-interval.

(D2) If $I = [v, u]$ is a d_k-interval and u covers w in P, then $w \in I$.

(D3) There are no d_k^--convex sets which differ only in the minimal elements.



Example Shapes (Young diagrams, left), shifted shapes (shifted Young diagrams, middle) and swivels (right) are d-complete posets.



Hook Lengths

Let P be a connected d-complete poset. For each $u \in P$, we define the hook length $h_P(u)$ inductively as follows:

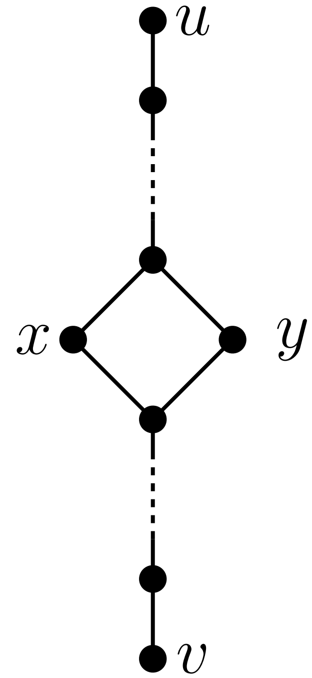
(a) If u is not the top of any d_k-interval, then we define

$$h_P(u) = \#\{w \in P : w \leq u\}.$$

(b) If u is the top of a d_k-interval $[v, u]$, then we define

$$h_P(u) = h_P(x) + h_P(y) - h_P(v),$$

where x and y are the sides of $[v, u]$.



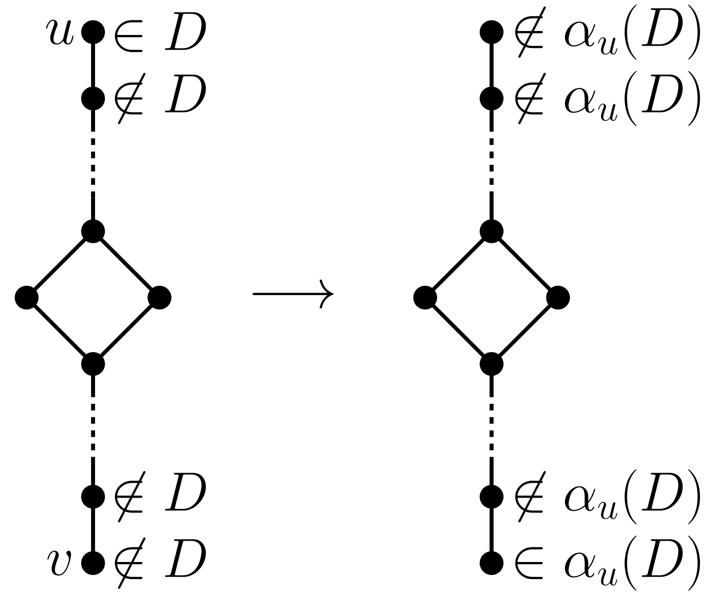
Also we can define the hook monomials $\boldsymbol{z}[H_P(u)]$.

Excited Diagrams for d-Complete Posets

Let P be a connected d-complete poset.

- We say that $u \in D$ is D-active if there is a d_k-interval $[v, u]$ with $v \notin D$ such that

$$z \in [v, u] \text{ and } \begin{cases} z \text{ is covered by } u \\ \text{or} \\ z \text{ covers } v \end{cases} \implies z \notin D.$$



- If $u \in D$ is D-active, then we define

$$\alpha_u(D) = D \setminus \{u\} \cup \{v\}.$$

Let F be an order filter of P.

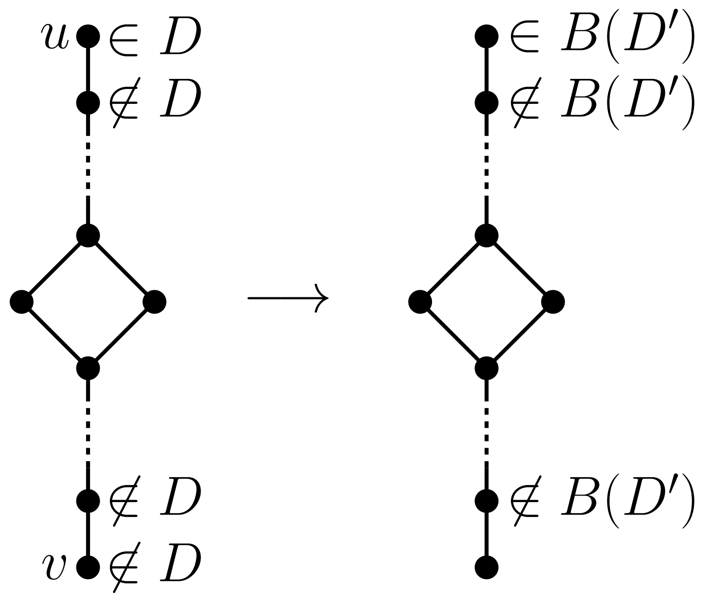
- We say that D is an excited diagram of F in P if D is obtained from F after a sequence of elementary excitations $D \to \alpha_u(D)$.

Excited Peaks for d-Complete Posets

Let P be a d-complete poset and F an order filter of P. To an excited diagram D of F in P, we associate a subset $B(D) \subset P$, called the subset of excited peaks of D, as follows:

(a) If $D = F$, then we define $B(F) = \emptyset$.

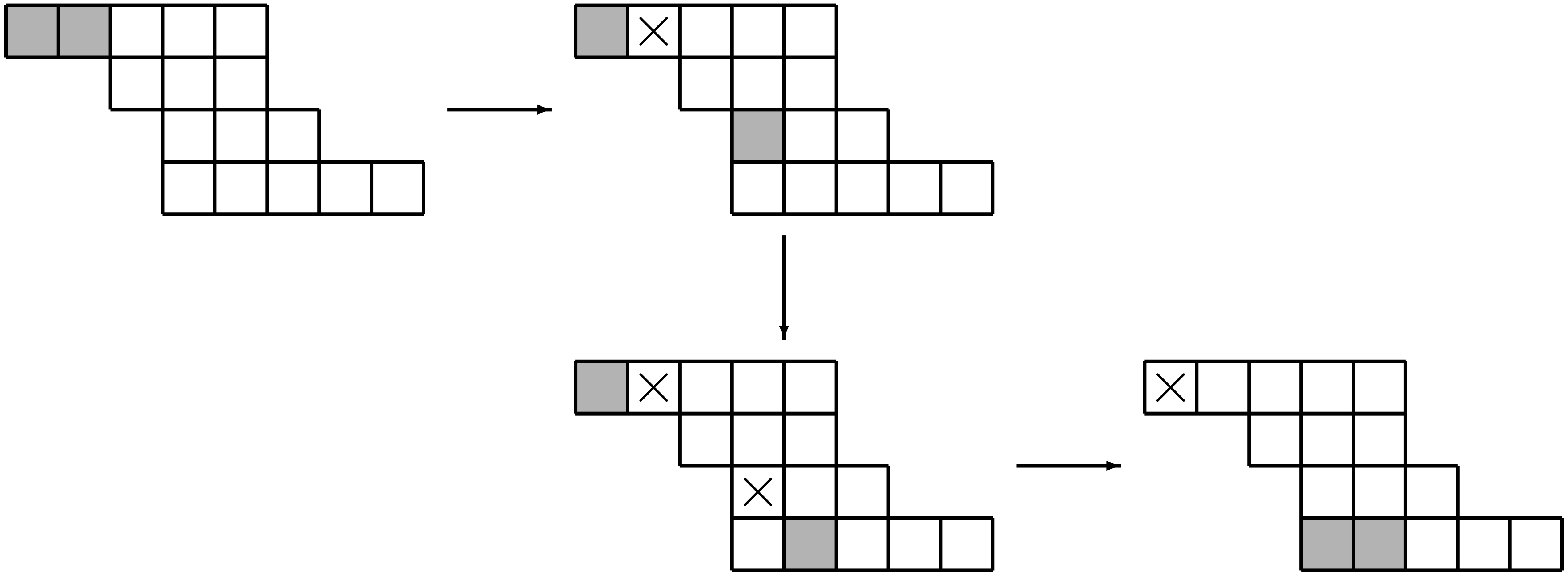
(b) If $D' = \alpha_u(D)$ is obtained from D by an elementary excitation at $u \in D$, then



$$B(\alpha_u(D)) = B(D) \setminus \left\{ z \in [v, u] : \begin{array}{l} z \text{ is covered by } u \\ \text{or } z \text{ covers } v \end{array} \right\} \cup \{v\},$$

where $[v, u]$ is the d_k-interval with top u.

Example If P is the Swivel and an order filter F has two elements, then there are 4 excited diagrams of F in P.



Here the shaded cells form an exited diagram and a cell with $\times$ is an excited peak.

Main Theorem

Theorem (Naruse–Okada) Let P be a connected d-complete poset and F an order filter of P. Then the univariate generating function of $(P \setminus F)$-partitions is given by

$$\sum_{\pi \in \mathcal{A}(P \setminus F)} q^{|\pi|} = \sum_{D} \frac{\prod_{v \in B(D)} q^{h_P(v)}}{\prod_{v \in P \setminus D} (1 - q^{h_P(v)})},$$

where D runs over all excited diagrams of F in P. More generally, the multivariate generating function of $(P \setminus F)$-partitions is given by

$$\sum_{\pi \in \mathcal{A}(P \setminus F)} \prod_{v \in P} \left(z_{c(v)} \right)^{\pi(v)} = \sum_{D} \frac{\prod_{v \in B(D)} \boldsymbol{z}[H_P(v)]}{\prod_{v \in P \setminus D} (1 - \boldsymbol{z}[H_P(v)])},$$

where D runs over all excited diagrams of F in P.

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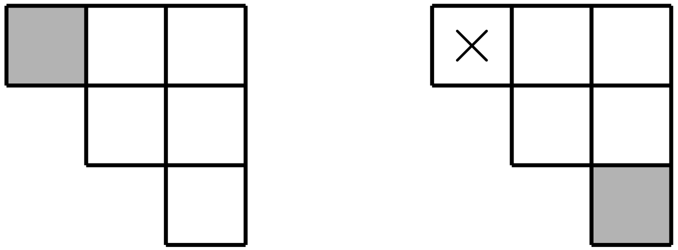
where D runs over all excited diagrams of F in P.

Remark

- If $F = \emptyset$, we recover Peterson–Proctor's hook formula, and our generalization provides an alternate proof.
- If $P = D(\lambda)$ and $F = D(\mu)$ are Young diagrams, then the above theorem reduces to Morales–Pak–Panova's skew hook formula after specializing $z_i = q$ $(i \in I)$.

Example If $P = S(3,2,1)$ and $F = S(1)$ are the shifted Young diagrams corresponding to strict partitions $(4,3,1)$ and (1) respectively, then we have

$$\sum_{\pi \in \mathcal{A}(S(3,2,1)\setminus S(1))} \boldsymbol{z}^{\pi}$$
$$= \frac{1}{(1-z_0z_{0'}z_1z_2)(1-z_0z_1z_2)(1-z_0z_{0'}z_1)(1-z_0z_1)(1-z_0)}$$
$$+ \frac{z_0z_{0'}z_1^2z_2}{(1-z_0z_{0'}z_1^2z_2)(1-z_0z_{0'}z_1z_2)(1-z_0z_1z_2)(1-z_0z_{0'}z_1)(1-z_0z_1)}$$
$$= \frac{1-z_0^2z_{0'}z_1^2z_2}{(1-z_0z_{0'}z_1^2z_2)(1-z_0z_{0'}z_1z_2)(1-z_0z_1z_2)(1-z_0z_{0'}z_1)(1-z_0z_1)(1-z_0)}.$$



Idea of Proof (1) — equivariant K-theory of partial flag variety

Let P be a connected d-complete poset. Then we can associate

- the Dynkin diagram Γ (the top tree of P),
- the Weyl group W,
- the fundamental weight λ_P corresponding to the color i_P of the maximum element of P,
- the set W^{λ_P} of minimum length coset representatives of W/W_{λ_P}, where W_{λ_P} is the stabilizer of λ_P.
- the Kac–Moody group $\mathcal{G}$ and its maximal torus $\mathcal{T}$,
- the maximal parabolic subgroup $\mathcal{P}_-$ corresponding to i_P,
- the Kashiwara's thick partial flag variety $\mathcal{X} =$ "$\mathcal{G}/\mathcal{P}_-$",
- the $\mathcal{T}$-equivariant K-theory $K_{\mathcal{T}}(\mathcal{X})$.

Idea of Proof (2) — equivariant K-theory of partial flag variety

Then we have

$$K_{\mathcal{T}}(\mathcal{X}) \cong \prod_{v \in W^{\lambda_P}} K_{\mathcal{T}}(\mathrm{pt})\, \xi^v \quad \text{(as } K_{\mathcal{T}}(\mathrm{pt})\text{-modules)},$$

and the localization maps

$$\begin{array}{rccc} \iota_w^* : & K_{\mathcal{T}}(\mathcal{X}) & \longrightarrow & K_{\mathcal{T}}(\mathrm{pt}) \cong \mathbb{Z}[\Lambda] \\ & \xi^v & \longmapsto & \xi^v|_w \end{array}$$

where Λ is the weight lattice. Also we can associate to each order filter F of P an element $w_F \in W^{\lambda_P}$.

Main Theorem follows from

$$\sum_{\pi \in \mathcal{A}(P \setminus F)} \boldsymbol{z}^{\pi} = \frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}} = \sum_{D} \frac{\prod_{v \in B(D)} \boldsymbol{z}[H_P(v)]}{\prod_{v \in P \setminus D} (1 - \boldsymbol{z}[H_P(v)])},$$

where $z_i = e^{\alpha_i}$ ($i \in I$).

Idea of Proof (3) — equivariant K-theory of partial flag variety

We can prove the first equality

$$\sum_{\pi \in \mathcal{A}(P \setminus F)} \boldsymbol{z}^{\pi} = \frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}}$$

by showing the both sides satisfy the same recurrence

$$Z_{P/F}(\boldsymbol{z}) = \frac{1}{1 - \boldsymbol{z}[P \setminus F]} \sum_{F'} (-1)^{\#(F' \setminus F) - 1} Z_{P/F'}(\boldsymbol{z}),$$

where F' runs over all order filters such that $F \subsetneq F' \subset P$ and $F' \setminus F$ is an antichain.

The second equality

$$\frac{\xi^{w_F}|_{w_P}}{\xi^{w_P}|_{w_P}} = \sum_{D} \frac{\prod_{v \in B(D)} \boldsymbol{z}[H_P(v)]}{\prod_{v \in P \setminus D} (1 - \boldsymbol{z}[H_P(v)])}$$

can be deduced from the Billey-type formula for equivariant K-theory.