Refined Enumeration of Vertically Symmetric Alternating Sign Matrices

Manjil P. Saikia (Universität Wien)

80th Séminaire Lotharingien de Combinatoire, Lyon, France

27 March, 2018

VIENNA DOCTORAL SCHOOL MATHEMATICS

ASMs

An alternating sign matrix (ASM) of size n is an $n \times n$ matrix with entries in the set $\{0, 1, -1\}$ such that

- all row and column sums are equal to 1,
- and the non-zero entries alternate in each row and column.

For instance, there are 7 ASMs of order 3, these are the six permutation matrices and the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

ASMs Enumeration

Mills, Robbins and Rumsey conjectured that the number of ASMs of size n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

This conjecture was later proved by Zeilberger and by Kuperberg, shortly after Zeilberger.

Symmetry Classes of ASMs

Richard Stanley suggested the study of symmetry classes of ASMs shortly after these objects were introduced. This led Robbins to conjecture various enumeration formulas for different classes of ASMs.

Kuperberg, Okada, Razumov - Stroganov and Behrend - Fischer - Konvalinka proved the conjectured formulas for these symmetry classes of ASMs.

The last remaining formula for the class of ASMs called Diagonally and Anti-Diagonally Symmetric ASMs (DASASMs) of odd order was completed in 2015.

Symmetry Classes of ASMs contd.

Among these symmetry classes are also the vertically symmetric alternating sign matrices (VSAMSs), which are symmetric under the vertical axis.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Boundary Conditions on ASMs

It is easy to see that there can be only one occurrence of 1 in the first row and column of any ASMs. This also suggests the study of some refined enumeration of these matrices.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

VSASMs

In the case of VASAMs, it turns out that in the second row of such matrices there are exactly two occurrences of 1,

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Fischer's Conjecture

Razumov and Stroganov has a formula counting the number of VSASMs with a fixed one in the first column.

Ilse Fischer had conjectured that the number of $(2n+1) \times (2n+1)$ VSASMs, where the first one in the second row is in the ith column is equal to

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!} \left(\prod_{j=1}^{n-1} \frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!} \right). \tag{1}$$

We will prove this conjecture.

Bijection between ASMs and Six Vertex Model

Kuperberg's proof of the ASM conjecture was by exploiting a bijection between the ASMs and a model in statistical physics, called the six-vertex model.

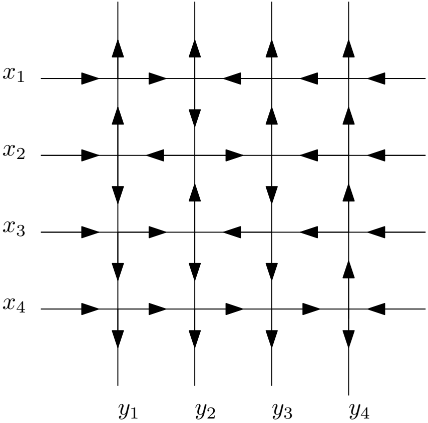

Figure: Six Vertex Model with Domain Wall Boundary Condition.

Bijection between ASMs and Six Vertex Model

A state of a corresponding six-vertex model is an orientation on the edges of this graph, such that both the in-degree and the out-degree of each vertex with degree 4 is 2.

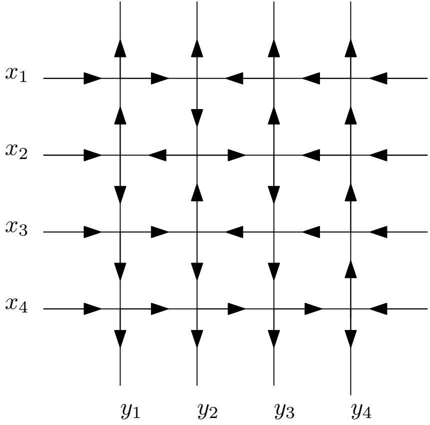

Figure: Six Vertex Model with Domain Wall Boundary Condition.

The Bijection

If we associate to each of the degree 4 vertex in a six-vertex state with a number, as given in the figure below

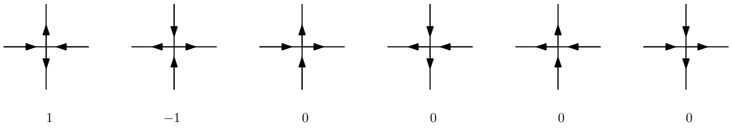

Figure: The corresponding states of the six-vertex model and the entries of an ASM.

then we obtain a matrix with entries in the set $\{0, 1, -1\}$.

Such a matrix will be an ASM, and we get a bijection between ASMs and states of the six-vertex model.

Example

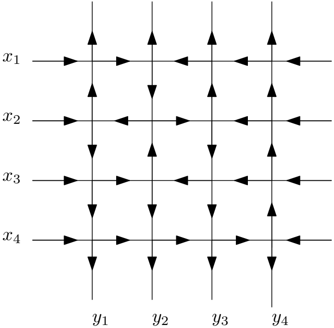

Figure: Six Vertex Model with Domain Wall Boundary Condition.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Weighted Enumeration

- We assign to each vertex v, a weight $w(v)$.
- Weight of a state C is $W(C) = \prod_{v \in C} w(v)$.
- Generating function or the partition function $Z_n = \sum_C W(C)$.
- Specializing the parameters in Z_n, we get enumeration results.

Our Weights

q is a parameter, which we will specialize later, $\overline{x} = \dfrac{1}{x} = x^{-1}$ and $\sigma(x) = x - \overline{x}$.

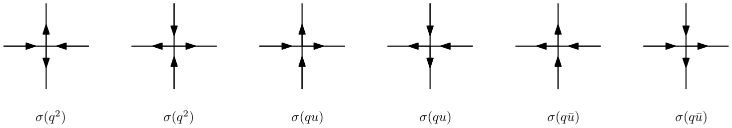

Figure: The weights of the vertices of an ASM with spectral parameter u.

Another type of ASM

In order to study VSASMs we need what are called U-turn domain boundary wall conditions.

The set of VASAMs is a subset of what are called U-turn ASMs or ASMs with U-turn boundary (UASMs).

We will explain this connection shortly.

UASMs

An U-turn ASM is an $2n \times n$ array which satisfies the usual properties of ASMs if one looks at it vertically.

However, if one looks at it horizontally then the 1's and -1's alternate if we start along an odd numbered row from left to right and then continue along the next even numbered row from right to left.

$$\begin{array}{cc} 0 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{array}$$

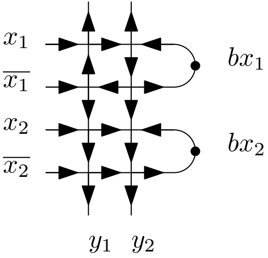

Figure: An U-turn ASM with the corresponding six-vertex state.

New weights

As can be seen from the figure, we add an additional parameter on the U-turns.

This gives rise to two new type of vertices whose corresponding weights are given below.

$\sigma(bu)$ $\qquad\qquad$ $\sigma(b\overline{u})$

Figure: Weights of the new vertices.

Partition Function

Tsuchiya was the first to consider a U-turn domain wall boundary condition, and gave a partition function for them.

$$Z_U(n; \mathbf{x}, \mathbf{y}) = \frac{\sigma(q^2)^n \prod_i (\sigma(b\overline{y_i})\sigma(q^2 x_i^2)) \prod_{i,j}(\sigma'(x_i\overline{y_j})\sigma'(x_i y_j))}{\prod_{i<j}(\sigma(\overline{x_i}x_j)\sigma(y_i\overline{y}_j)) \prod_{i\leq j}(\sigma(\overline{x_i x_j})\sigma(y_i y_j))} \times \det M_U(n; \mathbf{x}, \mathbf{y}), \tag{2}$$

where $\sigma'(x) = \sigma(qx)\sigma(q\overline{x})$ and M_U is an $n \times n$ matrix defined as

$$M_U(n; \mathbf{x}, \mathbf{y})_{i,j} = \frac{1}{\sigma'(x_i\overline{y}_j)} - \frac{1}{\sigma'(x_i y_j)}.$$

Refined Enumeration of USASMs

We assume

- The first U-turn is downward pointing.
- And the first row has no 1's, except at the U-turn.

Let $A_U(2n+1, i; w, s)$ be the total weight of the UASM whose unique 1 in the second row is in the ith column, and each -1 has a multiplicative weight of w and each upward orientation of an U-turn has multiplicative weight s.

Let us denote the partition function of such a configuration to be $Z_U(n; \mathbf{x}, \mathbf{y})$.

We will consider the case for $\mathbf{x} = (x, 1, \ldots, 1)$ and $\mathbf{y} = \mathbf{1}$.

Refined Enumeration of UASMs

If we consider only the first and second row of such a UASM, we see that the total weight of these rows will be

$$\sigma(qx)^n \sigma(q\bar{x})^{i-1} \sigma(q^2) \sigma(qx)^{n-i} \sigma(bx).$$

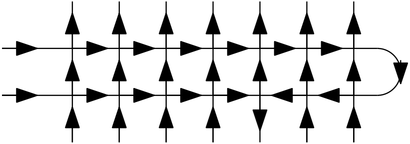

Figure: Weights of the first two.

Refined Enumeration of UASMs

If we consider only the first and second row of such a UASM, we see that the total weight of these rows will be

$$\sigma(qx)^n \sigma(q\overline{x})^{i-1} \sigma(q^2) \sigma(qx)^{n-i} \sigma(bx).$$

Let the number of -1's in the matrix under consideration be k and the number of upward turns be l, then we will see that the total weight of such a configuration would be

$$\sigma(q)^{2n^2-3n+2} \sigma(q^2)^n \sigma(qx)^{2n-1} \sigma(qb)^n \\ \times \left(\frac{\sigma(q^2)}{\sigma(q)}\right)^{2k} \left(\frac{\sigma(q\overline{x})}{\sigma(qx)}\right)^{i-1} \left(\frac{\sigma(\overline{q}b)}{\sigma(qb)}\right)^{l}. \tag{3}$$

Refined Enumeration of UASMs

From this we have the following

$$\sum_{i=1}^{n} A_U(2n+1, i; w, s) t^{i-1} = \frac{Z_U(n; (w, 1, \ldots, 1), \mathbf{1})}{\sigma(q)^{2n^2-3n+2} \sigma(q^2)^n \sigma(qx)^{2n-1} \sigma(qb)^n}, \tag{4}$$

where $w = \left(\frac{\sigma(q^2)}{\sigma(q)}\right)^2$, $s = \frac{\sigma(\overline{q}b)}{\sigma(qb)}$ and $t = \frac{\sigma(q\overline{x})}{\sigma(qx)}$.

Properties of Z_U

The partition function had been studied by Razumov and Stroganov. They consider the modified partition function

$$Z(n; \mathbf{x}, \mathbf{y}) = \frac{Z_U(n; \mathbf{x}, \mathbf{y})}{\prod_i (\sigma(b\overline{y_i})\sigma(q^2 x_i^2))}.$$

Let $f_U(n; u) = \sigma(u)^{4n-2}\sigma(u^2)Z(n; u, 1, \ldots, 1))$.

Properties of f

We have the following,

Lemma

1. *The function $u^{6n-2} f_U(n; u)$ is a polynomial of degree $6n-2$ in u^2.*
2. *The function $f_U(n; u)$ satisfies the relation*
$$f_U(n; \overline{u}) = -f_U(n; u).$$
3. *The function $f_U(n; u)$ satisfies the relation*
$$f_U(n; u) + f_U(n; a^2 u) + f_U(n; a^4 u) = 0.$$
4. *The Laurent polynomial $f_U(n; u)$ is divisible by $\sigma(u)^{4n-2}$ and by $\sigma(u^2)$.*

Properties of Z_U

Lemma

The function Z is proportional to the function

$$\varphi(2n,u)=\frac{(-1)^{2n-1}}{\sigma(q)\binom{4n-2}{2n-1}}\sum_{k=0}^{2n-1}\binom{2n-4/3}{2n-1-k}\binom{2n-2/3}{k}\times\sigma(u^{6n-2-6k}). \tag{5}$$

We also have the following

$$\frac{1}{A(n-1)}\sum_{i=1}^{n}A(n,i)t^{i-1}=\frac{\sigma(q)^{3n-2}\varphi(n;u)}{\sigma(qu)^{n-1}\sigma(u)^{2n-1}}, \tag{6}$$

where $A(n)$ is the number of $n\times n$ ASMs, and $A(n,i)$ is the number of $n\times n$ ASMs with the first 1 in the first row at the ith position.

Properties of Z_U

If $q = \exp(i\pi/3)$,

$$\frac{1}{A_U(2n-2;1,s)} \sum_{i=1}^{n} A_U(2n,i;1,s)t^{i-1} = \frac{\sigma(q)^{6n-3}\sigma(\overline{q}b)\sigma(b)\varphi(2n;u)}{2\sigma(u)^{4n-2}\sigma(u^2)\sigma(qu)^{2n-1}\sigma(qb)} \tag{7}$$

Generating Function

Combining everything from above, we get

Theorem

$$\frac{1}{A_U(2n-2;1,s)} \sum_{i=1}^{n} A_U(2n,i;1,s)t^{i-1} = \frac{1}{A(2n-1)} \frac{t+s}{t} \sum_{i=1}^{n} A(2n,i)t^{i-1}. \tag{8}$$

Some observations

- VSASMs occur only for odd order.
- We need only the first $n+1$ columns of the VSASM to know the full matrix.
- The middle column is an alternating row with 1 and -1.
- So, n columns are sufficient to know the whole matrix.
- Moreover, the first and last rows are always the same.

Transformation

We can transform a VSASM into an USASM in two steps:

- Delete the last row.
- Connect pairwise the alternating edges on the right most column of the $2n \times n$ matrix.

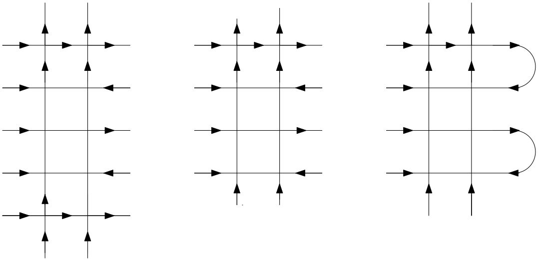

Figure: Transformation of a VSASM into an UASM.

Notice that all U-turns are downward pointing.

Proof

Theorem

The number of $(2n+1)\times(2n+1)$ VSASM with a 1 in the i-th position in it's second row is given by

$$\frac{(2n+i-2)!(4n-i-1)!}{2^{n-1}(4n-2)!(i-1)!(2n-i)!}\left(\prod_{j=1}^{n-1}\frac{(6j-2)!(2j-1)!}{(4j-1)!(4j-2)!}\right). \qquad (9)$$

For any VSASM, we shall have $s=0$,so if we equate the coefficients of t in the final equation earlier we get the following recursion

$$A(2n-1)A_V(2n,i;1,y)=A_V(2n-2;1,y)A(2n,i). \qquad (10)$$

VIENNA DOCTORAL SCHOOL MATHEMATICS

Proof

This gives us

$$A_V(2n+1, i) = \frac{A_V(2n-1)A(2n, i)}{A(2n-1)},$$

where $A_V(2n-1)$ is the number of $2n-1 \times 2n-1$ VSASMs, and $A_V(2n+1, i)$ is the number of $2n+1 \times 2n+1$ VSASMs where the first 1 in the second row is at the ith column.

The quantities on the right are all known, and we get the result.

VIENNA DOCTORAL SCHOOL MATHEMATICS

Remarks

In the light of Theorem 4, as already pointed out by Fischer, we also have the following result.

Theorem

$$A_V(2n+1; i) = A'_V(2n+1; i) + A'_V(2n+1; i+1),$$

where $A'_V(2n+1; i)$ *are the number of* $2n+1 \times 2n+1$ *VSASM with the unique* 1 *in the first column at the* i*-th position.*

A bijective proof of this result would be very interesting.

Thank you for your attention!