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INTERLACED RECTANGULAR PARKING FUNCTIONS

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ABSTRACT. The aim of this work is to extend the Grossman–Bizley [*Scripta Math.* **16** (1950), 207–212; *J. Inst. Actuar.* **80** (1954), 55–62] paradigm that allows the enumeration of Dyck paths in an $m \times n$-rectangle to a general $\mathbb{S}_m \times \mathbb{S}_n$-module context. We obtain an explicit formula for the the "bi-Frobenius" characteristic of what we call *interlaced* rectangular parking functions in an $m \times n$-rectangle. These are obtained by labeling the n vertical steps of an $m \times n$-Dyck path by the numbers from 1 to n, together with an independent labeling of its horizontal steps by integers from 1 to m. Our formula specializes to give the Frobenius characteristic of the $\mathbb{S}_n$-module of $m \times n$-parking functions in the general situation. Hence, it subsumes the result of Armstrong, Loehr and Warrington of [*Ann. Combin.* **20** (2016), 21–58], which furnishes such a formula for the special case where m and n are coprime integers.

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INTRODUCTION

The purpose of this paper is to extend the Grossman–Bizley enumeration formula (see [9, 14]) for the number of Dyck-like paths in an $m \times n$-rectangle to the context of parking functions. These are the south-east lattice paths that start from the north-west corner of the rectangle, end at its south-east corner, and stay below the line between these corners. To each such path we associate a family of parking functions, seen as labelings of the vertical steps of the path. We therefore consider enumeration problems for the global set of such functions, which is called the set of (m, n)-parking functions. More precisely, we obtain explicit formulas for the character (Pólya-enumeration) of the

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$\mathbb{S}_n$-module of (m,n)-parking function in the general context. Such formulas have already been established (see [1]) in the special "coprime" case, i.e., when m and n are coprime.

We then extend our approach to get formulas for bi-labeled paths. This means that we independently label south-steps by the numbers 1 to n, and east-steps by the numbers from 1 to m. We give an explicit formula (see 4.3) for the character of the resulting $\mathbb{S}_n \times \mathbb{S}_m$-module, thus characterizing its decomposition into irreducibles for the joint (commuting) actions of $\mathbb{S}_n$ and $\mathbb{S}_m$.

1. Rectangular Dyck paths

An **(m,n)-Dyck path** is a south-east lattice path, going from $(0,n)$ to $(m,0)$, which stays below the **(m,n)-diagonal**, which is the line segment joining $(0,n)$ to $(m,0)$. See Figure 1 for an example.

Figure 1. The $(10,5)$-Dyck path encoded as 76300

We encode such paths as decreasing integer sequences

$$\alpha = a_1 a_2 \cdots a_n, \qquad \text{with} \quad 0 \leq a_k \leq (n-k)\, m/n.$$

with each a_k giving the distance between the y-axis of the (unique) south-step that starts at level k. In other terms, α is an integer partition, with added 0-parts to make it of length n, lying inside the **(m,n)-staircase**

$$\delta_{m,n} := d_1 d_2 \cdots d_n, \qquad \text{with} \qquad d_k := \lfloor (n-k)\, m/n \rfloor.$$

Hence it makes sense to say that the **conjugate path** of an (m,n)-path α, denoted by α', is the (n,m)-path that corresponds to the conjugate partition. As an example, $\delta_{6,4} = 4310$. It is easy to check that $\delta_{kn,n} = \delta_{kn+1,n}$. We denote by $\mathcal{D}_{m,n}$, the set of (m,n)-Dyck paths, and by $C_{m,n}$ its cardinality. It follows from the observation that $\delta_{kn,n} = \delta_{kn+1,n}$ that we have the set equality

$$\mathcal{D}_{kn,n} = \mathcal{D}_{kn+1,n}. \tag{1.1}$$

Examples of values of $C_{m,n} = |\mathcal{D}_{m,n}|$ are given in the following table (observe the obvious symmetry in m and n).

$m \setminus n$	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	2	2	3	3	4	4	5	5
3	1	2	5	5	7	12	12	15	22
4	1	3	5	14	14	23	30	55	55
5	1	3	7	14	42	42	66	99	143
6	1	4	12	23	42	132	132	227	377
7	1	4	12	30	66	132	429	429	715

Observe that it is only when k takes the form $k = n - j\,b$, with $d = \gcd(m,n)$ and $(m,n) = (ad, bd)$, that we may have $(n-k)\,m/n$ lying in $\mathbb{N}$. Thus (m,n)-paths α may only **return** to the diagonal at such positions k. The set of such return positions clearly forms a subset of $\{1,\dots,d-1\}$, which is encoded in the usual manner[1] as a composition of d.

If m and n are coprime, the enumeration of (m,n)-Dyck path is given by the long known formula[2]

$$C_{m,n} = \frac{1}{m+n}\binom{m+n}{n}.$$

Observation (1.1), and a simple calculation, implies that the classical Catalan numbers (or more generally Fuss–Catalan numbers) can be obtained from this.

If m and n have a greatest common divisor d other than 1, the relevant formula is more complicated and seems to have escaped the attention of many until recently. In fact, it appears that it was first stated in 1950 by Grossman [14], and then proved by Bizley [9] in 1954. As we will see more clearly later, it is useful to recast this formula in terms of ring homomorphisms applied to symmetric functions. More specifically, for each fixed coprime pair (a,b), consider the ring homomorphism

$$\theta_{a,b} : \Lambda \longrightarrow \mathbb{Z}$$

defined by

$$\theta_{a,b}(p_k(\mathbf{x})) := \frac{1}{a+b}\binom{ak+bk}{ak},$$

where $\mathbf{x} = x_1, x_2, \dots$ is an infinite alphabet, and $p_k(\mathbf{x})$ stands for the classical **power sum symmetric function of degree k** in $\mathbf{x}$. Then, for $(m,n) = (ad, bd)$ with $d =$

[1] Recall that, to a composition $\gamma = (c_1,\dots,c_k)$, this correspondence associates the set of partial sums $S(\gamma) = \{s_1, s_2, \dots, s_{k-1}\}$, where $s_i = c_1 + c_2 + \cdots + c_i$, with $1 \le i < k$.

[2] which may be obtained by a classical cycle argument maybe due to Dvoretzky and Motzkin (see [12]), or even earlier to Łukasiewicz.

$\gcd(m,n)$, Grossman–Bizley formula may be very simply written as

$$C_{ad,bd} = \theta_{a,b}(h_d(\mathbf{x})), \tag{1.2}$$

where $h_d(\mathbf{x})$ stands[3] for the usual **complete homogeneous symmetric function of degree d**. Recall that, in generating function format, the link between power sum and complete homogeneous symmetric functions may be expressed as

$$\sum_{d=0}^{\infty} h_d(\mathbf{x})\, w^d = \exp\left(\sum_{k\geq 1} p_k(\mathbf{x})\, \frac{w^k}{k}\right). \tag{1.3}$$

Hence, in generating function terms, Formula (1.2) may be written as

$$\sum_{d=0}^{\infty} C_{ad,bd}\, w^d = \exp\left(\sum_{k\geq 1} \frac{1}{a+b}\binom{ak+bk}{ak}\frac{w^k}{k}\right). \tag{1.4}$$

This will be derived from a more general formula in the sequel (see Proposition 3). Bizley also showed (and we will see in the sequel that this generalizes as well) that the number $C'_{ad,bd}$ of **primitive (ad,bd)-Dyck paths** is given by

$$C'_{ad,bd} = \theta_{a,b}((-1)^{d-1} e_d(\mathbf{x})), \tag{1.5}$$

where $e_d(\mathbf{x})$ is the **elementary symmetric functions of degree d**. Recall that primitive paths are those that remain strictly below the diagonal. From this, it easily follows that one can enumerate the set of (m,n)-Dyck paths with returns to the diagonal encoded by a composition $\gamma = (c_1,\ldots,c_k)$ of d. These are the (m,n)-Dyck paths that go through the points $(a\, s_i, n - b\, s_i)$, with the notation of Footnote 1. The relevant enumeration formula is then

$$|\mathcal{D}_\gamma^{a/b}| = \theta_{a,b}((-1)^{d-k} e_{c_1}(\mathbf{x})\cdots e_{c_k}(\mathbf{x})), \tag{1.6}$$

where we write $\mathcal{D}_\gamma^{a/b}$ for the set of (m,n)-Dyck paths having returns to the diagonal exactly at the points specified by γ. Clearly, the set $\mathcal{D}_{m,n}$ of all (m,n)-Dyck paths decomposes as the disjoint union[4]

$$\mathcal{D}_{m,n} = \sum_{\gamma\models d} \mathcal{D}_\gamma^{a/b},$$

where $\gamma \models d$ means that γ is a composition of d. The set $\mathcal{D}'_{m,n}$ of primitive (ad,bd)-Dyck paths simply corresponds to the case of the one part composition $\gamma = (d)$, i.e.,

$$\mathcal{D}'_{m,n} = \mathcal{D}_{(d)}^{a/b}.$$

[3] We are using Macdonald's [21] notations here.

[4] We use summation to denote disjoint union.

2. (m,n)Parking functions

To each (m,n)-Dyck path $\alpha = a_1a_2\cdots a_n$, we associate the set $\mathcal{P}_\alpha$ of **α-parking functions**:

$$\mathcal{P}_\alpha := \{a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)} \mid \sigma \in \mathbb{S}_n\}.$$

For $\pi \in \mathcal{P}_\alpha$, we also say that α is the **shape** of π. Observe that α-parking functions may be identified with standard Young tableaux[5] of skew shape $(\alpha + 1^n)/\alpha$, where $\alpha + 1^n$ is the partition having parts $a_k + 1$. Indeed, if k sits in row j of $(\alpha + 1^n)/\alpha$, then one sets $\pi := b_1b_2\ldots b_n$, with $b_k := a_j$.

By definition, the symmetric group acts transitively on $\mathcal{P}_\alpha$. Indeed, for $\pi = b_1b_2\cdots b_n$ and $\sigma \in \mathbb{S}_n$, this action is defined by

$$\sigma \cdot \pi := b_{\sigma^{-1}(1)}b_{\sigma^{-1}(2)}\cdots b_{\sigma^{-1}(n)}.$$

The set of **(m,n)-parking functions**, denoted by $\mathcal{P}_{m,n}$, is the set of α-parking functions with α varying in the set of (m,n)-Dyck paths,

$$\mathcal{P}_{m,n} := \sum_{\alpha \in \mathcal{D}_{m,n}} \mathcal{P}_\alpha.$$

It clearly affords a permutation action of $\mathbb{S}_n$, whose orbits are the $\mathcal{P}_\alpha$'s. Obviously, the stabilizer of an (m,n)-Dyck path α (considered as a special (m,n)-parking-function) is the Young subgroup

$$\mathbb{S}_\rho := \mathbb{S}_{r_0} \times \mathbb{S}_{r_1} \times \cdots \times \mathbb{S}_{r_m},$$

where $\rho = \rho(\alpha) := (r_0, r_1, \ldots, r_m)$, with r_i equal to the number of occurrences of i in α. We may as well remove zero parts from $\rho(\alpha)$, since these parts play no role. The result is said to be the **riser composition** of α. It follows that the number of α-parking functions is given by the multinomial coefficient

$$|\mathcal{P}_\alpha| = \binom{n}{\rho(\alpha)} := \frac{n!}{r_0!\, r_1! \cdots r_k!}, \tag{2.1}$$

and thus

$$|\mathcal{P}_{m,n}| = \sum_{\alpha \in \mathcal{D}_{m,n}} \binom{n}{\rho(\alpha)}. \tag{2.2}$$

If m and n are coprime, (m,n)-parking functions may be seen to give canonical coset representatives of the subgroup $H := u\,\mathbb{Z}$, with $u = (1,1,\ldots,1)$, inside the abelian group $\mathbb{Z}_m^n$. Here, elements of $\mathbb{Z}_m^n$ correspond to general sequences of length n, with entries between 0 and $m-1$, whereas (m,n)-parking functions correspond to the special case for which such a sequence becomes an (m,n)-Dyck path when its entries are sorted (from smallest to largest). Indeed, it may be shown that each coset contains a unique (m,n)-parking function. These considerations have the following consequence.

[5] naturally using french notation.

Lemma 1 (Armstrong, Loehr, Warrington [1]). *For coprime m and n, the number of (m,n)-parking functions is*

$$|\mathcal{P}_{m,n}| = m^{n-1}. \tag{2.3}$$

If m and n are not coprime, cosets of H will contain up to d (>0) elements that are (m,n)-parking functions. Exploiting this fact, one can get an analog of Formula (1.2). We will not do this, since this actually follows from a finer result discussed in Section 3. Table 1 gives small explicit values in the general case. From now on, we set $(m,n) = (ad, bd)$ with a and b coprime (thus $d = \gcd(m,n)$).

$n \setminus m$	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	3	3	5	5	7	7	9
3	1	4	16	16	25	49	49	64
4	1	11	27	125	125	243	343	729
5	1	16	81	256	1296	1296	2401	4096
6	1	42	378	1184	3125	16807	16807	35328
7	1	64	729	4096	15625	46656	262144	262144

Table 1. Number of (m,n)-parking functions

3. Frobenius characteristic of the parking function representations

For a fixed integer composition ρ of n, consider the transitive permutation action of $\mathbb{S}_n$ on the set of **ρ-set partitions** of $\{1,2,\ldots,n\}$. These are the set partitions that have block sizes specified by ρ. One may consider this as a representation of $\mathbb{S}_n$, having dimension equal to the multinomial coefficient

$$\binom{n}{\rho} = \frac{n!}{\rho_1!\,\rho_2!\cdots\rho_k!}.$$

It is classical that the Frobenius transform[6] of the character of the resulting $\mathbb{S}_n$-module is $h_\rho(\mathbf{x}) := h_{\rho_1}(\mathbf{x})h_{\rho_2}(\mathbf{x})\cdots h_{\rho_k}(\mathbf{x})$. Recall that this means that the coefficients of the Schur function expansion of $h_\rho(\mathbf{x})$ correspond to multiplicities of irreducibles.

For a given (m,n)-Dyck path α of height n, the $\mathbb{S}_n$-action on α-parking functions is isomorphic to the action of $\mathbb{S}_n$ on ρ-set partitions, with $\rho = \rho(\alpha)$ equal to the riser-composition α. Thus $h_{\rho(\alpha)}(\mathbf{x})$ is the associated Frobenius characteristic.

[6]We simply say: Frobenius characteristic.

It follows from this that the Frobenius characteristic of the $\mathbb{S}_n$-action on (m,n)-parking functions, which we denote by $\mathcal{P}_{m,n}(\mathbf{x})$, can be calculated as follows

$$\mathcal{P}_{m,n}(\mathbf{x}) = \sum_{\alpha\in\mathcal{D}_{m,n}} h_{\rho(\alpha)}(\mathbf{x}). \tag{3.1}$$

As discussed in [1], and borrowing a presentation format inspired by [24], we have the following formulas.

Proposition 2 (Armstrong, Loehr, Warrington). *For coprime positive integers m and n, we have*

$$\mathcal{P}_{m,n}(\mathbf{x}) = \frac{1}{m}\sum_{\lambda\vdash n} m^{\ell(\lambda)}\,\frac{p_\lambda(\mathbf{x})}{z_\lambda} \tag{3.2}$$

$$= \frac{1}{m}\sum_{\lambda\vdash n} s_\lambda(1^m)\,s_\lambda(\mathbf{x}) \tag{3.3}$$

$$= \sum_{\lambda\vdash n} \frac{(m-1)\,(m-2)\cdots(m-\ell(\lambda)+1)}{d_1(\lambda)!\cdots d_k(\lambda)!}\,h_\lambda(\mathbf{x}),$$

where $d_i(\lambda)$ is the number of parts of size i in λ and $z_\lambda = \prod_{i\geq 1} i^{d_i(\lambda)d_i(\lambda)!}$.

From (3.3), we may calculate that the respective multiplicities of the trivial and the sign representation in $\mathcal{P}_{m,n}$ are given (as expected) by

$$\frac{1}{m}\,h_n(1^m) = \frac{1}{m+n}\binom{m+n}{n}, \qquad \text{and} \qquad \frac{1}{m}\,e_n(1^m) = \frac{1}{m}\binom{m}{n},$$

since these occur as coefficients of $s_n(\mathbf{x})$ and $s_{1^n}(\mathbf{x})$, respectively. More generally, the other multiplicities may be obtained using the classical evaluation (involving hook lengths)

$$s_\lambda(1^m) = \prod_{(i,j)\in\lambda} \frac{m+i-j}{(\lambda_j-i)+(\lambda'_i-j)+1}.$$

It is also worth recalling that the sign-twisted version[7] of $\mathcal{P}_{n,n}(\mathbf{x})$ is the Frobenius characteristic of the space of "diagonal harmonics" [15].

Formula for the non-coprime case. To get formulas for the non-coprime case, we generalize (1.2) in a natural manner as below. Once again, we assume that $(m,n) = (ad,bd)$ with (a,b) coprime, and consider γ a composition of d. We adapt our notations from Section 1 to parking functions. Consequently, we write $\mathcal{P}^{a/b}_\gamma$ for the set of (ad,bd)-parking function whose underlying path lies in the set $\mathcal{D}^{a/b}_\gamma$:

$$\mathcal{P}^{a/b}_\gamma := \sum_{\alpha\in\mathcal{D}^{a/b}_\gamma} \mathcal{P}_\alpha.$$

[7]This simply means that we replace $p_k(\mathbf{x})$ by $(-1)^{k-1}p_k(\mathbf{x})$ in (3.2).

Likewise, $\mathcal{P}'_{m,n}$ is the set of **primitive (m,n)-parking functions**, i.e., those whose underlying paths only touch the diagonal at both ends. Maintaining our previous conventions, we set

$$\mathcal{P}^{a/b}_{\gamma}(\mathbf{x}) := \sum_{\alpha\in\mathcal{D}^{a/b}_{\gamma}} h_{\rho(\alpha)}(\mathbf{x}), \qquad \text{and} \qquad \mathcal{P}'_{m.n}(\mathbf{x}) := \sum_{\alpha\in\mathcal{D}'_{m,n}} h_{\rho(\alpha)}(\mathbf{x}).$$

In the same spirit as previously, we consider a ring homomorphism $\Theta_{a,b} : \Lambda \longrightarrow \Lambda$ that sends degree d homogeneous symmetric functions to degree $n = bd$ homogeneous symmetric functions. Just as before, this homomorphism is characterized by its effect on the algebraic generators $p_k(\mathbf{x})$, setting

$$\Theta_{a,b}(p_k(\mathbf{x})) := \frac{1}{a}\sum_{\lambda\vdash ak}(ak)^{\ell(\lambda)}\,\frac{p_\lambda(\mathbf{x})}{z_\lambda},$$

For a degree d symmetric function $f_d(\mathbf{x})$, we also write $f^{a/b}_d(\mathbf{x})$ for the image of $f_d(\mathbf{x})$ under the homomorphism $\Theta_{a,b}$, i.e.,

$$f^{a/b}_d(\mathbf{x}) := \Theta_{a,b}(f_d(\mathbf{x})). \tag{3.4}$$

The following proposition extends the approach of Bizley (see [9]) for the enumeration of (m,n)-Dyck paths to the (m,n)-parking function Frobenius characteristic. Its proof makes use of the notion of **rank** of points along (m,n)-paths, which is simply defined as

$$\mathrm{rank}(x,y) := m\,y + n\,x.$$

Proposition 3. *Let $(m,n) = (ad, bd)$, with a and b coprime, and consider $\gamma = (c_1,\ldots,c_k)$ a composition of d. Then we have*

$$\mathcal{P}_{m,n}(\mathbf{x}) = \Theta_{a,b}(h_d(\mathbf{x})), \tag{3.5}$$

$$\mathcal{P}'_{m,n}(\mathbf{x}) = \Theta_{a,b}((-1)^{d-1}e_d(\mathbf{x})), \tag{3.6}$$

$$\mathcal{P}^{a/b}_{\gamma}(\mathbf{x}) = \Theta_{a,b}((-1)^{d-k}e_{c_1}(\mathbf{x})\cdots e_{c_k}(\mathbf{x})). \tag{3.7}$$

Proof. The proof is essentially an adaptation of Bizley's original proof, integrating symmetric functions arguments.

For the purpose of our argument, we consider the set $\mathcal{B}_{m,n}$ of all south-east lattice paths going from $(0,n)$ to $(m,0)$, which end with an east-step (without any condition relative to the diagonal). We think of these as length $m+n$ "words" in the letters y and x, which encode the successive steps, with y standing for a south-step and x for an east-step. Thus, the paths in $\mathcal{B}_{m,n}$ bijectively correspond to all possible words containing n copies of y and m copies of x, with final letter equal to x. It clearly follows that the number of such words/paths is

$$|\mathcal{B}_{m,n}| = \binom{m+n-1}{n}. \tag{3.8}$$

We bijectively label the n south-steps of such paths with the integers 1 to n, just as we earlier did for parking functions. This is to say that labels decrease along consecutive south-steps. The resulting set of labeled paths is denoted by $\mathcal{L}_{m,n}$. The symmetric group $\mathbb{S}_n$ acts on $\mathcal{L}_{m,n}$ by permuting labels, and the Frobenius characteristic of this (permutation) action is

$$\sum_{\alpha\in\mathcal{B}_{m,n}} h_{\rho(\alpha)}(\mathbf{x}) = \sum_{\lambda\vdash n}(m)^{\ell(\lambda)}\,\frac{p_\lambda(\mathbf{x})}{z_\lambda} = a\,p_d^{a/b}(\mathbf{x}). \tag{3.9}$$

To see this, observe that the stabilizer of an orbit of $\mathcal{L}_{m,n}$ (which corresponds to some fixed underlying path α) is the Young subgroup $\mathbb{S}_{r_1}\times\mathbb{S}_{r_2}\times\cdots\times\mathbb{S}_{r_j}$, where $\rho(\alpha) = (r_1, r_2, \ldots, r_j)$, and that the action is the action induced by the trivial action of $\mathbb{S}_{r_1}\times\mathbb{S}_{r_2}\times\cdots\times\mathbb{S}_{r_j}$. The second equality in (3.9) is by definition (3.4). The first equality may be derived from the Cauchy identity [11] as follows. Let $\mathbf{y} = y_1, y_2, \ldots$ denote a (second) infinite alphabet. We have

$$\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\frac{1}{1-x_iy_j} = \sum_{\lambda} h_\lambda(\mathbf{x})h_\lambda(\mathbf{y}) = \sum_{\lambda}\frac{1}{z_\lambda}p_\lambda(\mathbf{x})p_\lambda(\mathbf{y}). \tag{3.10}$$

Setting the first m variables y_j to t and the other ones to 0 in (3.10), and then taking the coefficient of t^n, we obtain

$$\sum_{\lambda\vdash n} h_\lambda(\mathbf{x})\binom{m}{m_1,\ldots,m_n,m-\ell(\lambda)} = \sum_{\lambda\vdash n}\frac{1}{z_\lambda}p_\lambda(\mathbf{x})m^{\ell(\lambda)}, \tag{3.11}$$

with $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$. The left-hand side of (3.11) is precisely $\sum_{\alpha\in\mathcal{B}_{m,n}} h_{\rho(\alpha)}(\mathbf{x})$.

We next consider *highest rank points* along a path α in $\mathcal{B}_{m,n}$. Namely, these are the points (i,j) on the path for which $\mathrm{rank}(i,j)$ reaches its maximal value. To simplify our discussion, we remove the point $(m,0)$ from those considered. Clearly, a path α may have more than one highest rank point, but the number of such points is at most d. We denote by $\mathcal{B}^t_{m,n}$ (respectively $\mathcal{D}^t_{m,n}$) the subset of $\mathcal{B}_{m,n}$ (respectively $\mathcal{D}_{m,n}$) consisting of paths with exactly t highest points.

We then consider *cyclic permutations* of $\alpha = \ell_1\cdots\ell_{m+n}$ (where either $\ell_i = x$ or $\ell_i = y$) in the following sense. Choosing any occurrence of x in α, say at ℓ_i, we cut the path after this x, and build a new word β by switching the two resulting components of α:

$$\alpha = \ell_1\cdots\ell_i\,\ell_{i+1}\cdots\ell_{m+n} \quad\mapsto\quad \beta = \ell_{i+1}\cdots\ell_{m+n}\,\ell_1\cdots\ell_i.$$

Observe that the number of highest rank points is invariant under such cyclic permutations, and that these highest rank points can be brought to the diagonal by some cyclic permutation. Moreover, the riser-composition of α is cyclically preserved. Hence, the Frobenius characteristic is left unchanged, and it equals $h_{\rho(\alpha)}(\mathbf{x})$ for both the action of $\mathbb{S}_n$ on labelings with underlying path α or β. Figure 2 illustrates the notion of highest point (two in this case, represented by blue dots) and the procedure of cyclic permutation (the cut ℓ_i appears as a black cross).

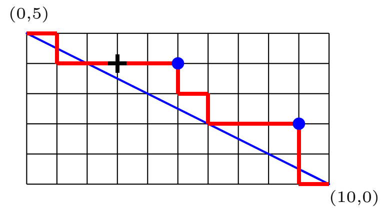

FIGURE 2. Cyclic permutation of an element of $\mathcal{B}_{10,5}$ with 2 highest points

The key point is the following: the cyclic permutation allows us to build a bijection

$$\mathcal{D}^t_{m,n} \times [m] \xrightarrow{\sim} \mathcal{B}^t_{m,n} \times [t]. \tag{3.12}$$

Consider $(\alpha, j) \in \mathcal{D}^t_{m,n} \times [m]$. By cutting α after its j-th east step, and performing the corresponding cyclic permutation, we get an element γ of $\mathcal{B}^t_{m,n}$. We may keep track of the final point of α, which corresponds to one of the t highest points of γ (call it h). The reverse bijection consists in applying the cyclic permutation to γ in position h: we get back α, and we keep track of the final point of γ as the index j.

Now recall from our hypothesis that $(m, n) = (ad, bd)$ with (a, b) coprime. Denote by $\mathcal{P}^t_{ad,bd}(\mathbf{x})$ the Frobenius characteristic of the parking functions associated with (ad, bd)-Dyck paths having exactly t contact points with the diagonal. From bijection (3.12), we get that $\sum_{t=1}^{d}(da/t)\,\mathcal{P}^t_{ad,bd}(\mathbf{x})$ is the Frobenius characteristic of $\mathcal{L}_{ad,bd}$. Consequently, because of (3.9) and after simplification, we obtain

$$\sum_{t=1}^{d} \frac{1}{t}\,\mathcal{P}^t_{ad,bd}(\mathbf{x}) = \frac{1}{d}\,p_d^{a/b}(\mathbf{x}). \tag{3.13}$$

Since the case $t = 1$ of $\mathcal{P}^t_{ad,bd}(\mathbf{x})$ corresponds to the Frobenius characteristic of the primitive (ad, bd)-Dyck path, we clearly have

$$\mathcal{P}^t_{ad,bd}(\mathbf{x}) = \sum_{\gamma \models_t d} \mathcal{P}'_{ac_1,bc_1}(\mathbf{x})\,\mathcal{P}'_{ac_2,bc_2}(\mathbf{x}) \cdots \mathcal{P}'_{ac_t,bc_t}(\mathbf{x}), \tag{3.14}$$

where the sum is over length t compositions $\gamma = (c_1, c_2, \ldots, c_t)$ of d. In other terms, if we set

$$\mathcal{P}'_{a,b}(\mathbf{x}; z) := \sum_{j=1}^{\infty} \mathcal{P}'_{aj,bj}(\mathbf{x})\,z^j,$$

then $\mathcal{P}^t_{ak,bk}(\mathbf{x})$ is the coefficient of z^k in $\left(\mathcal{P}'_{a,b}(\mathbf{x};z)\right)^t$. We may thus argue that Equation (3.13) says that $\frac{1}{k}\,p_k^{a/b}(\mathbf{x})$ is the coefficient of z^k in $\log(1/(1-\mathcal{P}'_{a,b}(\mathbf{x};z))$, so that

$$\mathcal{P}'_{a,b}(\mathbf{x};z) = 1-\exp\Big(-\sum_{k=1}^{\infty} p_k^{a/b}(\mathbf{x})\,z^k/k\Big) \tag{3.15}$$

$$= \Theta_{a,b}\left(\sum_{d=1}^{\infty}(-1)^{d-1}e_d(\mathbf{x})\,z^d\right), \tag{3.16}$$

which is equivalent to (3.6), which in turn readily implies (3.7). Clearly,

$$\mathcal{P}_{ad,bd}(\mathbf{x}) = \sum_{t>0}\mathcal{P}^t_{ak,bk}(\mathbf{x}),$$

so that

$$\mathcal{P}_{a,b}(\mathbf{x};z) = \frac{1}{1-\mathcal{P}'_{a,b}(\mathbf{x};z)} = \exp\Big(\sum_{k=1}^{\infty} p_k^{a/b}(\mathbf{x})\,z^k/k\Big) \tag{3.17}$$

$$= \Theta_{a,b}\left(\sum_{d=0}^{\infty} h_d(\mathbf{x})\,z^d\right), \tag{3.18}$$

which concludes the proof of Proposition 3. □

For example, for any coprime a and b, we get

$$\mathcal{P}_{2a,2b}(\mathbf{x}) = \frac{1}{2a}h_{2b}[2a\,\mathbf{x}] + \frac{1}{2}\left(\frac{1}{a}h_b[a\,\mathbf{x}]\right)^2,$$

$$\mathcal{P}'_{2a,2b}(\mathbf{x}) = \frac{1}{2a}h_{2b}[2a\,\mathbf{x}] - \frac{1}{2}\left(\frac{1}{a}h_b[a\,\mathbf{x}]\right)^2,$$

which give the Frobenius characteristic that correspond to $(2a,2b)$-Dyck paths and primitive $(2a,2b)$-Dyck paths, respectively. The notation $h_n[m\,\mathbf{x}]$ refers to the well-known plethystic substitution (see for example [19]). To get explicit formulas for the number of (m,n)-parking functions, we need simply compute the scalar product $\langle\mathcal{P}_{m,n}(\mathbf{x}), h_1^n(\mathbf{x})\rangle$. Likewise for $\mathcal{P}'_{m,n}$ or $\mathcal{P}^{a/b}_\gamma$.

4. Bi-Frobenius characteristic

As before, let $(m,n)=(ad,bd)$ with a and b coprime. We can now consider the "riser-step" bi-Frobenius characteristic of (m,n)-parking function, which may be defined/calculated to be

$$\mathcal{P}_{m,n}(\mathbf{x},\mathbf{y}) := \sum_{\alpha\in\mathcal{D}_{m,n}} h_{\rho(\alpha)}(\mathbf{x})\,h_{\rho(\alpha')}(\mathbf{y}), \tag{4.1}$$

where, as before, $\mathbf{y} = y_1, y_2, \dots$ stands for another denumerable alphabet of variables. Hence, $\alpha(\mathbf{x})$ encodes the "riser structure" of α, whereas $\alpha'(\mathbf{y})$ encodes its "step structure" (which are the risers of the conjugate path α'). Clearly this bi-Frobenius characteristic affords the symmetry

$$\mathcal{P}_{m,n}(\mathbf{x},\mathbf{y}) = \mathcal{P}_{n,m}(\mathbf{y},\mathbf{x}). \tag{4.2}$$

Once again, there is a Bizley-like formula for $\mathcal{P}_{m,n}(\mathbf{x},\mathbf{y})$, which subsumes (up to some calculations) all of our previous results. This formula is the subject of the following theorem.

Theorem 4. *For coprime positive integers a and b, we have*

$$\sum_{d=0}^{\infty} \mathcal{P}_{ad,bd}(\mathbf{x},\mathbf{y})\ z^d = \exp\left(\sum_{k\geq 1} p_k^{a/b}(\mathbf{x},\mathbf{y}) z^k/k\right), \tag{4.3}$$

where we set

$$p_k^{a/b}(\mathbf{x},\mathbf{y}) := \sum_{j=1}^{k} \frac{k}{j} \sum_{\rho\models_j bk,\ \sigma\models_j ak} h_\rho(\mathbf{x})\, h_\sigma(\mathbf{y}), \tag{4.4}$$

and where, as before, we write $\rho \models_j n$ to say that ρ is a composition of n having j parts.

Proof. The proof of Theorem 4 uses a refinement of the argument used to prove Proposition 3. We introduce the set $\mathcal{C}_{m,n}$ of lattice paths in $\mathcal{B}_{m,n}$ with the additional condition that they start with a south step. By definition, *corners* of a south-east lattice path are the points that lie between an east-step and a following south-step. We consider (m,n)-Dyck paths α having t highest points and j corners, and modify the argument of Proposition 3 by restricting cuts to points that lie at one of the j corners. In the same way as (3.12), we get a bijection

$$\mathcal{D}_{m,n}^{t,j} \times [j] \xrightarrow{\sim} \mathcal{C}_{m,n}^{t,j} \times [t], \tag{4.5}$$

where the superscripts t, j indicate a restriction to paths with exactly t highest points and j corners.

Set

$$\mathcal{P}_{m,n}^{t,j}(\mathbf{x},\mathbf{y}) := \sum_{\alpha\in\mathcal{D}_{m,n}^{j,j}} h_{\rho(\alpha)}(\mathbf{x})\, h_{\rho(\alpha')}(\mathbf{y}).$$

Bijection (4.5) implies

$$\frac{1}{t}\mathcal{P}_{m,n}^{t,j}(\mathbf{x},\mathbf{y}) = \frac{1}{j} \sum_{\rho\models_j n,\ \sigma\models_j m} h_\rho(\mathbf{x})\, h_\sigma(\mathbf{y}).$$

Summing both sides over j (writing (m,n) as (ad,bd)), we obtain

$$\sum_{t=1}^{d} \frac{1}{t}\, \mathcal{P}_{ad,bd}^{t}(\mathbf{x},\mathbf{y}) = \sum_{j=1}^{d} \frac{1}{j} \sum_{\rho\models_j bd,\ \sigma\models_j ad} h_\rho(\mathbf{x})\, h_\sigma(\mathbf{y}) = \frac{1}{d}\, p_d^{a/b}(\mathbf{x},\mathbf{y}). \tag{4.6}$$

Then (4.3) is deduced from (4.6) just as (3.15) was deduced from (3.13). □

An alternate formula for $p_k^{a/b}(\mathbf{x},\mathbf{y})$, which involves much less terms, is easily seen to be

$$p_k^{a/b}(\mathbf{x},\mathbf{y}) := k\sum_{j=1}^{k}\frac{1}{j}\sum_{\mu\vdash_j bk,\ \nu\vdash_j ak}\binom{j}{\lambda(\mu)}\binom{j}{\lambda(\nu)}h_\mu(\mathbf{x})\,h_\nu(\mathbf{y}). \tag{4.7}$$

The second sum is now over j-part partitions, with $\lambda(\mu)$ denoting the partition of j that indicates the multiplicities of the parts of μ (likewise for ν), and $\binom{j}{\lambda(\,.\,)}$ stands for the corresponding multinomial coefficient. Hence, the above formula is simply obtained by collecting the compositions that have the same parts, up to re-ordering, in (4.4).

Observe that we get back Proposition 3 from Theorem 4 if we take the usual symmetric function scalar product with $h_n(\mathbf{y})$ on each side of (4.6). Indeed, we first recall that this scalar product is such that

$$\langle h_\mu(\mathbf{y}), h_n(\mathbf{y})\rangle = 1,$$

for any partition (or composition) of n. This implies that $\langle -, h_n(\mathbf{y})\rangle$ is a ring homomorphism. Hence, we need only prove that

$$\langle p_k^{a/b}(\mathbf{x},\mathbf{y}), h_n(\mathbf{y})\rangle = k\sum_{j=1}^{k}\frac{1}{j}\sum_{\mu\vdash_j bk,\ \nu\vdash_j ak}\binom{j}{\lambda(\mu)}\binom{j}{\lambda(\nu)}h_\rho(\mathbf{x}) \tag{4.8}$$

reduces to

$$p_k^{a/b}(\mathbf{x}) = \frac{1}{a}\sum_{\lambda\vdash bk}(ak)^{\ell(\lambda)}\,\frac{p_\lambda(\mathbf{x})}{z_\lambda}. \tag{4.9}$$

This is obtained as follows. The right-hand side of (4.8) is equal to

$$k\sum_{t,j}\frac{1}{j}\sum_{\gamma\in\mathcal{C}_{m,n}^{t,j}} h_{\rho(\gamma)}(\mathbf{x}),$$

whereas the right-hand side of (4.9) is equal to

$$\frac{k}{m}\sum_{\gamma\in\mathcal{B}_{m,n}} h_{\rho(\gamma)}(\mathbf{x}).$$

Because of bijections (3.12) and (4.5), these two expressions are equal (and equal to $k\sum_t \frac{1}{t}\sum_{\alpha\in\mathcal{D}_{m,n}^t} h_{\rho(\gamma)}(\mathbf{x})$).

5. Further considerations

Extensions of these considerations, linked to several interesting questions (see [3, 4, 7, 16, 17]), take into account parameters on parking functions such as "area" and "dinv". To formulate the analogous results, one needs to work with an algebra of operators on symmetric functions isomorphic to the elliptic Hall algebra studied in [10, 13, 23]. In this framework, the homomorphism $\Theta_{a,b}$ sends a symmetric function to an operator on symmetric functions. In turn, formulas are obtained by applying the resulting operator to the symmetric function 1.

In this light, it is worth observing that the image under $\Theta_{a,b}$ of other symmetric functions gives rise to significant formulas. The interesting feature of these formulas is that their Schur function expansion has positive integer coefficients. It is usual to say that they are "Schur-positive". This is the case for hook Schur functions[8] $s_{(k|j)}(\mathbf{x})$, for which we can easily show h-positivity of $\Theta_{a,b}((-1)^j s_{(k|j)}(\mathbf{x}))$, which implies Schur-positivity. Indeed, one easily verifies that the symmetric function $(-1)^j s_{(k|j)}(\mathbf{x})$ expands with positive integer coefficients in the basis $(-1)^{|\mu|-\ell(\mu)}\, e_\mu(\mathbf{x})$; and we have seen that $\Theta_{a,b}((-1)^{j-1} e_j(\mathbf{x}))$ expands with positive integer coefficients in the basis $h_\nu(\mathbf{x})$. Hence, application of the homomorphism $\Theta_{a,b}$ to $(-1)^j s_{(k|j)}(\mathbf{x})$ gives rise to an h-positive expression. This expression is also Schur-positive, since any $h_\nu(\mathbf{x})$ is.

Extensive experiments suggest that, for all μ, $\Theta_{a,b}((-1)^{\iota(\mu)} s_\mu(\mathbf{x}))$ is Schur-positive, where $\iota(\mu)$ is the number of cells (i,j) of the diagram of μ, such that $j > i$. Moreover, all of this seems to carry over to the study of the bi-Frobenius characteristic. An intriguing question is to expand the elliptic Hall algebra techniques to cover these bi-Frobenius characteristic. The hope is that this would lead to more explicit formulas for three parameter expressions such as

$$\sum_{\alpha\in\mathcal{D}_{m,n}} q^{\mathrm{area}(\alpha)} \alpha(\mathbf{x};t)\alpha'(\mathbf{y};r), \tag{5.1}$$

were $\alpha(\mathbf{x};t)$ is an LLT-polynomial calculated using the dinv-statistic on (m,n)-parking functions (see [7] for more details on all this):

$$\alpha(\mathbf{x};t) := \sum_{\pi\mathcal{P}_\alpha} t^{\mathrm{dinv}(\pi)}\, s_{\mathbf{co}(\pi)}(\mathbf{x}),$$

where $\mathbf{co}(\pi)$ is a composition that encodes "descents" of the parking function π. Recall that composition-indexed Schur functions may be defined by a suitable adaptation of the Jacobi–Trudi identity. Up to a sign-twist, Expression (5.1) specializes to the right-hand side of (4.1). It is known that the LLT-polynomial $\alpha(\mathbf{x};t)$ is Schur-positive.

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