# Branched continued fractions

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Séminaire Lotharingien de Combinatoire

# Plan

1. General theory

2. Application to ratios of generalized hypergeometric functions

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- Viennot (1983): consequence for total positivity

# $m$-Dyck paths

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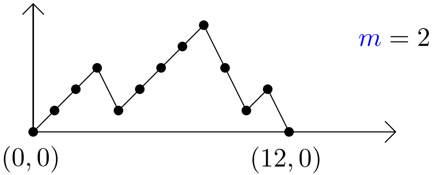

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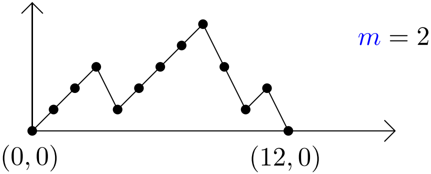


**Theorem (Fuss, 1795)**

*There are* $\frac{1}{(m+1)n+1}\binom{(m+1)n}{n}$ *$m$-Dyck paths of length $n$*

# Stieltjes continued fractions

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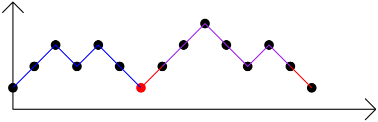

$$f_k(t) = 1 + \quad {\color{blue} f_k(t)} \quad {\color{purple} f_{k+1}(t)} \quad {\color{red} t}$$

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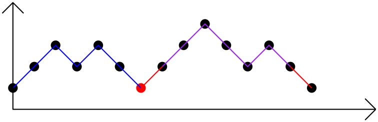

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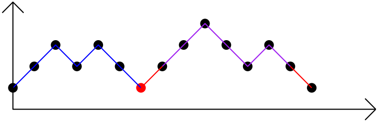

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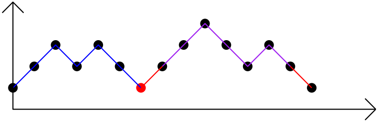
**Theorem (Flajolet, 1980)**

*We have:* $f_0(t) = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{t}{\ddots}}}$

Define $f_k(t)$ as the generating function for Dyck path starting and ending at height $k$, staying at height $\geq k$ with weight $\alpha_i$ for a fall from height $i$



$$f_k(t) = 1 + \quad f_k(t) \quad f_{k+1}(t)\alpha_k t$$

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# From $m$-Dyck Paths to $m$-branched continued fractions

Define $f_{m,k}(t)$ as the generating function for $m$-Dyck path starting and ending at height $k$ and staying at height $\geq k$ with weight $\alpha_i$ for a fall from height $i$

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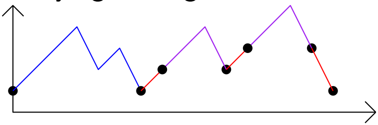

$$f_{m,k}(t) = 1 + \quad f_{m,k}(t)\, f_{m,k+1}(t) \cdots f_{m,k+m} \alpha_{k+m} t$$

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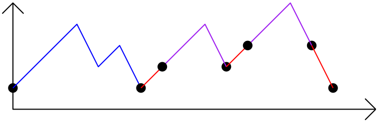

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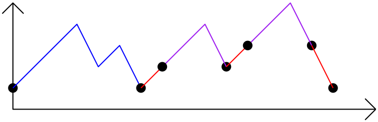

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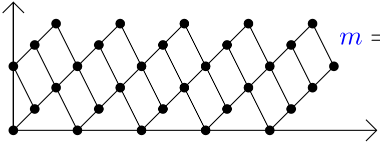
## Theorem (P.–Sokal–Zhu, 2018)

*We have:* $f_{m,0}(t) =$

$$\cfrac{1}{1 - \cfrac{\alpha_m t}{\left(1 - \cfrac{\alpha_{m+1} t}{(1 - \frac{\alpha_{m+2} t}{\ddots}) \cdots (1 - \frac{\alpha_{2m+2} t}{\ddots})}\right) \cdots \left(1 - \cfrac{\alpha_{2m} t}{(1 - \frac{\alpha_{2m+1} t}{\ddots}) \cdots (1 - \frac{\alpha_{3m} t}{\ddots})}\right)}}$$

# Total positivity

$m$-Dyck paths live in a proper sub-graph of $\mathbb{Z}^2$. This sub-graph is planar



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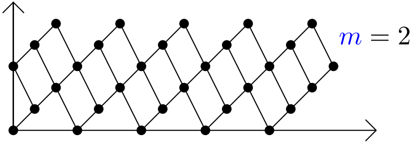


**Theorem (P.–Sokal–Zhu, 2018)**

*Let $(a_n)_{n\geq 0}$ be a sequence. If the generating function of $(a_n)_{n\geq 0}$ has a m-branched continued fraction with non-negative coefficients, then $(a_n)_{n\geq 0}$ is Hankel totally positive.*

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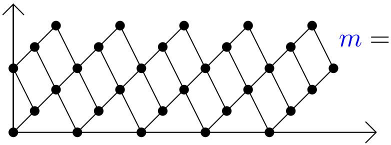


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The proof relies on Lindstrom–Gessel–Viennot theorem

# Some (proved) examples

The following sequences are Hankel totally positive as they have a nonnegative branched continued fractions:

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- $(P_n(x))_{n\geq 0}$, where $P_n(x)$ is the number of non crossing tree counted with respect to the outer degree of the root, has a 2-branched continued fraction

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# Another point of view: a recurrence relation

Starting from $f_k(t) = 1 + f_k(t) \cdots f_{k+m} \alpha_{k+2} t$, define

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**Principle (Euler, 1746; P.–Sokal–Zhu, 2018)**

A pair of sequences $(g_i(t))_{n \geq -1}$ and $(\alpha_i)_{i \geq m}$ satisfying $(*)$ gives a $m$ branched continued fraction for $g_0(t)/g_{-1}(t)$

# Example of $n!^m$

Define $g_{-1}(t) = 1$ and

$$g_{(m+1)i+k}(t) = \sum_{n\geq 0} (n!)^m \frac{(n+1)^{m+1} \ldots (n+i)^{m+1} (n+i+1)^k}{i!^{m+1-k} (i+1)^k} t^n$$

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Checking $(*)$ is purely computational.

# Generalisation of Gauss continued fraction

Define the generalized hypergeometric function

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**Theorem (P.–Sokal–Zhu, 2018)**

*Any ratio*

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Ideas of the proof:

- define an (explicit) sequence of functions $(g_i(t))_{i \geq 0}$
- prove that (*) holds for this sequence, by (proving and) using new contiguous relations for ${}_pF_q$

# Total positivity for ratios of ${}_mF_0$

Let $(A_n(a_1,\ldots,a_p))_{n\geq 0}$ be the sequence with generating function

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Applications:

- $n!^m$
- $(mn)!$
- $(2n-1)!!^m$

# Some conjectures and open questions

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- Generalized Genocchi numbers
- find all contiguous relations for ${}_pF_q$
- find a combinatorial interpretation of ratios of ${}_pF_0$ in terms of permutations

Thank you for your attention!