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A GENERALIZATION OF SCHUR'S P- AND Q-FUNCTIONS

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Dedicated to Christian Krattenthaler on the occasion of his 60th birthday

Abstract. We introduce and study a generalization of Schur's P-/Q-functions associated with a polynomial sequence, which can be viewed as "Macdonald's ninth variation" for P-/Q-functions. This variation includes as special cases Schur's P-/Q-functions, Ivanov's factorial P-/Q-functions and the $t = -1$ specialization of Hall–Littlewood functions associated with the classical root systems. We establish several identities and properties such as generalizations of Schur's original definition of Schur's Q-functions, a Cauchy-type identity, a generalization of the Józefiak–Pragacz–Nimmo formula for skew Q-functions, and a Pieri-type rule for multiplication.

Contents

1. Introduction

Schur (S-)functions and Schur P-/Q-functions are two important families of symmetric functions, and they appear in several parallel situations. For example, in the representation theory of the symmetric groups, Schur functions describe the characters of irreducible linear representations, while Schur Q-functions describe the characters of irreducible projective representations (see [22]). In cohomology theory, Schur functions represent the Schubert classes of Grassmannians, while Schur Q-functions represent the

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Schubert classes of Lagrangian Grassmannians (see [20]). Moreover, some identities for Schur functions have their counterparts for Schur P-/Q-functions.

There are several generalizations, variations or deformations of Schur functions, such as Hall–Littlewood functions, Macdonald functions and factorial Schur functions. The generalization relevant to this paper is Macdonald's ninth variation ([11], see also [16]) associated with a polynomial sequence, which is defined as follows.

Let $\mathcal{F} = \{f_d\}_{d=0}^{\infty}$ be a sequence of polynomials $f_d(u) \in K[u]$, where K is a ground field of characteristic 0, such that $\deg f_d = d$ for $d \geq 0$. Given a partition λ of length $l \leq n$, we define the *generalized Schur function* $s^{\mathcal{F}}_{\lambda}(x_1, \dots, x_n)$ as the ratio of two alternants,

$$s^{\mathcal{F}}_{\lambda}(x_1, \dots, x_n) = \frac{\det \left(f_{\lambda_j+n-j}(x_i) \right)_{1 \leq i,j \leq n}}{\det \left(f_{n-j}(x_i) \right)_{1 \leq i,j \leq n}}, \tag{1.1}$$

where $\lambda_{l+1} = \cdots = \lambda_n = 0$. The original Schur functions $s_{\lambda}(\boldsymbol{x})$ are recovered by setting $f_d(u) = u^d$ for $d \geq 0$. The factorial Schur functions $s_{\lambda}(\boldsymbol{x}|\boldsymbol{a})$ with factorial parameters $\boldsymbol{a} = (a_0, a_1, \dots)$ are obtained by taking $f_d(u) = (u|\boldsymbol{a})^d = \prod_{i=0}^{d-1}(u - a_i)$. Moreover, classical group characters are special cases of generalized Schur functions. For example, if the polynomial sequence $\mathcal{F} = \{f_d\}_{d=0}^{\infty}$ is defined by

$$f_d(x + x^{-1}) = \frac{x^{d+1} - x^{-d-1}}{x - x^{-1}} \quad (d \geq 0),$$

then it is not difficult to see that the generalized Schur function $s^{\mathcal{F}}_{\lambda}(x_1 + x_1^{-1}, \dots, x_n + x_n^{-1})$ equals the irreducible character of the symplectic group $\mathbf{Sp}_{2n}(\mathbb{C})$ with highest weight λ.

Generalized Schur functions share many of the same properties as the original Schur functions. For example, they satisfy the modified Jacobi–Trudi identity and the Giambelli identity,

$$\begin{aligned} s^{\mathcal{F}}_{\lambda}(x_1, \dots, x_n) &= \det \left(s^{\mathcal{F}}_{(\lambda_i - i + j)}(x_j, \dots, x_n) \right)_{1 \leq i,j \leq l} \\ &= \det \left(s^{\mathcal{F}}_{(\alpha_i | \beta_j)}(x_1, \dots, x_n) \right)_{1 \leq i,j \leq r}, \end{aligned}$$

where λ is a partition of length $l \leq n$ and $\lambda = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r)$ in Frobenius notation.

The aim of this paper is to introduce and study the "ninth variation" of Schur P-/Q-functions, which we call generalized P-functions associated with polynomial sequences. We define generalized P-functions in terms of Nimmo-type formula and derive Pfaffian identities and basic properties by following a linear algebraic approach similar to [19].

We use the following terminology on polynomial sequences.

Definition 1.1. Let $\mathcal{F} = \{f_d\}_{d=0}^{\infty}$ be a sequence of polynomials $f_d(u) \in K[u]$. We say that $\mathcal{F}$ is *admissible* if it satisfies the conditions

$$f_0(u) = 1, \quad \deg f_d = d \quad (d \geq 1). \tag{1.2}$$

An admissible sequence $\mathcal{F}$ is called *constant-term free* if $f_d(0) = 0$ for any $d \geq 1$.

In this article, a *partition of length l* is a weakly decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_l)$ of *positive* integers. We write $l = l(\lambda)$ and $|\lambda| = \sum_{i=1}^{l} \lambda_i$. A partition λ of length l is called *strict* if $\lambda_1 > \cdots > \lambda_l$. The empty sequence $\emptyset$ is the unique strict partition of length 0.

For a sequence $\boldsymbol{x} = (x_1, \dots, x_n)$ of n indeterminates, we put

$$A(\boldsymbol{x}) = \left(\frac{x_j - x_i}{x_j + x_i} \right)_{1 \le i,j \le n}, \quad \Delta(\boldsymbol{x}) = \prod_{1 \le i < j \le n} \frac{x_j - x_i}{x_j + x_i}. \tag{1.3}$$

Now we give a definition of generalized P-functions associated with polynomial sequences in terms of a Nimmo-type formula (see [18, (A13)]).

Definition 1.2. Let n be a positive integer and $\boldsymbol{x} = (x_1, \dots, x_n)$ a sequence of n indeterminates. For an admissible sequence $\mathcal{F} = \{f_d\}_{d=0}^{\infty}$ of polynomials and a sequence $\alpha = (\alpha_1, \dots, \alpha_r)$ of nonnegative integers, let $V_\alpha^{\mathcal{F}}(\boldsymbol{x})$ be the $n \times r$ matrix given by

$$V_\alpha^{\mathcal{F}}(\boldsymbol{x}) = \left(f_{\alpha_j}(x_i) \right)_{1 \le i \le n, 1 \le j \le r}.$$

Given a strict partition λ of length l, we define the corresponding *generalized P-function* $P_\lambda^{\mathcal{F}}(\boldsymbol{x})$ associated with $\mathcal{F}$ by putting

$$P_\lambda^{\mathcal{F}}(\boldsymbol{x}) = \begin{cases} \dfrac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & V_\lambda^{\mathcal{F}}(\boldsymbol{x}) \\ -{}^t V_\lambda^{\mathcal{F}}(\boldsymbol{x}) & O \end{pmatrix}, & \text{if } n + l \text{ is even,} \\ \dfrac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & V_{\lambda^0}^{\mathcal{F}}(\boldsymbol{x}) \\ -{}^t V_{\lambda^0}^{\mathcal{F}}(\boldsymbol{x}) & O \end{pmatrix}, & \text{if } n + l \text{ is odd,} \end{cases} \tag{1.4}$$

where $\lambda^0 = (\lambda_1, \dots, \lambda_l, 0)$. We simply write $V_\alpha(\boldsymbol{x})$ and $P_\lambda(\boldsymbol{x})$ for $V_\alpha^{\mathcal{F}}(\boldsymbol{x})$ and $P_\lambda^{\mathcal{F}}(\boldsymbol{x})$ if there is no confusion, e.g., in the proofs.

Note that (see Proposition A.1)

$$\Delta(\boldsymbol{x}) = \begin{cases} \operatorname{Pf} A(\boldsymbol{x}), & \text{if } n \text{ is even,} \\ \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & \mathbf{1} \\ -{}^t\mathbf{1} & 0 \end{pmatrix}, & \text{if } n \text{ is odd,} \end{cases}$$

where $\mathbf{1}$ is the all-one column vector of appropriate size. Hence our definition (1.4) can be regarded as a counterpart of the definition (1.1) of generalized Schur functions.

Example 1.3.
(1) It follows from Nimmo's formula [18, (A13)] that we recover the original Schur P-function $P_\lambda(\boldsymbol{x})$ and Schur Q-function $Q_\lambda(\boldsymbol{x})$ by setting $f_d(u) = u^d$ and $f_d(u) = 2u^d$, respectively.
(2) It follows from the Nimmo-type formula [7, Theorem 3.2] that Ivanov's factorial P-function $P_\lambda(\boldsymbol{x}|\boldsymbol{a})$ and Q-function $Q_\lambda(\boldsymbol{x}|\boldsymbol{a})$ are obtained by taking $f_d(u) = (u|\boldsymbol{a})^d$ and $f_d(u) = 2(u|\boldsymbol{a})^d$, respectively.
(3) As we will see in Section 7, our generalized P-functions include the $t = -1$ specializations of Hall–Littlewood functions associated with the root systems of type B, C and D.

Ikeda and Naruse [4], and Nakagawa and Naruse [17] introduced other generalizations of factorial P- and Q-functions from the viewpoint of Schubert calculus. In a very recent paper [2], Foley and King give a combinatorial generalization of Schur Q-functions in terms of shifted tableaux and prove several Pfaffian formulas.

The organization and main results of this paper are as follows. In Section 2, we relate our definition of generalized P-functions (Definition 1.2) with generalizations of two other definitions of Schur P-/Q-functions. Namely, we prove that $P^{\mathcal{F}}_{\lambda}(\boldsymbol{x})$ is also obtained by setting $t = -1$ in the generalized Hall–Littlewood function associated with a polynomial sequence (see Theorem 2.3), and that $P^{\mathcal{F}}_{\lambda}(\boldsymbol{x})$ is expressed as the Pfaffian of the skew-symmetric matrix with entries $P^{\mathcal{F}}_{(\lambda_i,\lambda_j)}(\boldsymbol{x})$ (see Theorem 2.6). In Section 3, we introduce the notion of generalized dual P-functions $\widehat{P}^{\mathcal{F}}_{\lambda}(\boldsymbol{x})$ and prove a corresponding Cauchy-type identity. In Section 4, we define generalized skew P-functions $P^{\mathcal{F}}_{\lambda/\mu,p}(\boldsymbol{x})$ in terms of a Józefiak–Pragacz–Nimmo-type Pfaffian and prove that $P^{\mathcal{F}}_{\lambda/\mu,p}(\boldsymbol{x})$ appears as the coefficient of $P^{\mathcal{F}}_{\mu}(\boldsymbol{y})$ in the expansion of $P^{\mathcal{F}}_{\lambda}(\boldsymbol{x},\boldsymbol{y})$ (see Theorem 4.2). In Section 5, we consider the modified Pieri coefficients in the expansion of the product $P^{\mathcal{F}}_{\mu}(\boldsymbol{x}) \cdot Q_{(r)}(\boldsymbol{x})$ and obtain a determinant formula for the generating function of modified Pieri coefficients (see Theorem 5.3). Section 6 focuses on Ivanov's factorial P-/Q-functions. We derive a determinant formula for the factorial skew P-function in one variable (see Theorem 6.5), and an explicit product formula for the generating function of modified Pieri coefficients (see Theorem 6.6). In Section 7, we show that the Hall–Littlewood functions at $t = -1$ associated with the classical root systems can be written as generalized P-functions associated with certain polynomial sequences (see Theorem 7.2). Appendix A collects some Schur-type Pfaffian evaluations and useful formulas.

2. Several expressions for generalized P-functions

In this section, we give several expressions for generalized P-functions associated with an admissible polynomial sequence, and we study their basic properties.

2.1. Hall–Littlewood-type expression.

In this subsection, we prove that our generalized P-functions are obtained as the $t = -1$ specialization of Hall–Littlewood-type functions.

We begin with the following proposition.

Proposition 2.1. *Let $\mathcal{F}$ be an admissible sequence of polynomials and $\boldsymbol{x} = (x_1, \dots, x_n)$. Then we have:*

(1) *For the empty partition $\emptyset$, we have $P^{\mathcal{F}}_{\emptyset}(\boldsymbol{x}) = 1$.*
(2) *If λ is a strict partition of length $> n$, then we have $P^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) = 0$.*

Proof. By using the definition (1.4), we can derive (1) from the Pfaffian evaluations (A.3) and (A.4), and (2) from Proposition A.5. □

We define a generalization of Hall–Littlewood polynomials associated with an admissible polynomial sequence.

Definition 2.2. Let n be a positive integer and $\boldsymbol{x} = (x_1, \dots, x_n)$. Given a partition λ of length $l \leq n$, we regard λ as a sequence $(\lambda_1, \dots, \lambda_l, 0, \dots, 0)$ of length n, and define a polynomial $v^{(n)}_{\lambda}(t)$ by putting

$$v^{(n)}_{\lambda}(t) = \prod_{k \geq 0} [m_k]_t!,$$

where $m_k = \{i : 1 \leq i \leq n,\ \lambda_i = k\}$ and $[m]_t! = \prod_{j=1}^{m}(1-t^j)/(1-t)$. For an admissible polynomial sequence $\mathcal{F}$ and a partition λ of length $\leq n$, we define the *generalized Hall–Littlewood function* $\mathbb{P}_\lambda^{\mathcal{F}}(\boldsymbol{x};t)$ corresponding to λ by putting

$$\mathbb{P}_\lambda^{\mathcal{F}}(\boldsymbol{x};t) = \frac{1}{v_\lambda^{(n)}(t)} \sum_{w \in S_n} w\left(\prod_{i=1}^{n} f_{\lambda_i}(x_i) \prod_{1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j}\right), \tag{2.1}$$

where S_n is the symmetric group acting on $K(t)[x_1, \ldots, x_n]$ by permuting variables. We write $\mathbb{P}_\lambda(\boldsymbol{x};t)$ for $\mathbb{P}_\lambda^{\mathcal{F}}(\boldsymbol{x};t)$ when there is no confusion.

Setting $f_d(u) = u^d$ for $d \geq 0$, we recover the original Hall–Littlewood polynomials. The following is the main theorem of this subsection.

Theorem 2.3. *For an admissible sequence $\mathcal{F}$ and a strict partition λ of length $l \leq n$, we have*

$$P_\lambda^{\mathcal{F}}(\boldsymbol{x}) = \mathbb{P}_\lambda^{\mathcal{F}}(\boldsymbol{x};-1). \tag{2.2}$$

Note that Equation (2.2) with $f_d(u) = u^d$ is the definition of Schur P-function adopted in [12, III.8]. For the sake of completeness and later use, we give a proof of this theorem, which follows the argument in [18, Appendix]. As a first step, we show the following lemma.

Lemma 2.4. *For a strict partition λ of length $l \leq n$, we have*

$$\mathbb{P}_\lambda^{\mathcal{F}}(\boldsymbol{x};-1) = \sum_{u \in S_n/S_{n-l}} u\left(\prod_{i=1}^{l} f_{\lambda_i}(x_i) \prod_{\substack{1 \leq i < j \leq n \\ i \leq l}} \frac{x_i + x_j}{x_i - x_j}\right) \tag{2.3}$$

$$= \frac{1}{(n-l)!} \sum_{u \in S_n} u\left(\prod_{i=1}^{l} f_{\lambda_i}(x_i) \prod_{\substack{1 \leq i < j \leq n \\ i \leq l}} \frac{x_i + x_j}{x_i - x_j}\right), \tag{2.4}$$

where S_{n-l} is the symmetric group on the last $n-l$ variables $x_{l+1}, \ldots, x_n$.

Proof. Since $f_0(u) = 1$ and the product $\prod_{1 \leq i < j \leq n, i \leq l}(x_i - tx_j)/(x_i - x_j)$ is invariant under S_{n-l}, we have

$$\mathbb{P}_\lambda(\boldsymbol{x};t)$$
$$= \frac{1}{v_\lambda^{(n)}(t)} \sum_{w' \in S_n/S_{n-l}} w'\left(\prod_{i=1}^{l} f_{\lambda_i}(x_i) \prod_{\substack{1 \leq i < j \leq n \\ i \leq l}} \frac{x_i - tx_j}{x_i - x_j} \sum_{w'' \in S_{n-l}} w''\left(\prod_{l+1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j}\right)\right).$$

By using (see [10, Theorem 2.8])

$$\sum_{w'' \in S_{n-l}} w''\left(\prod_{l+1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j}\right) = [n-l]_t! = v_\lambda^{(n)}(t),$$

we have

$$\mathbb{P}_\lambda(\boldsymbol{x};t) = \sum_{w' \in S_n/S_{n-l}} w' \left(\prod_{i=1}^{l} f_{\lambda_i}(x_i) \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{x_i - t x_j}{x_i - x_j} \right).$$

By specializing $t = -1$, we obtain (2.3), from which (2.4) follows. □

Proof of Theorem 2.3. Since $\Delta(\boldsymbol{x})$ is alternating in $x_1, \dots, x_n$, it follows from (2.3) that

$$\mathbb{P}_\lambda(\boldsymbol{x};-1) = \frac{(-1)^{\binom{n}{2}+\binom{n-l}{2}}}{\Delta(\boldsymbol{x})} \sum_{v \in S_n/S_{n-l}} \operatorname{sgn}(v) v \left(\prod_{i=1}^{l} f_{\lambda_i}(x_i) \prod_{l+1 \le i < j \le n} \frac{x_j - x_i}{x_j + x_i} \right).$$

Since $\prod_{l+1 \le i < j \le n} (x_j - x_i)/(x_j + x_i)$ is invariant under the symmetric group S_l acting on the first l variables $x_1, \dots, x_l$, we have

$$\begin{aligned}
&\mathbb{P}_\lambda(\boldsymbol{x};-1) \\
&= \frac{(-1)^{\binom{n}{2}+\binom{n-l}{2}}}{\Delta(\boldsymbol{x})} \sum_{v' \in S_n/(S_l \times S_{n-l})} \operatorname{sgn}(v') v' \left(\sum_{v'' \in S_l} \operatorname{sgn}(v'') v'' \left(\prod_{i=1}^{l} f_{\lambda_i}(x_i) \right) \prod_{l+1 \le i < j \le n} \frac{x_j - x_i}{x_j + x_i} \right) \\
&= \frac{(-1)^{\binom{n}{2}+\binom{n-l}{2}}}{\Delta(\boldsymbol{x})} \sum_{v' \in S_n/(S_l \times S_{n-l})} \operatorname{sgn}(v') v' \left(\det \left(f_{\lambda_j}(x_i) \right)_{1 \le i,j \le l} \prod_{l+1 \le i < j \le n} \frac{x_j - x_i}{x_j + x_i} \right).
\end{aligned}$$

We take $R = \{ u \in S_n : u(1) < \cdots < u(l), u(l+1) < \cdots < u(n) \}$ as a complete set of coset representatives of $S_n/(S_l \times S_{n-l})$. We note that the correspondence $u \mapsto \{u(l+1), \dots, u(n)\}$ gives a bijection between the coset representatives R and the set $\binom{[n]}{n-l}$ of all $(n-l)$-element subsets of $[n] = \{1, \dots, n\}$.

First we consider the case where $n-l$ is even. In this case, by using Schur's Pfaffian evaluation (A.3), we have

$$\mathbb{P}_\lambda(\boldsymbol{x};-1) = \frac{(-1)^{\binom{n}{2}+\binom{n-l}{2}}}{\Delta(\boldsymbol{x})} \sum_{u \in S_n/S_l \times S_{n-l}} \operatorname{sgn}(u) u \Big(\det V_\lambda(\boldsymbol{x}_{[l]}) \operatorname{Pf} A(\boldsymbol{x}_{[n] \setminus [l]}) \Big),$$

where $[l] = \{1, \dots, l\}$, $[n] \setminus [l] = \{l+1, \dots, n\}$ and $\boldsymbol{x}_J = (x_{j_1}, \dots, x_{j_m})$ for $J = \{j_1, \dots, j_m\}$ with $j_1 < \cdots < j_m$. On the other hand, by applying Proposition A.5 (a Pfaffian version of the Laplace expansion) to the matrices $Z = A(\boldsymbol{x})$ and $W = V_\lambda(\boldsymbol{x})$, we obtain

$$\operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & V_\lambda(\boldsymbol{x}) \\ -{}^t V_\lambda(\boldsymbol{x}) & O \end{pmatrix} = \sum_{I \in \binom{[n]}{n-l}} (-1)^{\Sigma(I) - \binom{n}{2}} \operatorname{Pf} A(\boldsymbol{x}_I) \det V_\lambda(\boldsymbol{x}_{[n] \setminus I}),$$

where $\Sigma(I) = \sum_{i \in I} i$. Since $n-l$ is even, we can see that, if $u \in R$ corresponds to $I \in \binom{[n]}{n-l}$, then the inversion number of u is given by

$$\operatorname{inv}(u) = \binom{n+1}{2} - \binom{l+1}{2} - \Sigma(I) \equiv \binom{n}{2} - \binom{l}{2} + \Sigma(I) \mod 2.$$

Hence we have

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & V_\lambda(\boldsymbol{x}) \\ -{}^t V_\lambda(\boldsymbol{x}) & O \end{pmatrix} = \sum_{u \in S_n/(S_l \times S_{n-l})} (-1)^{\binom{l}{2}} \operatorname{sgn}(u) u \Big(\det V_\lambda(\boldsymbol{x}_{[l]}) \operatorname{Pf} A(\boldsymbol{x}_{[n]\setminus[l]}) \Big),$$

and

$$\mathbb{P}_\lambda(\boldsymbol{x}; -1) = \frac{(-1)^{\binom{n}{2}+\binom{n-l}{2}+\binom{l}{2}}}{\Delta(\boldsymbol{x})} \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & V_\lambda(\boldsymbol{x}) \\ -{}^t V_\lambda(\boldsymbol{x}) & O \end{pmatrix}.$$

Now we can use the relation $\binom{n}{2} - \binom{l}{2} - \binom{n-l}{2} = l(n-l) \equiv 0 \bmod 2$ to complete the proof of (2.2) in the case where $n-l$ is even.

Next we consider the case where $n-l$ is odd. In this case, by using (A.4), we see that

$$\mathbb{P}_\lambda(\boldsymbol{x}; -1) = \frac{(-1)^{\binom{n}{2}+\binom{n-l}{2}}}{\Delta(\boldsymbol{x})} \sum_{u \in S_n/(S_l \times S_{n-l})} \operatorname{sgn}(u) u \left(\det V_\lambda(\boldsymbol{x}_{[l]}) \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}_{[n]\setminus[l]}) & \mathbf{1} \\ -{}^t\mathbf{1} & 0 \end{pmatrix} \right),$$

where $\mathbf{1}$ is the all-one column vector. On the other hand, by applying Proposition A.5 to the matrices

$$Z = \begin{pmatrix} A(\boldsymbol{x}) & \mathbf{1} \\ -{}^t\mathbf{1} & 0 \end{pmatrix}, \quad W = \begin{pmatrix} V_\lambda(\boldsymbol{x}) \\ O \end{pmatrix},$$

we see that

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & \mathbf{1} & V_\lambda(\boldsymbol{x}) \\ -{}^t\mathbf{1} & 0 & O \\ -{}^t V_\lambda(\boldsymbol{x}) & O & O \end{pmatrix} = \sum_I (-1)^{\Sigma(I) - \binom{n+1}{2}} \operatorname{Pf} Z(I) \det W([n+1] \setminus I; [l]),$$

where I runs over all $(n+1-l)$-element subsets of $[n+1]$. If $n+1 \notin I$, then we have

$$\det W([n+1] \setminus I; [l]) = \det \begin{pmatrix} V_\lambda(\boldsymbol{x}_{[n]\setminus I}) \\ O \end{pmatrix} = 0.$$

Hence we have

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & \mathbf{1} & V_\lambda(\boldsymbol{x}) \\ -{}^t\mathbf{1} & 0 & O \\ -{}^t V_\lambda(\boldsymbol{x}) & O & O \end{pmatrix} = \sum_{I \in \binom{[n]}{n-l}} (-1)^{\Sigma(I \cup \{n+1\}) - \binom{n+1}{2}} \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}_I) & \mathbf{1} \\ -{}^t\mathbf{1} & 0 \end{pmatrix} \det V_\lambda(\boldsymbol{x}_{[n]\setminus I}).$$

Since $n-l$ is odd, we see that, if $u \in R$ corresponds to $I \in \binom{[n]}{n-l}$, then we have

$$\operatorname{inv}(u) = \binom{n+1}{2} - \binom{l+1}{2} - \Sigma(I) \equiv \binom{n+1}{2} - \binom{l}{2} - \Sigma(I \cup \{n+1\}) \bmod 2.$$

Moreover, by permuting rows/columns we have

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & \mathbf{1} & V_\lambda(\boldsymbol{x}) \\ -{}^t\mathbf{1} & 0 & O \\ -{}^t V_\lambda(\boldsymbol{x}) & O & O \end{pmatrix} = (-1)^l \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & V_{\lambda^0}(\boldsymbol{x}) \\ -{}^t V_{\lambda^0}(\boldsymbol{x}) & O \end{pmatrix}.$$

Hence we have

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & V_{\lambda^0}(\boldsymbol{x}) \\ -{}^t V_{\lambda^0}(\boldsymbol{x}) & O \end{pmatrix}$$

$$= \sum_{u \in S_n/(S_l \times S_{n-l})} (-1)^{\binom{l}{2}+l} \operatorname{sgn}(u) u \Bigg(\operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}_{[n]\setminus[l]}) & \mathbf{1} \\ -{}^t\mathbf{1} & 0 \end{pmatrix} \det V_\lambda(\boldsymbol{x}_{[l]}) \Bigg),$$

and

$$\mathbb{P}_\lambda(\boldsymbol{x}; -1) = \frac{(-1)^{\binom{n}{2}+\binom{n-l}{2}+\binom{l}{2}+l}}{\Delta(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & V_{\lambda^0}(\boldsymbol{x}) \\ -{}^tV_{\lambda^0}(\boldsymbol{x}) & O \end{pmatrix}.$$

Now we can complete the proof in the case where $n - l$ is odd by using the congruence relation $\binom{n}{2} - \binom{l}{2} - \binom{n-l}{2} = l(n-l) \equiv l \bmod 2$. □

By combining Theorem 2.3 and Lemma 2.4, we obtain the following corollary.

Corollary 2.5. *For a strict partition λ of length $l \le n$, we have*

$$P^{\mathcal{F}}_\lambda(\boldsymbol{x}) = \sum_{u \in S_n/S_{n-l}} u \left(\prod_{i=1}^{l} f_{\lambda_i}(x_i) \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{x_i + x_j}{x_i - x_j} \right) \tag{2.5}$$

$$= \frac{1}{(n-l)!} \sum_{u \in S_n} u \left(\prod_{i=1}^{l} f_{\lambda_i}(x_i) \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{x_i + x_j}{x_i - x_j} \right). \tag{2.6}$$

2.2. **Schur-type Pfaffian formula.** In this subsection, we use the definition (1.4) and a Pfaffian version of the Sylvester formula (Proposition A.4) to derive a Schur-type Pfaffian formula for $P^{\mathcal{F}}_\lambda(\boldsymbol{x})$, which generalizes (a part of) Schur's original definition of Schur Q-functions [22, §35] and a similar formula for factorial Q-functions [7, Theorem 9.1]. We use the following conventions:

$$P^{\mathcal{F}}_{(0)}(\boldsymbol{x}) = 1, \tag{2.7}$$

$$P^{\mathcal{F}}_{(s,r)}(\boldsymbol{x}) = -P^{\mathcal{F}}_{(r,s)}(\boldsymbol{x}), \quad P^{\mathcal{F}}_{(r,0)}(\boldsymbol{x}) = -P^{\mathcal{F}}_{(0,r)}(\boldsymbol{x}) = P^{\mathcal{F}}_{(r)}(\boldsymbol{x}), \quad P^{\mathcal{F}}_{(0,0)}(\boldsymbol{x}) = 0, \tag{2.8}$$

where r and s are positive integers.

Theorem 2.6. *Let $\mathcal{F}$ be an admissible sequence. For a sequence $\alpha = (\alpha_1, \dots, \alpha_r)$ of nonnegative integers, let $S^{\mathcal{F}}_\alpha(\boldsymbol{x})$ be the $r \times r$ skew-symmetric matrix defined by*

$$S^{\mathcal{F}}_\alpha(\boldsymbol{x}) = \left(P^{\mathcal{F}}_{(\alpha_i, \alpha_j)}(\boldsymbol{x}) \right)_{1 \le i,j \le r}. \tag{2.9}$$

Then, for a strict partition λ of length l, we have

$$P^{\mathcal{F}}_\lambda(\boldsymbol{x}) = \begin{cases} \operatorname{Pf} S^{\mathcal{F}}_\lambda(\boldsymbol{x}), & \text{if } l \text{ is even}, \\ \operatorname{Pf} S^{\mathcal{F}}_{\lambda^0}(\boldsymbol{x}), & \text{if } l \text{ is odd}, \end{cases} \tag{2.10}$$

where $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\lambda^0 = (\lambda_1, \dots, \lambda_l, 0)$.

In order to prove this theorem, we can use the same argument as in [19, Theorem 4.1 (3) and Remark 4.3]. As we will see in Proposition 2.7, the generalized P-functions do not have the stability property, so we cannot reduce the proof to the case where n is even.

Proof. By applying Proposition A.4 to the matrix X given by

$$X = \begin{cases} \begin{pmatrix} A(\boldsymbol{x}) & V_\lambda(\boldsymbol{x}) \\ -{}^tV_\lambda(\boldsymbol{x}) & O \end{pmatrix}, & \text{if } n \text{ is even and } l \text{ is even}, \\ \begin{pmatrix} A(\boldsymbol{x}) & V_{\lambda^0}(\boldsymbol{x}) \\ -{}^tV_{\lambda^0}(\boldsymbol{x}) & O \end{pmatrix}, & \text{if } n \text{ is even and } l \text{ is odd}, \\ \begin{pmatrix} A(\boldsymbol{x}) & \mathbf{1} & V_\lambda(\boldsymbol{x}) \\ -{}^t\mathbf{1} & 0 & O \\ -{}^tV_\lambda(\boldsymbol{x}) & O & O \end{pmatrix}, & \text{if } n \text{ is odd and } l \text{ is even}, \\ \begin{pmatrix} A(\boldsymbol{x}) & \mathbf{1} & V_\lambda(\boldsymbol{x}) & 0 \\ -{}^t\mathbf{1} & 0 & O & -1 \\ -{}^tV_\lambda(\boldsymbol{x}) & O & O & O \\ 0 & 1 & O & 0 \end{pmatrix}, & \text{if } n \text{ is odd and } l \text{ is odd}. \end{cases}$$

If n is even, then we have

$$\frac{\operatorname{Pf} X}{\operatorname{Pf} X([n])} = P_\lambda(\boldsymbol{x}), \quad \frac{\operatorname{Pf} X([n] \cup \{n+i, n+j\})}{\operatorname{Pf} X([n])} = P_{(\lambda_i,\lambda_j)}(\boldsymbol{x}).$$

If n is odd and l is even, then, by permuting rows/columns, we see that

$$\frac{\operatorname{Pf} X}{\operatorname{Pf} X([n+1])} = P_\lambda(\boldsymbol{x}), \quad \frac{\operatorname{Pf} X([n+1] \cup \{n+1+i, n+1+j\})}{\operatorname{Pf} X([n+1])} = P_{(\lambda_i,\lambda_j)}(\boldsymbol{x}).$$

If n is odd and l is odd, then by expanding the Pfaffians along the last row/column, we have

$$\frac{\operatorname{Pf} X}{\operatorname{Pf} X([n+1])} = P_\lambda(\boldsymbol{x}), \quad \frac{\operatorname{Pf} X([n+1] \cup \{n+1+i, n+1+(l+1)\})}{\operatorname{Pf} X([n+1])} = P_{(\lambda_i)}(\boldsymbol{x}),$$

and by permuting rows/columns we see that

$$\frac{\operatorname{Pf} X([n+1] \cup \{n+1+i, n+1+j\})}{\operatorname{Pf} X([n+1])} = P_{(\lambda_i,\lambda_j)}(\boldsymbol{x}).$$

Now Theorem 2.6 follows immediately from Proposition A.4. □

2.3. **Stability.** The Schur P-functions have the stability property (see [12, III, (2.5)])

$$P_\lambda(x_1, \dots, x_n, 0) = P_\lambda(x_1, \dots, x_n).$$

Our generalizations $P^{\mathcal{F}}_\lambda(\boldsymbol{x})$ do not have the stability property in general. For example we can show that

$$P^{\mathcal{F}}_{(r)}(x_1, \dots, x_n, 0) = P^{\mathcal{F}}_{(r)}(x_1, \dots, x_n) + (-1)^n f_r(0)$$

for $r \geq 1$. The following "mod 2 stability property" was given by [3, Proposition 8.1] for factorial P-functions.

Proposition 2.7. *Let $\mathcal{F}$ be an admissible sequence, and λ a strict partition.*

(1) *In general, we have*

$$P^{\mathcal{F}}_\lambda(x_1, \dots, x_n, 0, 0) = P^{\mathcal{F}}_\lambda(x_1, \dots, x_n).$$

(2) *If $\mathcal{F}$ is constant-term free, then we have*

$$P_\lambda^{\mathcal{F}}(x_1, \dots, x_n, 0) = P_\lambda^{\mathcal{F}}(x_1, \dots, x_n).$$

Proof. (1) Let $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\widetilde{\boldsymbol{x}} = (x_1, \dots, x_n, x_{n+1})$. It follows from the definition (1.4) that

$$P_\lambda(x_1, \dots, x_n, x_{n+1}, 0) = \frac{1}{(-1)^{n+1}\Delta(\widetilde{\boldsymbol{x}})} \operatorname{Pf} \begin{pmatrix} A(\widetilde{\boldsymbol{x}}) & -\mathbf{1} & V_{\lambda^*}(\widetilde{\boldsymbol{x}}) \\ {}^t\mathbf{1} & 0 & V_{\lambda^*}(0) \\ -{}^tV_{\lambda^*}(\widetilde{\boldsymbol{x}}) & -{}^tV_{\lambda^*}(0) & O \end{pmatrix},$$

where $\lambda^* = \lambda$ or λ^0 according to whether $n+l$ is even or odd, and $V_\alpha(0)$ is the row vector $(f_{\alpha_1}(0), \dots, f_{\alpha_r}(0))$. Hence, if we put $x_{n+1} = 0$ in the above formula, we have

$$P_\lambda(x_1, \dots, x_n, 0, 0) = \frac{1}{(-1)^{n+1} \cdot (-1)^n \Delta(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & -\mathbf{1} & -\mathbf{1} & V_{\lambda^*}(\boldsymbol{x}) \\ {}^t\mathbf{1} & 0 & -1 & V_{\lambda^*}(0) \\ {}^t\mathbf{1} & 1 & 0 & V_{\lambda^*}(0) \\ -{}^tV_{\lambda^*}(\boldsymbol{x}) & -{}^tV_{\lambda^*}(0) & -{}^tV_{\lambda^*}(0) & O \end{pmatrix}.$$

By subtracting the $(n+1)$st row/column from the $(n+2)$nd row/column and then by expanding the resulting Pfaffian along the $(n+2)$nd row/column, we see that

$$P_\lambda(x_1, \dots, x_n, 0, 0) = \frac{1}{(-1)^{n+1} \cdot (-1)^n \Delta(\boldsymbol{x})} \cdot (-1) \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & V_{\lambda^*}(\boldsymbol{x}) \\ -{}^tV_{\lambda^*}(\boldsymbol{x}) & O \end{pmatrix} = P_\lambda(\boldsymbol{x}).$$

(2) By the definition (1.4), we have

$$P_\lambda(x_1, \dots, x_n, 0) = \frac{1}{(-1)^n \Delta(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & -\mathbf{1} & V_{\lambda^*}(\boldsymbol{x}) \\ {}^t\mathbf{1} & 0 & V_{\lambda^*}(0) \\ -{}^tV_{\lambda^*}(\boldsymbol{x}) & -{}^tV_{\lambda^*}(0) & O \end{pmatrix},$$

where $\lambda^* = \lambda^0$ or λ according to whether $n+l$ is even or odd. If $n+l$ is even, by adding the $(n+1)$st row/column to the last row/column and then by expanding the resulting Pfaffian along the last row/column, we see that $P_\lambda(\boldsymbol{x}, 0) = P_\lambda(\boldsymbol{x})$. If $n+l$ is odd, then by permuting rows/columns, we obtain $P_\lambda(\boldsymbol{x}, 0) = P_\lambda(\boldsymbol{x})$. □

2.4. **Relation with generalized Schur functions.** We conclude this section by proving a relation between generalized P-functions and generalized Schur functions.

Proposition 2.8. *Let $\mathcal{F}$ be an admissible sequence and $\boldsymbol{x} = (x_1, \dots, x_n)$. For a partition μ of length $m \le n$, let $\mu + \delta_n$ be the strict partition obtained from $(\mu_1 + n - 1, \mu_2 + n - 2, \dots, \mu_{n-1} + 1, \mu_n)$ by removing 0s, where $\mu_{m+1} = \cdots = \mu_n = 0$. Then we have*

$$P_{\mu+\delta_n}^{\mathcal{F}}(\boldsymbol{x}) = \prod_{1 \le i < j \le n} (x_i + x_j) \cdot s_\mu^{\mathcal{F}}(\boldsymbol{x}),$$

where $s_\mu^{\mathcal{F}}(\boldsymbol{x})$ is the generalized Schur function given by (1.1).

Proof. Since the strict partition $\mu + \delta_n$ has length $n-1$ or n, it follows from (2.6) that

$$P_{\mu+\delta_n}^{\mathcal{F}}(\boldsymbol{x}) = \sum_{w \in S_n} w \left(\prod_{i=1}^n f_{\mu_i + n - i}(x_i) \prod_{1 \le i < j \le n} \frac{x_i + x_j}{x_i - x_j} \right)$$

$$= \prod_{1 \le i < j \le n} \frac{x_i + x_j}{x_i - x_j} \sum_{w \in S_n} \operatorname{sgn}(w) w \left(\prod_{i=1}^{n} f_{\mu_i + n - i}(x_i) \right)$$
$$= \prod_{1 \le i < j \le n} (x_i + x_j) \cdot s^{\mathcal{F}}_{\mu}(\boldsymbol{x}). \qquad \square$$

3. Dual P-functions and Cauchy-type identity

In this section, we introduce the dual of generalized P-functions and prove a Cauchy-type identity for generalized P-functions.

3.1. **Dual sequences.** For a nonzero formal power series $g(v) = \sum_{i=0}^{\infty} b_i v^i \in K[[v]]$, the *order* $\operatorname{ord} g$ of g is defined to be the minimum integer k such that $b_k \neq 0$. Let $\langle \ , \ \rangle : K[u] \times K[[v]] \to K$ be the non-degenerate bilinear pairing defined by

$$\langle u^i, v^j \rangle = \begin{cases} 1, & \text{if } i = j = 0, \\ 1/2, & \text{if } i = j > 0, \\ 0, & \text{if } i \neq j. \end{cases}$$

Lemma 3.1. *Let $\mathcal{F} = \{f_d\}_{d=0}^{\infty}$ be an admissible sequence of polynomials.*

(1) *Let $\widehat{\mathcal{F}} = \{\widehat{f}_d\}_{d=0}^{\infty}$ be a sequence of formal power series $\widehat{f}_d(v) \in K[[v]]$ satisfying* $\operatorname{ord} \widehat{f}_d = d$ *for $d \ge 0$. Then $\langle f_k, \widehat{f}_l \rangle = \delta_{k,l}$ for any k, $l \ge 0$ if and only if*
$$\sum_{k=0}^{\infty} f_k(u) \widehat{f}_k(v) = \frac{1 + uv}{1 - uv}. \tag{3.1}$$

(2) *There exists a unique sequence $\widehat{\mathcal{F}} = \{\widehat{f}_d\}_{d=0}^{\infty}$ satisfying* $\operatorname{ord} \widehat{f}_d = d$ *for $d \ge 0$ and the equivalent conditions in* (1)*. We call such a sequence $\widehat{\mathcal{F}}$ the* dual *of $\mathcal{F}$.*

(3) *If $\widehat{\mathcal{F}} = \{\widehat{f}_d\}_{d=0}^{\infty}$ is the dual of $\mathcal{F}$, then $\mathcal{F}$ is constant-term free if and only if $\widehat{f}_0 = 1$.*

Proof. (1), (2) We write $f_d(u) = \sum_{i \ge 0} a_{d,i} u^i$ and $\widehat{f}_d(v) = \sum_{i \ge 0} b_{d,i} v^i$. Since $\deg f_d = d$ and $\operatorname{ord} \widehat{f}_d = d$, we have $a_{d,i} = 0$ for $i \ge d+1$ and $b_{d,i} = 0$ for $i \le d-1$. We define $b'_{d,i}$ by putting

$$b'_{d,i} = \begin{cases} b_{d,0}, & \text{if } i = 0, \\ b_{d,i}/2, & \text{if } i > 0. \end{cases}$$

Fix a nonnegative integer N and consider two $(N+1) \times (N+1)$ matrices $A = (a_{i,j})_{0 \le i,j \le N}$ and $B' = (b'_{i,j})_{0 \le i,j \le N}$. Then it is easy to see that

(a) $\langle f_k, \widehat{f}_l \rangle = \delta_{k,l}$ for all $0 \le k, l \le N$ if and only if $A \cdot {}^t B' = I_{N+1}$;
(b) $\sum_{k \ge 0} f_k(u) \widehat{f}_k(v) = (1 + uv)/(1 - uv)$ in the quotient ring $K[[u,v]]/(u^{N+1}, v^{N+1})$ if and only if ${}^t A \cdot B' = I_{N+1}$.

The claims (1) and (2) follow from this observation.

(3) If $f_d(0) = 0$ for any $d \ge 1$, then by substituting $u = 0$ in (3.1) we obtain $\widehat{f}_0(v) = 1$. Conversely, if $\widehat{f}_0 = 1$, then $\langle f_d, \widehat{f}_0 \rangle = \delta_{d,0}$ is equal to the constant term of f_d. $\square$

For example, if $f_d(u) = u^d$ for $d \ge 0$, then the dual of $\mathcal{F}$ is given by

$$\widehat{f}_d(v) = \begin{cases} 1, & \text{if } d = 0, \\ 2v^d, & \text{if } d \ge 1. \end{cases}$$

See Lemma 6.1 for the dual of the sequence $\{(u|\boldsymbol{a})^d\}_{d=0}^{\infty}$ of factorial monomials.

3.2. **Generating functions of generalized P-functions.** For a sequence of variables $\boldsymbol{x} = (x_1, \dots, x_n)$ and another variable z, we put

$$\Pi_z(\boldsymbol{x}) = \prod_{i=1}^{n} \frac{1 + x_i z}{1 - x_i z}. \tag{3.2}$$

Then the generating functions of Schur Q-functions $Q_{(r)}(\boldsymbol{x})$ and $Q_{(r,s)}(\boldsymbol{x})$ are expressed as

$$\sum_{r \ge 0} Q_{(r)}(\boldsymbol{x}) z^r = \Pi_z(\boldsymbol{x}), \tag{3.3}$$

$$\sum_{r,s \ge 0} Q_{(r,s)}(\boldsymbol{x}) z^r w^s = \frac{z-w}{z+w} \big(\Pi_z(\boldsymbol{x}) \Pi_w(\boldsymbol{x}) - 1 \big), \tag{3.4}$$

respectively (see [12, III, (8.1)] and [23, p. 117]). We can generalize these generating functions in terms of the dual sequence $\widehat{\mathcal{F}}$.

Proposition 3.2. *Let $\mathcal{F} = \{f_d\}_{d=0}^{\infty}$ be an admissible sequence of polynomials, and $\widehat{\mathcal{F}} = \{\widehat{f}_d\}_{d=0}^{\infty}$ the dual sequence of $\mathcal{F}$. Then we have:*

(1) *The generating function of generalized P-functions $P^{\mathcal{F}}_{(r)}(\boldsymbol{x})$ is given by*

$$\sum_{r \ge 0} P^{\mathcal{F}}_{(r)}(\boldsymbol{x}) \widehat{f}_r(z) = \begin{cases} \Pi_z(\boldsymbol{x}), & \text{if } n \text{ is odd}, \\ \Pi_z(\boldsymbol{x}) + \widehat{f}_0(z) - 1, & \text{if } n \text{ is even}, \end{cases} \tag{3.5}$$

under the convention (2.7).

(2) *The generating function of generalized P-functions $P^{\mathcal{F}}_{(r,s)}(\boldsymbol{x})$ is given by*

$$\begin{aligned} &\sum_{r,s \ge 0} P^{\mathcal{F}}_{(r,s)}(\boldsymbol{x}) \widehat{f}_r(z) \widehat{f}_s(w) \\ &= \begin{cases} \dfrac{z-w}{z+w} \big(\Pi_z(\boldsymbol{x}) \Pi_w(\boldsymbol{x}) - 1 \big) + (\widehat{f}_0(w) - 1) \Pi_z(\boldsymbol{x}) - (\widehat{f}_0(z) - 1) \Pi_w(\boldsymbol{x}), & \text{if } n \text{ is odd}, \\ \dfrac{z-w}{z+w} \big(\Pi_z(\boldsymbol{x}) \Pi_w(\boldsymbol{x}) - 1 \big), & \text{if } n \text{ is even}, \end{cases} \end{aligned} \tag{3.6}$$

under the convention (2.8).

If we set $f_d(u) = (u|\boldsymbol{a})^d$ (factorial monomial) with $a_0 = 0$, then it follows from Lemma 6.1 that the formulas (3.5) and (3.6) reduce to the formulas given in [7, Theorem 8.2] and [7, Theorem 8.4] for factorial P-functions.

Proof. The idea of the proof is similar to that of [19, Theorem 4.1 (1) and (2)]. Let $B_z(\boldsymbol{x})$ be the column vector with ith entry $(1+x_iz)/(1-x_iz)$. By (3.1) we have

$$\sum_{r\geq 0}\widehat{f}_r(z)V_{(r)}(\boldsymbol{x})=B_z(\boldsymbol{x}),\quad \sum_{r\geq 1}\widehat{f}_r(z)V_{(r)}(\boldsymbol{x})=B_z(\boldsymbol{x})-\widehat{f}_0(z)\mathbf{1}.$$

(1) If n is odd, then by the definition (1.4) and the multilinearity of Pfaffians, we have

$$\sum_{r\geq 0}P_{(r)}(\boldsymbol{x})\widehat{f}_r(z)=\frac{1}{\Delta(\boldsymbol{x})}\operatorname{Pf}\begin{pmatrix}A(\boldsymbol{x}) & B_z(\boldsymbol{x})\\ -{}^tB_z(\boldsymbol{x}) & O\end{pmatrix},$$

which equals $\Pi_z(\boldsymbol{x})$ by (A.5). Similarly, if n is even, then we have

$$\sum_{r\geq 0}P_{(r)}(\boldsymbol{x})\widehat{f}_r(z)=\widehat{f}_0(z)+\frac{1}{\Delta(\boldsymbol{x})}\operatorname{Pf}\begin{pmatrix}A(\boldsymbol{x}) & B_z(\boldsymbol{x})-\widehat{f}_0(z)\mathbf{1} & \mathbf{1}\\ -{}^tB_z(\boldsymbol{x})+\widehat{f}_0(z){}^t\mathbf{1} & 0 & 0\\ -{}^t\mathbf{1} & 0 & 0\end{pmatrix}.$$

By adding the last row/column multiplied by $\widehat{f}_0(z)$ to the second to last row/column, and by then using (A.6) with $w=0$, we obtain

$$\sum_{r\geq 0}P_{(r)}(\boldsymbol{x})\widehat{f}_r(z)=\widehat{f}_0(z)+\frac{1}{\Delta(\boldsymbol{x})}\operatorname{Pf}\begin{pmatrix}A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & \mathbf{1}\\ -{}^tB_z(\boldsymbol{x}) & 0 & 0\\ -{}^t\mathbf{1} & 0 & 0\end{pmatrix}=\widehat{f}_0(z)+(\Pi_z(\boldsymbol{x})-1).$$

(2) If n is even, then we have

$$\sum_{r,s\geq 0}P_{(r,s)}(\boldsymbol{x})\widehat{f}_r(z)\widehat{f}_s(w)=\frac{1}{\Delta(\boldsymbol{x})}\operatorname{Pf}\begin{pmatrix}A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x})\\ -{}^tB_z(\boldsymbol{x}) & 0 & 0\\ -{}^tB_w(\boldsymbol{x}) & 0 & 0\end{pmatrix},$$

and this equals $(z-w)/(z+w)\cdot(\Pi_z(\boldsymbol{x})\Pi_w(\boldsymbol{x})-1)$ by (A.6). If n is odd, then we have

$$\begin{aligned}
&\sum_{r,s\geq 0}P_{(r,s)}(\boldsymbol{x})\widehat{f}_r(z)\widehat{f}_s(w)\\
&=\sum_{r,s>0}P_{(r,s)}(\boldsymbol{x})\widehat{f}_r(z)\widehat{f}_s(w)+\sum_{r\geq 0}P_{(r,0)}(\boldsymbol{x})\widehat{f}_r(z)\widehat{f}_0(w)+\sum_{s\geq 0}P_{(0,s)}(\boldsymbol{x})\widehat{f}_0(z)\widehat{f}_s(w)\\
&=\frac{1}{\Delta(\boldsymbol{x})}\operatorname{Pf}\begin{pmatrix}A(\boldsymbol{x}) & B_z(\boldsymbol{x})-\widehat{f}_0(z)\mathbf{1} & B_w(\boldsymbol{x})-\widehat{f}_0(w)\mathbf{1} & \mathbf{1}\\ -{}^tB_z(\boldsymbol{x})+\widehat{f}_0(z){}^t\mathbf{1} & 0 & 0 & 0\\ -{}^tB_w(\boldsymbol{x})+\widehat{f}_0(w){}^t\mathbf{1} & 0 & 0 & 0\\ -{}^t\mathbf{1} & 0 & 0 & 0\end{pmatrix}\\
&\quad+\widehat{f}_0(w)\cdot\frac{1}{\Delta(\boldsymbol{x})}\operatorname{Pf}\begin{pmatrix}A(\boldsymbol{x}) & B_z(\boldsymbol{x})-\widehat{f}_0(z)\mathbf{1}\\ -{}^tB_z(\boldsymbol{x})+\widehat{f}_0(z){}^t\mathbf{1} & 0\end{pmatrix}\\
&\quad-\widehat{f}_0(z)\cdot\frac{1}{\Delta(\boldsymbol{x})}\operatorname{Pf}\begin{pmatrix}A(\boldsymbol{x}) & B_w(\boldsymbol{x})-\widehat{f}_0(w)\mathbf{1}\\ -{}^tB_w(\boldsymbol{x})+\widehat{f}_0(w){}^t\mathbf{1} & 0\end{pmatrix}.
\end{aligned}$$

By adding the last row/column multiplied by $\widehat{f}_0(z)$ (respectively $\widehat{f}_0(w)$) to the third (respectively second) to last row/column in the first Pfaffian, we see that

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) - \widehat{f}_0(z)\mathbf{1} & B_w(\boldsymbol{x}) - \widehat{f}_0(w)\mathbf{1} & \mathbf{1} \\ -{}^tB_z(\boldsymbol{x}) + \widehat{f}_0(z){}^t\mathbf{1} & 0 & 0 & 0 \\ -{}^tB_w(\boldsymbol{x}) + \widehat{f}_0(w){}^t\mathbf{1} & 0 & 0 & 0 \\ -{}^t\mathbf{1} & 0 & 0 & 0 \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) & \mathbf{1} \\ -{}^tB_z(\boldsymbol{x}) & 0 & 0 & 0 \\ -{}^tB_w(\boldsymbol{x}) & 0 & 0 & 0 \\ -{}^t\mathbf{1} & 0 & 0 & 0 \end{pmatrix}.$$

By using the multilinearity of Pfaffians, we have

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) - \widehat{f}_0(z)\mathbf{1} \\ -{}^tB_z(\boldsymbol{x}) + \widehat{f}_0(z){}^t\mathbf{1} & 0 \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) \\ -{}^tB_z(\boldsymbol{x}) & 0 \end{pmatrix} - \widehat{f}_0(z)\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & \mathbf{1} \\ -{}^t\mathbf{1} & 0 \end{pmatrix},$$

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_w(\boldsymbol{x}) - \widehat{f}_0(w)\mathbf{1} \\ -{}^tB_w(\boldsymbol{x}) + \widehat{f}_0(w){}^t\mathbf{1} & 0 \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_w(\boldsymbol{x}) \\ -{}^tB_w(\boldsymbol{x}) & 0 \end{pmatrix} - \widehat{f}_0(w)\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & \mathbf{1} \\ -{}^t\mathbf{1} & 0 \end{pmatrix}.$$

Hence we can use (A.7), (A.5) and (A.4) to evaluate these Pfaffians and complete the proof. □

3.3. **Dual P-functions and Cauchy-type identity.** In this section, we introduce the dual P-functions and prove a Cauchy-type identity.

Definition 3.3. Let $\mathcal{F} = \{f_d\}_{d=0}^{\infty}$ be an admissible sequence and $\widehat{\mathcal{F}} = \{\widehat{f}_d\}_{d=0}^{\infty}$ the dual of $\mathcal{F}$. For a sequence $\alpha = (\alpha_1, \dots, \alpha_r)$ of nonnegative integers, let $\widehat{V}^{\mathcal{F}}_{\alpha}(\boldsymbol{x})$ be the $n \times r$ matrix given by

$$\widehat{V}^{\mathcal{F}}_{\alpha}(\boldsymbol{x}) = \left(\widehat{f}_{\alpha_j}(x_i)\right)_{1 \le i \le n, 1 \le j \le r}.$$

Given a strict partition λ of length l, we define the *generalized dual P-functions* $\widehat{P}^{\mathcal{F}}_{\lambda}(\boldsymbol{x})$ by putting

$$\widehat{P}^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) = \begin{cases} \dfrac{1}{\Delta(\boldsymbol{x})}\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & \widehat{V}^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) \\ -{}^t\widehat{V}^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) & O \end{pmatrix}, & \text{if } n+l \text{ is even}, \\ \dfrac{1}{\Delta(\boldsymbol{x})}\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & \widehat{V}^{\mathcal{F}}_{\lambda^0}(\boldsymbol{x}) \\ -{}^t\widehat{V}^{\mathcal{F}}_{\lambda^0}(\boldsymbol{x}) & O \end{pmatrix}, & \text{if } n+l \text{ is odd}, \end{cases} \tag{3.7}$$

where $\lambda^0 = (\lambda_1, \dots, \lambda_l, 0)$.

If $f_d(u) = u^d$ for $d \ge 0$, then $\widehat{f}_d(v) = 2v^d$ for $d \ge 0$. Hence $P^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) = P_{\lambda}(\boldsymbol{x})$ is the original Schur P-function, and the dual P-function $\widehat{P}^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) = Q_{\lambda}(\boldsymbol{x})$ is the original Schur Q-function.

By the same arguments as in Section 2, we obtain the following proposition.

Proposition 3.4. *Suppose that $\mathcal{F} = \{f_d\}_{d=0}^{\infty}$ is a constant-term free admissible sequence with dual $\widehat{\mathcal{F}} = \{\widehat{f}_d\}_{d=0}^{\infty}$. Then we have:*

(1) *For a strict partition of length $l \le n$, we have*

$$\widehat{P}^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) = \frac{1}{(n-l)!} \sum_{u \in S_n} u \left(\prod_{i=1}^{l} \widehat{f}_{\lambda_i}(x_i) \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{x_i + x_j}{x_i - x_j} \right). \tag{3.8}$$

(2) *For a sequence $\alpha = (\alpha_1, \dots, \alpha_r)$ of nonnegative integers, let $\widehat{S}^{\mathcal{F}}_{\alpha}(\boldsymbol{x})$ be the $r \times r$ skew-symmetric matrix defined by*

$$\widehat{S}^{\mathcal{F}}_{\alpha}(\boldsymbol{x}) = \left(\widehat{P}^{\mathcal{F}}_{(\alpha_i, \alpha_j)}(\boldsymbol{x}) \right)_{1 \le i,j \le r},$$

where we use the same convention as (2.8). Then, for a strict partition λ of length l, we have

$$\widehat{P}^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) = \begin{cases} \operatorname{Pf} \widehat{S}^{\mathcal{F}}_{\lambda}(\boldsymbol{x}), & \text{if } l \text{ is even,} \\ \operatorname{Pf} \widehat{S}^{\mathcal{F}}_{\lambda^0}(\boldsymbol{x}), & \text{if } l \text{ is odd.} \end{cases} \tag{3.9}$$

(3) *For a strict partition λ, we have*

$$\widehat{P}^{\mathcal{F}}_{\lambda}(x_1, \dots, x_n, 0) = \widehat{P}^{\mathcal{F}}_{\lambda}(x_1, \dots, x_n). \tag{3.10}$$

Proof. By Lemma 3.1 (3), we have $\widehat{f}_0 = 1$ for a constant-term free admissible sequence $\mathcal{F}$. Hence the proofs of Theorems 2.3, 2.6 and Proposition 2.7 (2) work literally in this dual setting. □

For Schur's P- and Q-functions we have the following Cauchy-type identity (see [12, III, (8.13)], and [19, Theorem 5.1] for a linear algebraic proof):

$$\sum_{\lambda} P_{\lambda}(\boldsymbol{x}) Q_{\lambda}(\boldsymbol{y}) = \prod_{i,j=1}^{n} \frac{1 + x_i y_j}{1 - x_i y_j},$$

where $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{y} = (y_1, \dots, y_n)$, and the summation is taken over all strict partitions of length $\le n$. We can use the notion of dual P-functions to formulate a Cauchy-type identity for generalized P-functions.

Theorem 3.5. *Let $\mathcal{F}$ be an admissible sequence of polynomials, and let $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ be two sequences of indeterminates. Then we have*

$$\sum_{\lambda} P^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) \widehat{P}^{\mathcal{F}}_{\lambda}(\boldsymbol{y}) = \prod_{i,j=1}^{n} \frac{1 + x_i y_j}{1 - x_i y_j},$$

where λ runs over all strict partitions.

Proof. We use the same argument as in the proof of [19, Theorem 5.1]. Apply a Pfaffian version of the Cauchy–Binet formula (A.16) to the matrices

$$A = A(\boldsymbol{x}), \quad B = A(\boldsymbol{y}), \quad S = \left(f_k(x_i) \right)_{1 \le i \le n, k \ge 0}, \quad T = \left(\widehat{f}_k(y_i) \right)_{1 \le i \le n, k \ge 0}.$$

Strict partitions λ are in bijection with subsets of $\mathbb{N}$ satisfying $\#I = n \bmod 2$ via the correspondence $\lambda \mapsto I = \{\lambda_1, \dots, \lambda_{l(\lambda)}\}$ or $\{\lambda_1, \dots, \lambda_{l(\lambda)}, 0\}$. Furthermore, we have

$$P^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) = \frac{(-1)^{\binom{\#I}{2}}}{\Delta(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & S([n];I) \\ -{}^t S([n];I) & O \end{pmatrix},$$

$$\widehat{P}^{\mathcal{F}}_{\lambda}(\boldsymbol{y}) = \frac{(-1)^{\binom{\#I}{2}}}{\Delta(\boldsymbol{y})} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{y}) & T([n];I) \\ -{}^t T([n];I) & O \end{pmatrix},$$

where $S([n];I)$ and $T([n];I)$ are the submatrices of S and T, respectively, obtained by picking up the columns indexed by I. Since the (i,j) entry of $S {}^t T$ is equal to $(1 + x_i y_j)/(1 - x_i y_j)$ by (3.1), we can complete the proof by using the Pfaffian evaluation (A.5). $\square$

4. Generalized skew P-functions

In this section, we introduce generalized skew P-functions in terms of a Józefiak–Pragacz–Nimmo-type Pfaffian, and study their properties.

4.1. Józefiak–Pragacz–Nimmo-type formula.

First we define generalized skew P-functions associated with an admissible sequence.

Definition 4.1. Let $\mathcal{F} = \{f_d\}_{d=0}^{\infty}$ be an admissible sequence. For a pair of nonnegative integers r and k, we define a symmetric polynomial $R^{\mathcal{F}}_{r/k}(\boldsymbol{x})$ by the relation

$$P^{\mathcal{F}}_{(r)}(x_1, \dots, x_n, y) = \sum_{k=0}^{\infty} R^{\mathcal{F}}_{r/k}(x_1, \dots, x_n) f_k(y). \tag{4.1}$$

For two sequences $\alpha = (\alpha_1, \dots, \alpha_r)$ and $\beta = (\beta_1, \dots, \beta_s)$ of nonnegative integers, let $M_{\alpha/\beta}(\boldsymbol{x})$ be the $r \times s$ matrix given by

$$M^{\mathcal{F}}_{\alpha/\beta}(\boldsymbol{x}) = \left(R^{\mathcal{F}}_{\alpha_i/\beta_{s+1-j}}(\boldsymbol{x}) \right)_{1 \le i \le r, 1 \le j \le s}. \tag{4.2}$$

For a pair of strict partitions λ of length l and μ of length m and a positive integer p, we define the *generalized skew P-function* $P^{\mathcal{F}}_{\lambda/\mu,p}(\boldsymbol{x})$ by putting

$$P^{\mathcal{F}}_{\lambda/\mu,p}(\boldsymbol{x}) = \begin{cases} \operatorname{Pf} \begin{pmatrix} S^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) & M^{\mathcal{F}}_{\lambda/\mu}(\boldsymbol{x}) \\ -{}^t M^{\mathcal{F}}_{\lambda/\mu}(\boldsymbol{x}) & O \end{pmatrix}, & \text{if } l \equiv p \text{ and } m \equiv p \bmod 2, \\ \operatorname{Pf} \begin{pmatrix} S^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) & M^{\mathcal{F}}_{\lambda/\mu^0}(\boldsymbol{x}) \\ -{}^t M^{\mathcal{F}}_{\lambda/\mu^0}(\boldsymbol{x}) & O \end{pmatrix}, & \text{if } l \equiv p \text{ and } m \not\equiv p \bmod 2, \\ \operatorname{Pf} \begin{pmatrix} S^{\mathcal{F}}_{\lambda^0}(\boldsymbol{x}) & M^{\mathcal{F}}_{\lambda^0/\mu}(\boldsymbol{x}) \\ -{}^t M^{\mathcal{F}}_{\lambda^0/\mu}(\boldsymbol{x}) & O \end{pmatrix}, & \text{if } l \not\equiv p \text{ and } m \equiv p \bmod 2, \\ \operatorname{Pf} \begin{pmatrix} S^{\mathcal{F}}_{\lambda^0}(\boldsymbol{x}) & M^{\mathcal{F}}_{\lambda^0/\mu^0}(\boldsymbol{x}) \\ -{}^t M^{\mathcal{F}}_{\lambda^0/\mu^0}(\boldsymbol{x}) & O \end{pmatrix}, & \text{if } l \not\equiv p \text{ and } m \not\equiv p \bmod 2, \end{cases} \tag{4.3}$$

where $S^{\mathcal{F}}_{\alpha}(\boldsymbol{x})$ is the skew-symmetric matrix defined in (2.9), and $\lambda^0 = (\lambda_1, \dots, \lambda_l, 0)$, $\mu^0 = (\mu_1, \dots, \mu_m, 0)$.

The main result of this section is the following theorem, which is a generalization of the Józefiak–Pragacz–Nimmo formula for Schur P/Q functions (see [21, Theorem 1] and [18, (2.22)]).

Theorem 4.2. *For two sequences of variables $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_p)$, we have*

$$P^{\mathcal{F}}_{\lambda}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\mu} P^{\mathcal{F}}_{\lambda/\mu,p}(\boldsymbol{x}) P^{\mathcal{F}}_{\mu}(\boldsymbol{y}), \tag{4.4}$$

where μ runs over all strict partitions.

We postpone the proof of this theorem to the next subsection. Before the proof, we derive several properties of generalized skew P-functions from the definition (4.3). We begin with the following property of $R^{\mathcal{F}}_{r/k}(\boldsymbol{x})$.

Lemma 4.3. *For a positive integer r, the generalized P-function $P^{\mathcal{F}}_{(r)}(\boldsymbol{x}, y)$ has degree at most r in y. Hence we have $R^{\mathcal{F}}_{r/k}(\boldsymbol{x}) = 0$ unless $r \geq k$.*

Proof. The coefficient of z^r in $\Pi_z(\boldsymbol{x}, y) = \prod_{i=1}^{n} (1 + x_i z)/(1 - x_i z) \cdot (1 + yz)/(1 - yz)$ has degree at most r in y. On the other hand, since $\operatorname{ord} \widehat{f}_r = r$, the coefficient of z^r in $\sum_{r \geq 0} P^{\mathcal{F}}_{(r)}(\boldsymbol{x}, y) \widehat{f}_r(z)$ is a linear combination of $P^{\mathcal{F}}_{(0)}(\boldsymbol{x}, y), \dots, P^{\mathcal{F}}_{(r)}(\boldsymbol{x}, y)$ with nonzero coefficient for $P^{\mathcal{F}}_{(r)}(\boldsymbol{x}, y)$. Hence, by using (3.5) and the induction on r, we can conclude that $P^{\mathcal{F}}_{(r)}(\boldsymbol{x}, y)$ has degree at most r in y. □

This lemma can be used to prove the following vanishing property of generalized skew P-functions. For a strict partition λ, we define its shifted diagram $S(\lambda)$ by putting

$$S(\lambda) = \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq l(\lambda),\ i \leq j \leq \lambda_i + i - 1\}.$$

For two strict partitions λ and μ, we write $\lambda \supset \mu$ if $S(\lambda) \supset S(\mu)$.

Proposition 4.4. *Let $\mathcal{F}$ be an admissible sequence. For two strict partitions λ and μ, we have $P^{\mathcal{F}}_{\lambda/\mu,p}(\boldsymbol{x}) = 0$ unless $\lambda \supset \mu$.*

Proof. Suppose that there exists an index k such that $\lambda_k < \mu_k$. Then, if $i \geq k$ and $j \leq k$, we have $\lambda_i \leq \lambda_k < \mu_k \leq \mu_j$ and $R_{\lambda_i/\mu_j}(\boldsymbol{x}) = 0$ by Lemma 4.3. Hence the skew-symmetric matrices X appearing in the definition (4.3) of $P^{\mathcal{F}}_{\lambda/\mu,p}(\boldsymbol{x})$ are of the form

$$X = \begin{pmatrix} Z & W \\ -{}^t W & O \end{pmatrix},$$

where W has k columns and at most $(k-1)$ nonzero rows. Since all the $k \times k$ minors of W vanish, it follows from the Laplace expansion (Proposition A.5) that $P^{\mathcal{F}}_{\lambda/\mu,p}(\boldsymbol{x}) = \operatorname{Pf} X = 0$.
□

If the polynomial sequence $\mathcal{F}$ is constant-term free, then the skew P-function $P^{\mathcal{F}}_{\lambda/\mu,p}$ is independent of p and some formulas have simple forms.

Proposition 4.5. *Suppose that $\mathcal{F}$ is a constant-term free admissible sequence.*

(1) *$R^{\mathcal{F}}_{0/0}(\boldsymbol{x}) = 1$ and $R^{\mathcal{F}}_{r/0}(\boldsymbol{x}) = P^{\mathcal{F}}_{(r)}(\boldsymbol{x})$ for a positive integer r.*

(2) *For two strict partitions λ of length l and μ of length m, we have*

$$P^{\mathcal{F}}_{\lambda/\mu,p}(\boldsymbol{x}) = \begin{cases} \operatorname{Pf}\begin{pmatrix} S^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) & M^{\mathcal{F}}_{\lambda/\mu}(\boldsymbol{x}) \\ -{}^t M^{\mathcal{F}}_{\lambda/\mu}(\boldsymbol{x}) & O \end{pmatrix}, & \text{if } l \equiv m \bmod 2, \\ \operatorname{Pf}\begin{pmatrix} S^{\mathcal{F}}_{\lambda}(\boldsymbol{x}) & M^{\mathcal{F}}_{\lambda/\mu^0}(\boldsymbol{x}) \\ -{}^t M^{\mathcal{F}}_{\lambda/\mu^0}(\boldsymbol{x}) & O \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} S^{\mathcal{F}}_{\lambda^0}(\boldsymbol{x}) & M^{\mathcal{F}}_{\lambda^0/\mu}(\boldsymbol{x}) \\ -{}^t M^{\mathcal{F}}_{\lambda^0/\mu}(\boldsymbol{x}) & O \end{pmatrix}, & \\ & \text{if } l \not\equiv m \bmod 2. \end{cases}$$

In particular, $P^{\mathcal{F}}_{\lambda/\mu,p}(\boldsymbol{x})$ is independent of p. We write $P^{\mathcal{F}}_{\lambda/\mu}(\boldsymbol{x})$ in this case.

Proof. (1) Substituting $y = 0$ in the definition (4.1), we obtain

$$P_{(r)}(\boldsymbol{x}, 0) = \sum_{k=0}^{r} R_{r/k}(\boldsymbol{x}) f_k(0).$$

Since $P_{(r)}(\boldsymbol{x}, 0) = P_{(r)}(\boldsymbol{x})$ by the stability property (Proposition 2.7 (2)), and $f_r(0) = \delta_{r,0}$ by assumption, we have $R_{r/0}(\boldsymbol{x}) = P_{(r)}(\boldsymbol{x})$.

(2) By using (1), we see that

$$\begin{pmatrix} S_{\lambda}(\boldsymbol{x}) & M_{\lambda/\mu^0}(\boldsymbol{x}) \\ -{}^t M_{\lambda/\mu^0}(\boldsymbol{x}) & O \end{pmatrix} = \begin{pmatrix} S_{\lambda^0}(\boldsymbol{x}) & M_{\lambda^0/\mu}(\boldsymbol{x}) \\ -{}^t M_{\lambda^0/\mu}(\boldsymbol{x}) & O \end{pmatrix}.$$

It remains to show that, if $l(\lambda) \not\equiv p$ and $l(\mu) \not\equiv p \bmod 2$, then

$$\operatorname{Pf}\begin{pmatrix} S_{\lambda^0}(\boldsymbol{x}) & M_{\lambda^0/\mu^0}(\boldsymbol{x}) \\ -{}^t M_{\lambda^0/\mu^0}(\boldsymbol{x}) & O \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} S_{\lambda}(\boldsymbol{x}) & M_{\lambda/\mu}(\boldsymbol{x}) \\ -{}^t M_{\lambda/\mu}(\boldsymbol{x}) & O \end{pmatrix}.$$

Let $T_{\lambda}(\boldsymbol{x})$ (respectively $T_{\lambda/0}(\boldsymbol{x})$) be the column vector with ith entry $P_{(\lambda_i)}(\boldsymbol{x})$ (respectively $R_{\lambda_i/0}(\boldsymbol{x})$). Since $R_{0/0}(\boldsymbol{x}) = 1$ and $R_{0/k}(\boldsymbol{x}) = 0$ for $k \geq 1$, we have

$$\begin{pmatrix} S_{\lambda^0}(\boldsymbol{x}) & M_{\lambda^0/\mu^0}(\boldsymbol{x}) \\ -{}^t M_{\lambda^0/\mu^0}(\boldsymbol{x}) & O \end{pmatrix} = \begin{pmatrix} S_{\lambda}(\boldsymbol{x}) & T_{\lambda}(\boldsymbol{x}) & T_{\lambda/0}(\boldsymbol{x}) & M_{\lambda/\mu}(\boldsymbol{x}) \\ -{}^t T_{\lambda}(\boldsymbol{x}) & 0 & 1 & O \\ -{}^t T_{\lambda/0}(\boldsymbol{x}) & -1 & 0 & O \\ -{}^t M_{\lambda/\mu}(\boldsymbol{x}) & O & O & O \end{pmatrix}.$$

Since $T_{\lambda}(\boldsymbol{x}) = T_{\lambda/0}(\boldsymbol{x})$ by (1), we add the $(l+1)$st row/column multiplied by -1 to the $(l+2)$nd row/column and expand the resulting Pfaffian along the $(l+2)$nd row/column to obtain

$$\operatorname{Pf}\begin{pmatrix} S_{\lambda^0}(\boldsymbol{x}) & M_{\lambda^0/\mu^0}(\boldsymbol{x}) \\ -{}^t M_{\lambda^0/\mu^0}(\boldsymbol{x}) & O \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} S_{\lambda}(\boldsymbol{x}) & M_{\lambda/\mu}(\boldsymbol{x}) \\ -{}^t M_{\lambda/\mu}(\boldsymbol{x}) & O \end{pmatrix}. \qquad \square$$

If $\boldsymbol{x}$ consists of a single variable, then we have the following proposition.

Proposition 4.6. *Let $\mathcal{F}$ be an admissible sequence and $\boldsymbol{x} = (x)$ a single variable.*

(1) *For two strict partitions λ and μ, we have $P^{\mathcal{F}}_{\lambda/\mu,p}(x) = 0$ unless $l(\lambda) - l(\mu) \leq 2$.*

(2) *If $\mathcal{F}$ is constant-term free, then we have $P^{\mathcal{F}}_{\lambda/\mu}(x) = 0$ unless $l(\lambda) - l(\mu) \leq 1$, and*

$$P^{\mathcal{F}}_{\lambda/\mu}(x) = \det\left(R^{\mathcal{F}}_{\lambda_i/\mu_j}(x)\right)_{1 \leq i,j \leq l(\lambda)}. \tag{4.5}$$

Proof. Put $l = l(\lambda)$ and $m = l(\mu)$.

(1) Since $P_{(\lambda_i,\lambda_j)}(x) = 0$ for $\lambda_i > \lambda_j > 0$ by Proposition 2.1, we have

$$P_{\lambda/\mu,p}(x) = \begin{cases} \operatorname{Pf}\begin{pmatrix} O_l & M_{\lambda/\mu}(x) \\ -{}^tM_{\lambda/\mu}(x) & O_m \end{pmatrix}, & \text{if } l \equiv p \text{ and } m \equiv p \bmod 2, \\ \operatorname{Pf}\begin{pmatrix} O_l & M_{\lambda/\mu^0}(x) \\ -{}^tM_{\lambda/\mu^0}(x) & O_{m+1} \end{pmatrix}, & \text{if } l \equiv p \text{ and } m \not\equiv p \bmod 2, \\ \operatorname{Pf}\begin{pmatrix} O_l & T_\lambda(x) & M_{\lambda/\mu}(x) \\ -{}^tT_\lambda(x) & 0 & O \\ -{}^tM_{\lambda/\mu}(x) & O & O_m \end{pmatrix}, & \text{if } l \not\equiv p \text{ and } m \equiv p \bmod 2, \\ \operatorname{Pf}\begin{pmatrix} O_l & T_\lambda(x) & T_{\lambda/0}(x) & M_{\lambda/\mu}(x) \\ -{}^tT_\lambda(x) & 0 & 1 & O \\ -{}^tT_{\lambda/0}(x) & -1 & 0 & O \\ -{}^tM_{\lambda/\mu}(x) & O & O & O_m \end{pmatrix}, & \text{if } l \not\equiv p \text{ and } m \not\equiv p \bmod 2, \end{cases}$$

where $T_\lambda(x)$ (respectively $T_{\lambda/0}(x)$) is the column vector with ith entry $P_{(\lambda_i)}(x)$ (respectively $R_{\lambda_i/0}(x)$). By using Proposition A.5, we see that $P_{\lambda/\mu}(x) = 0$ unless

$$\begin{cases} l = m, & \text{if } l \equiv p \text{ and } m \equiv p \bmod 2, \\ l = m+1, & \text{if } l \equiv p \text{ and } m \not\equiv p \bmod 2, \\ l = m+1, & \text{if } l \not\equiv p \text{ and } m \equiv p \bmod 2, \\ l = m \text{ or } m+2, & \text{if } l \not\equiv p \text{ and } m \not\equiv p \bmod 2. \end{cases}$$

Here we note that $m \le l \le m+2$ if and only if $l \le (l+m+2)/2$ and $m+1 \le (l+m+2)/2$.

(2) By using Proposition 4.5 (2) and Proposition A.5, we have

$$P^{\mathcal{F}}_{\lambda/\mu,p}(x) = \begin{cases} \operatorname{Pf}\begin{pmatrix} O_l & M^{\mathcal{F}}_{\lambda/\mu}(x) \\ -{}^tM^{\mathcal{F}}_{\lambda/\mu}(x) & O \end{pmatrix} = (-1)^{\binom{l}{2}} \det M_{\lambda/\mu}(x), & \text{if } l = m, \\ \operatorname{Pf}\begin{pmatrix} O_l & M^{\mathcal{F}}_{\lambda/\mu^0}(\boldsymbol{x}) \\ -{}^tM^{\mathcal{F}}_{\lambda/\mu^0}(\boldsymbol{x}) & O \end{pmatrix} = (-1)^{\binom{l}{2}} \det M_{\lambda/\mu^0}(x), & \text{if } l = m+1. \end{cases}$$

By permuting columns of $M_{\lambda/\mu}(x)$ and $M_{\lambda/\mu^0}(x)$, we obtain (4.5). □

4.2. **Proof of Theorem 4.2.** We give a proof of Theorem 4.2 by using the same idea as in the proof of [19, Theorem 6.1] for Schur Q-functions. A key is played by the following proposition, which interpolates the Nimmo-type formula (1.4) ($n = 0$ case) and the Schur-type formula (2.10) ($p = 0$ case).

Proposition 4.7. *Let $\mathcal{F}$ be an admissible polynomial sequence and $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_p)$ two sequences of indeterminates. For a nonnegative integer sequence $\alpha = (\alpha_1, \dots, \alpha_r)$, let $N^{\mathcal{F}}_\alpha(\boldsymbol{x}|\boldsymbol{y})$ be the $r \times p$ matrix defined by*

$$N^{\mathcal{F}}_\alpha(\boldsymbol{x}|\boldsymbol{y}) = \left(P^{\mathcal{F}}_{\alpha_i}(\boldsymbol{x}, y_j)\right)_{1 \le i \le r, 1 \le j \le p}.$$

Then, for a strict partition λ of length l, we have

$$P^{\mathcal{F}}_\lambda(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \dfrac{1}{\Delta(\boldsymbol{y})} \operatorname{Pf} \begin{pmatrix} S^{\mathcal{F}}_\lambda(\boldsymbol{x}) & N^{\mathcal{F}}_\lambda(\boldsymbol{x}|\boldsymbol{y}) \\ -{}^t N^{\mathcal{F}}_\lambda(\boldsymbol{x}|\boldsymbol{y}) & A(\boldsymbol{y}) \end{pmatrix}, & \text{if } l+p \text{ is even,} \\ \dfrac{1}{\Delta(\boldsymbol{y})} \operatorname{Pf} \begin{pmatrix} S^{\mathcal{F}}_{\lambda^0}(\boldsymbol{x}) & N^{\mathcal{F}}_{\lambda^0}(\boldsymbol{x}|\boldsymbol{y}) \\ -{}^t N^{\mathcal{F}}_{\lambda^0}(\boldsymbol{x}|\boldsymbol{y}) & A(\boldsymbol{y}) \end{pmatrix}, & \text{if } l+p \text{ is odd,} \end{cases} \tag{4.6}$$

where $\lambda^0 = (\lambda_1, \dots, \lambda_l, 0)$, and $S^{\mathcal{F}}_\alpha(\boldsymbol{x})$ and $A(\boldsymbol{y})$ are given by (2.9) *and* (1.3), *respectively.*

We denote by $P'_\lambda(\boldsymbol{x}|\boldsymbol{y})$ the right-hand side of (4.6). First we prove that $P'_\lambda(\boldsymbol{x}|\boldsymbol{y})$ satisfies a Schur-type Pfaffian formula.

Lemma 4.8. *For a strict partition λ of length l, we have*

$$P'_\lambda(\boldsymbol{x}|\boldsymbol{y}) = \begin{cases} \operatorname{Pf}\left(P'_{(\lambda_i,\lambda_j)}(\boldsymbol{x}|\boldsymbol{y}) \right)_{1 \le i,j \le l}, & \text{if } l \text{ is even,} \\ \operatorname{Pf}\left(P'_{(\lambda_i,\lambda_j)}(\boldsymbol{x}|\boldsymbol{y}) \right)_{1 \le i,j \le l+1}, & \text{if } l \text{ is odd,} \end{cases}$$

where $\lambda_{l+1} = 0$ if l is odd, and we use a convention similar to (2.8).

Proof. The proof is similar to that of Theorem 2.6, so we omit the details. We apply a Pfaffian analogue of the Sylvester formula (Proposition A.4) to the following matrix X (after permuting rows/columns):

$$X = \begin{cases} \begin{pmatrix} S_\lambda(\boldsymbol{x}) & N_\lambda(\boldsymbol{x}|\boldsymbol{y}) \\ -{}^t N_\lambda(\boldsymbol{x}|\boldsymbol{y}) & A(\boldsymbol{y}) \end{pmatrix}, & \text{if } l \text{ is even and } p \text{ is even,} \\ \begin{pmatrix} S_{\lambda^0}(\boldsymbol{x}) & N_{\lambda^0}(\boldsymbol{x}|\boldsymbol{y}) \\ -{}^t N_{\lambda^0}(\boldsymbol{x}|\boldsymbol{y}) & A(\boldsymbol{y}) \end{pmatrix}, & \text{if } l \text{ is odd and } p \text{ is even,} \\ \begin{pmatrix} S_\lambda(\boldsymbol{x}) & T_\lambda(\boldsymbol{x}) & N_\lambda(\boldsymbol{x}|\boldsymbol{y}) \\ -{}^t T_\lambda(\boldsymbol{x}) & 0 & {}^t\mathbf{1} \\ -{}^t N_\lambda(\boldsymbol{x}|\boldsymbol{y}) & -\mathbf{1} & A(\boldsymbol{y}) \end{pmatrix}, & \text{if } l \text{ is even and } p \text{ is even,} \\ \begin{pmatrix} S_\lambda(\boldsymbol{x}) & T_\lambda(\boldsymbol{x}) & \mathbf{0} & N_\lambda(\boldsymbol{x}|\boldsymbol{y}) \\ -{}^t T_\lambda(\boldsymbol{x}) & 0 & 1 & O \\ 0 & -1 & 0 & {}^t\mathbf{1} \\ -{}^t N_\lambda(\boldsymbol{x}|\boldsymbol{y}) & O & -{}^t\mathbf{1} & A(\boldsymbol{y}) \end{pmatrix}, & \text{if } l \text{ is odd and } p \text{ is odd,} \end{cases}$$

where $T_\lambda(\boldsymbol{x})$ is the column vector with ith entry $P_{(\lambda_i)}(\boldsymbol{x})$. □

Proof of Proposition 4.7. By comparing Theorem 2.6 with Lemma 4.8, the proof is reduced to showing

$$P_{(r)}(\boldsymbol{x},\boldsymbol{y}) = P'_{(r)}(\boldsymbol{x}|\boldsymbol{y}), \quad P_{(r,s)}(\boldsymbol{x},\boldsymbol{y}) = P'_{(r,s)}(\boldsymbol{x}|\boldsymbol{y}).$$

By considering the generating functions and using Proposition 3.2, it is enough to prove

$$\sum_{r \ge 0} P'_{(r)}(\boldsymbol{x}|\boldsymbol{y}) \widehat{f}_r(z) = \begin{cases} \Pi_z(\boldsymbol{x},\boldsymbol{y}), & \text{if } n+p \text{ is odd,} \\ \Pi_z(\boldsymbol{x},\boldsymbol{y}) + \widehat{f}_0(z) - 1, & \text{if } n+p \text{ is even,} \end{cases} \tag{4.7}$$

and

$$\sum_{r,s\geq 0} P'_{(r,s)}(\boldsymbol{x}|\boldsymbol{y})\widehat{f}_r(z)\widehat{f}_s(w)$$
$$= \begin{cases} \dfrac{z-w}{z+w}\big(\Pi_z(\boldsymbol{x},\boldsymbol{y})\Pi_w(\boldsymbol{x},\boldsymbol{y})-1\big) + (\widehat{f}_0(w)-1)\Pi_z(\boldsymbol{x},\boldsymbol{y}) - (\widehat{f}_0(z)-1)\Pi_w(\boldsymbol{x},\boldsymbol{y}), & \\ & \text{if } n+p \text{ is odd,} \\ \dfrac{z-w}{z+w}\big(\Pi_z(\boldsymbol{x},\boldsymbol{y})\Pi_w(\boldsymbol{x},\boldsymbol{y})-1\big), & \text{if } n+p \text{ is even.} \end{cases} \tag{4.8}$$

First we prove (4.7). We put

$$F_z(\boldsymbol{x}) = \sum_{r\geq 0} P_{(r)}(\boldsymbol{x})\widehat{f}_r(\boldsymbol{x}), \quad F_z^+(\boldsymbol{x}) = \sum_{r\geq 1} P_{(r)}(\boldsymbol{x})\widehat{f}_r(\boldsymbol{x}) = F_z(\boldsymbol{x}) - \widehat{f}_0(z).$$

Let $U_z(\boldsymbol{x}|\boldsymbol{y})$ (respectively $U_z^+(\boldsymbol{x}|\boldsymbol{y})$) be the row vector with jth entry $\sum_{r\geq 0} P_{(r)}(\boldsymbol{x},y_j)\widehat{f}_r(z)$ (respectively $\sum_{r\geq 1} P_{(r)}(\boldsymbol{x},y_j)\widehat{f}_r(z)$). Then, by the definition of $P'_{(r)}(\boldsymbol{x})$ and the multilinearity of Pfaffians, we have

$$\sum_{r\geq 0} P'_{(r)}(\boldsymbol{x}|\boldsymbol{y})\widehat{f}_r(z) = \begin{cases} \dfrac{1}{\Delta(\boldsymbol{y})}\operatorname{Pf}\begin{pmatrix} 0 & U_z(\boldsymbol{x}|\boldsymbol{y}) \\ -{}^tU_z(\boldsymbol{x}|\boldsymbol{y}) & A(\boldsymbol{y}) \end{pmatrix}, & \text{if } p \text{ is odd,} \\ \widehat{f}_0(z) + \dfrac{1}{\Delta(\boldsymbol{y})}\operatorname{Pf}\begin{pmatrix} 0 & F_z^+(\boldsymbol{x}) & U_z^+(\boldsymbol{x}|\boldsymbol{y}) \\ -F_z^+(\boldsymbol{x}) & 0 & {}^t\boldsymbol{1} \\ -{}^tU^+z(\boldsymbol{x}|\boldsymbol{y}) & -\boldsymbol{1} & A(\boldsymbol{y}) \end{pmatrix}, & \text{if } p \text{ is even.} \end{cases}$$

By adding the 2nd row/column multiplied by $\widehat{f}_0(z)$ to the 1st row/column, we see that

$$\operatorname{Pf}\begin{pmatrix} 0 & F_z^+(\boldsymbol{x}) & U_z^+(\boldsymbol{x}|\boldsymbol{y}) \\ -F_z^+(\boldsymbol{x}) & 0 & {}^t\boldsymbol{1} \\ -{}^tU^+z(\boldsymbol{x}|\boldsymbol{y}) & -\boldsymbol{1} & A(\boldsymbol{y}) \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} 0 & F_z^+(\boldsymbol{x}) & U_z(\boldsymbol{x}|\boldsymbol{y}) \\ -F_z^+(\boldsymbol{x}) & 0 & {}^t\boldsymbol{1} \\ -{}^tU_z(\boldsymbol{x}|\boldsymbol{y}) & -\boldsymbol{1} & A(\boldsymbol{y}) \end{pmatrix}.$$

By Proposition 3.2, we have

$$U_z(\boldsymbol{x}|\boldsymbol{y}) = \begin{cases} \Pi_z(\boldsymbol{x}){}^tB_z(\boldsymbol{y}), & \text{if } n \text{ is even,} \\ \Pi_z(\boldsymbol{x}){}^tB_z(\boldsymbol{y}) + (\widehat{f}_0(z)-1){}^t\boldsymbol{1}, & \text{if } n \text{ is odd.} \end{cases} \tag{4.9}$$

$$F_z^+(\boldsymbol{x}) = \begin{cases} \Pi_z(\boldsymbol{x}) - 1, & \text{if } n \text{ is even,} \\ \Pi_z(\boldsymbol{x}) - \widehat{f}_0(z), & \text{if } n \text{ is odd,} \end{cases} \tag{4.10}$$

where $B_z(\boldsymbol{x})$ is the column vector with ith entry $(1+x_iz)/(1-x_iz)$. Now we distinguish four cases according to the parity of p and n.

If p is odd and n is even, then, by using (A.5), we have

$$\sum_{r\geq 0} P'_{(r)}(\boldsymbol{x}|\boldsymbol{y})\widehat{f}_r(z) = \frac{1}{\Delta(\boldsymbol{y})}\cdot\Pi_z(\boldsymbol{x})\operatorname{Pf}\begin{pmatrix} 0 & {}^tB_z(\boldsymbol{y}) \\ -B_z(\boldsymbol{y}) & A(\boldsymbol{y}) \end{pmatrix} = \Pi_z(\boldsymbol{x})\Pi_z(\boldsymbol{y}).$$

If p is odd and n is odd, then, by using the multilinearity of Pfaffians and (A.5), (A.4), we have

$$\sum_{r\geq 0} P'_{(r)}(\boldsymbol{x}|\boldsymbol{y})\widehat{f}_r(z) = \frac{1}{\Delta(\boldsymbol{y})}\left[\Pi_z(\boldsymbol{x})\,\mathrm{Pf}\begin{pmatrix} 0 & {}^tB_z(\boldsymbol{y}) \\ -B_z(\boldsymbol{y}) & A(\boldsymbol{y})\end{pmatrix} + (\widehat{f}_0(z)-1)\,\mathrm{Pf}\begin{pmatrix} 0 & {}^t\mathbf{1} \\ -\mathbf{1} & A(\boldsymbol{y})\end{pmatrix}\right]$$
$$= \Pi_z(\boldsymbol{x},\boldsymbol{y}) + \widehat{f}_0(z) - 1.$$

If p is even and n is even, then, by using the multilinearity and the expansion of Pfaffians, (A.5) (with $w=0$) and (A.4), we have

$$\sum_{r\geq 0} P'_{(r)}(\boldsymbol{x}|\boldsymbol{y})\widehat{f}_r(z)$$
$$= \widehat{f}_0(z)$$
$$+ \frac{1}{\Delta(\boldsymbol{y})}\left[(\Pi_z(\boldsymbol{x})-1)\,\mathrm{Pf}\begin{pmatrix} 0 & 1 & {}^t\mathbf{0} \\ -1 & 0 & {}^t\mathbf{1} \\ 0 & -\mathbf{1} & A(\boldsymbol{y})\end{pmatrix} + \Pi_z(\boldsymbol{x})\,\mathrm{Pf}\begin{pmatrix} 0 & 0 & {}^tB_z(\boldsymbol{y}) \\ 0 & 0 & {}^t\mathbf{1} \\ -B_z(\boldsymbol{y}) & -\mathbf{1} & A(\boldsymbol{y})\end{pmatrix}\right]$$
$$= \Pi_z(\boldsymbol{x},\boldsymbol{y}) + \widehat{f}_0(z) - 1.$$

If p is even and n is odd, then similarly we have

$$\sum_{r\geq 0} P'_{(r)}(\boldsymbol{x}|\boldsymbol{y})\widehat{f}_r(z) = \widehat{f}_0(z) + \frac{1}{\Delta(\boldsymbol{y})}\left[(\Pi_z(\boldsymbol{x})-\widehat{f}_0(z))\,\mathrm{Pf}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & {}^t\mathbf{1} \\ 0 & -\mathbf{1} & A(\boldsymbol{y})\end{pmatrix}\right.$$
$$\left. + \Pi_z(\boldsymbol{x})\,\mathrm{Pf}\begin{pmatrix} 0 & 0 & {}^tB_z(\boldsymbol{y}) \\ 0 & 0 & {}^t\mathbf{1} \\ -B_z(\boldsymbol{y}) & -\mathbf{1} & A(\boldsymbol{y})\end{pmatrix}\right]$$
$$= \Pi_z(\boldsymbol{x},\boldsymbol{y}).$$

Next we prove (4.8). We put

$$G_{z,w}(\boldsymbol{x}) = \sum_{r,s\geq 0} P_{(r,s)}(\boldsymbol{x})\widehat{f}_r(z)\widehat{f}_s(w), \quad G^{++}_{z,w}(\boldsymbol{x}) = \sum_{r,s\geq 1} P_{(r,s)}(\boldsymbol{x})\widehat{f}_r(z)\widehat{f}_s(w).$$

By the definition of $P'_{(r,s)}(\boldsymbol{x})$ and the multilinearity of Pfaffians, we see that, if p is even, then

$$\sum_{r,s\geq 0} P'_{(r,s)}(\boldsymbol{x}|\boldsymbol{y})\widehat{f}_r(z)\widehat{f}_s(w) = \frac{1}{\Delta(\boldsymbol{y})}\,\mathrm{Pf}\begin{pmatrix} 0 & G_{z,w}(\boldsymbol{x}) & U_z(\boldsymbol{x}|\boldsymbol{y}) \\ -G_{z,w}(\boldsymbol{x}) & 0 & U_w(\boldsymbol{x}|\boldsymbol{y}) \\ -{}^tU_z(\boldsymbol{x}|\boldsymbol{y}) & -{}^tU_w(\boldsymbol{x}|\boldsymbol{y}) & A(\boldsymbol{y})\end{pmatrix},$$

and, if p is odd, then

$$\sum_{r,s\geq 0} P'_{(r,s)}(\boldsymbol{x}|\boldsymbol{y})\widehat{f}_r(z)\widehat{f}_s(w)$$
$$= \sum_{r,s\geq 1} P'_{(r,s)}(\boldsymbol{x}|\boldsymbol{y})\widehat{f}_r(z)\widehat{f}_s(w) + \sum_{r\geq 1} P'_{(r,0)}(\boldsymbol{x}|\boldsymbol{y})\widehat{f}_r(z)\widehat{f}_0(w) + \sum_{s\geq 1} P'_{(0,s)}(\boldsymbol{x}|\boldsymbol{y})\widehat{f}_0(z)\widehat{f}_s(w)$$

$$
= \frac{1}{\Delta(\boldsymbol{y})} \operatorname{Pf} \begin{pmatrix} 0 & G^{++}_{z,w}(\boldsymbol{x}) & F^{+}_{z}(\boldsymbol{x}) & U^{+}_{z}(\boldsymbol{x}|\boldsymbol{y}) \\ -G^{++}_{z,w}(\boldsymbol{x}) & 0 & F^{+}_{w}(\boldsymbol{x}) & U^{+}_{w}(\boldsymbol{x}|\boldsymbol{y}) \\ -F^{+}_{z}(\boldsymbol{x}) & -F^{+}_{w}(\boldsymbol{x}) & 0 & {}^{t}\boldsymbol{1} \\ -{}^{t}U^{+}_{z}(\boldsymbol{x}|\boldsymbol{y}) & -{}^{t}U^{+}_{w}(\boldsymbol{x}|\boldsymbol{y}) & -\boldsymbol{1} & A(\boldsymbol{y}) \end{pmatrix}
$$

$$
+ \widehat{f}_0(w) \cdot \frac{1}{\Delta(\boldsymbol{y})} \operatorname{Pf} \begin{pmatrix} 0 & U^{+}_{z}(\boldsymbol{x}|\boldsymbol{y}) \\ -{}^{t}U^{+}_{z}(\boldsymbol{x}|\boldsymbol{y}) & A(\boldsymbol{y}) \end{pmatrix}
$$

$$
- \widehat{f}_0(z) \cdot \frac{1}{\Delta(\boldsymbol{y})} \operatorname{Pf} \begin{pmatrix} 0 & U^{+}_{w}(\boldsymbol{x}|\boldsymbol{y}) \\ -{}^{t}U^{+}_{w}(\boldsymbol{x}|\boldsymbol{y}) & A(\boldsymbol{y}) \end{pmatrix}.
$$

Here we note

$$
G_{z,w}(\boldsymbol{x}) = G^{++}_{z,w}(\boldsymbol{x}) + \widehat{f}_0(w) F^{+}_{z}(\boldsymbol{x}) - \widehat{f}_0(z) F^{+}_{w}(\boldsymbol{x}),
$$

and

$$
U_z(\boldsymbol{x}|\boldsymbol{y}) = U^{+}_{z}(\boldsymbol{x}|\boldsymbol{y}) + \widehat{f}_0(z)\,{}^{t}\boldsymbol{1}, \quad U_w(\boldsymbol{x}|\boldsymbol{y}) = U^{+}_{w}(\boldsymbol{x}|\boldsymbol{y}) + \widehat{f}_0(w)\,{}^{t}\boldsymbol{1}.
$$

By adding the 3rd row/column multiplied by $\widehat{f}_0(z)$ (respectively $\widehat{f}_0(w)$) to the 1st (respectively 2nd) row/column in the first Pfaffian, we see that

$$
\operatorname{Pf} \begin{pmatrix} 0 & G^{++}_{z,w}(\boldsymbol{x}) & F^{+}_{z}(\boldsymbol{x}) & U^{+}_{z}(\boldsymbol{x}|\boldsymbol{y}) \\ -G^{++}_{z,w}(\boldsymbol{x}) & 0 & F^{+}_{w}(\boldsymbol{x}) & U^{+}_{w}(\boldsymbol{x}|\boldsymbol{y}) \\ -F^{+}_{z}(\boldsymbol{x}) & -F^{+}_{w}(\boldsymbol{x}) & 0 & {}^{t}\boldsymbol{1} \\ -{}^{t}U^{+}_{z}(\boldsymbol{x}|\boldsymbol{y}) & -{}^{t}U^{+}_{w}(\boldsymbol{x}|\boldsymbol{y}) & -\boldsymbol{1} & A(\boldsymbol{y}) \end{pmatrix}
= \operatorname{Pf} \begin{pmatrix} 0 & G_{z,w}(\boldsymbol{x}) & F^{+}_{z}(\boldsymbol{x}) & U_{z}(\boldsymbol{x}|\boldsymbol{y}) \\ -G_{z,w}(\boldsymbol{x}) & 0 & F^{+}_{w}(\boldsymbol{x}) & U_{w}(\boldsymbol{x}|\boldsymbol{y}) \\ -F^{+}_{z}(\boldsymbol{x}) & -F^{+}_{w}(\boldsymbol{x}) & 0 & {}^{t}\boldsymbol{1} \\ -{}^{t}U_{z}(\boldsymbol{x}|\boldsymbol{y}) & -{}^{t}U_{w}(\boldsymbol{x}|\boldsymbol{y}) & -\boldsymbol{1} & A(\boldsymbol{y}) \end{pmatrix}.
$$

By Proposition 3.2, we have

$$
G_{z,w}(\boldsymbol{x}) = \begin{cases} \dfrac{z-w}{z+w}\bigl(\Pi_z(\boldsymbol{x})\Pi_w(\boldsymbol{x}) - 1\bigr), & \text{if } n \text{ is even}, \\ \dfrac{z-w}{z+w}\bigl(\Pi_z(\boldsymbol{x})\Pi_w(\boldsymbol{x}) - 1\bigr) + (\widehat{f}_0(w) - 1)\Pi_z(\boldsymbol{x}) - (\widehat{f}_0(z) - 1)\Pi_w(\boldsymbol{x}), & \text{if } n \text{ is odd}. \end{cases} \tag{4.11}
$$

Now we distinguish four cases according to the parity of p and n, and in each case we evaluate Pfaffians by using (4.9), (4.10) and (4.11) together with Propositions A.1 and A.2. The rest of the proof is done by straightforward computation, so we omit it. □

Now we are in the position to give a proof of Theorem 4.2.

Proof of Theorem 4.2. We put $l = l(\lambda)$.

First we consider the case where $l \equiv p \bmod 2$. Then we apply the Pfaffian analogue of the Cauchy–Binet formula (A.15) to the matrices given by

$$
A = S_\lambda(\boldsymbol{x}), \quad S = \bigl(R_{\lambda_i/k}(\boldsymbol{x})\bigr)_{1 \le i \le l, k \ge 0},
$$
$$
B = A(\boldsymbol{y}), \quad T = \bigl(f_k(x_i)\bigr)_{1 \le i \le p, k \ge 0}.
$$

Strict partitions μ are in bijection with subsets of $\mathbb{N}$ satisfying $\# I = p \bmod 2$ via the correspondence $\mu \mapsto I = \{\mu_1, \dots, \mu_{l(\mu)}\}$ or $\{\mu_1, \dots, \mu_{l(\mu)}, 0\}$, and we see that

$$\operatorname{Pf} \begin{pmatrix} S_\lambda(\boldsymbol{x}) & S([l]; I) \\ -{}^tS([l]; I) & O \end{pmatrix} = P_{\lambda/\mu,p}(\boldsymbol{x}),$$

$$\operatorname{Pf} \begin{pmatrix} A(\boldsymbol{y}) & T([p]; I) \\ -{}^tT([p]; I) & O \end{pmatrix} = (-1)^{\binom{\# I}{2}} \Delta(\boldsymbol{y}) P_\mu(\boldsymbol{y}).$$

By (4.1), the (i,j) entry of $S{}^tT$ is equal to

$$\sum_{k \ge 0} R_{\lambda_i/k}(\boldsymbol{x}) f_k(y_j) = P_{(\lambda_i)}(\boldsymbol{x}, y_j).$$

Hence, by applying (A.15), we have

$$\sum_\mu P_{\lambda/\mu,p}(\boldsymbol{x}) P_\mu(\boldsymbol{y}) = \frac{1}{\Delta(\boldsymbol{y})} \operatorname{Pf} \begin{pmatrix} S_\lambda(\boldsymbol{x}) & N_\lambda(\boldsymbol{x}, \boldsymbol{y}) \\ -{}^tN_\lambda(\boldsymbol{x}, \boldsymbol{y}) & A(\boldsymbol{y}) \end{pmatrix}.$$

If $l \not\equiv p \bmod 2$, then we apply (A.15) to the matrices given by

$$A = S_{\lambda^0}(\boldsymbol{x}), \quad S = \left(R_{\lambda_i/k}(\boldsymbol{x}) \right)_{1 \le i \le l+1, k \ge 0},$$
$$B = A(\boldsymbol{y}), \quad T = \left(f_k(y_i) \right)_{1 \le i \le p, k \ge 0}.$$

Then, by an argument similar to above, we obtain

$$\sum_\mu P_{\lambda/\mu,p}(\boldsymbol{x}) P_\mu(\boldsymbol{y}) = \frac{1}{\Delta(\boldsymbol{y})} \operatorname{Pf} \begin{pmatrix} S_{\lambda^0}(\boldsymbol{x}) & N_{\lambda^0}(\boldsymbol{x}, \boldsymbol{y}) \\ -{}^tN_{\lambda^0}(\boldsymbol{x}, \boldsymbol{y}) & A(\boldsymbol{y}) \end{pmatrix}.$$

Now the proof of Theorem 4.2 can be completed by using Proposition 4.7. □

5. Pieri-type rule

In this section, we give a Pieri-type rule for the product of any generalized P-function $P^{\mathcal{F}}_\lambda(\boldsymbol{x})$ with a Schur Q-function $Q_{(r)}(\boldsymbol{x})$ corresponding to a one-row partition.

5.1. **The ring of Schur P- and Q-functions.** Let $\Gamma^{(n)}$ be the subring of the ring of symmetric polynomials $\Lambda^{(n)} = K[x_1, \dots, x_n]^{S_n}$ defined by

$$\Gamma^{(n)} = \{ f \in K[x_1, \dots, x_n]^{S_n} : f(t, -t, x_3, \dots, x_n) \text{ is independent of } t \}.$$

Then it is known that Schur P-functions $\{ P_\lambda(\boldsymbol{x}) : \lambda \in \mathcal{S}^{(n)} \}$ form a basis of $\Gamma^{(n)}$, where $\mathcal{S}^{(n)}$ is the set of all strict partitions of length $\le n$ (see [20, Theorem 2.11]).

We give a relation between two families of generalized P-functions associated with different admissible sequences.

Proposition 5.1. *Let $\mathcal{F} = \{f_d\}_{d=0}^\infty$ and $\mathcal{G} = \{g_d\}_{d=0}^\infty$ be two admissible sequences. For a strict partition λ of length $l \le n$, the generalized P-function $P^{\mathcal{F}}_\lambda(\boldsymbol{x})$ associated with $\mathcal{F}$ can be written as a K-linear combination of the generalized P-functions $P^{\mathcal{G}}_\mu(\boldsymbol{x})$ associated with $\mathcal{G}$ in the following form:*

$$P^{\mathcal{F}}_\lambda(\boldsymbol{x}) = a_{\lambda,\lambda} P^{\mathcal{G}}_\lambda(\boldsymbol{x}) + \sum_{\mu \subsetneq \lambda} a_{\lambda,\mu} P^{\mathcal{G}}_\mu(\boldsymbol{x}),$$

where $a_{\lambda,\lambda} \neq 0$, and μ runs over all strict partitions satisfying $\mu \subsetneq \lambda$.

Proof. We write $f_k = \sum_{i=0}^{\infty} a_{k,l} g_l$ for $k \geq 0$. Then by the assumption (1.2) we have $a_{k,l} = 0$ for $k < l$, $a_{k,k} \neq 0$, and $a_{0,0} = 1$.

If $n + l$ is even, then by using the multilinear and alternating property of Pfaffians we have

$$\begin{aligned} P_\lambda^{\mathcal{F}}(\boldsymbol{x}) &= \frac{1}{\Delta(\boldsymbol{x})} \sum_{\alpha \in \mathbb{N}^l} a_{\lambda_1,\alpha_1} \dots a_{\lambda_l,\alpha_l} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & V_\alpha^{\mathcal{G}}(\boldsymbol{x}) \\ -{}^t V_\alpha^{\mathcal{G}}(\boldsymbol{x}) & O \end{pmatrix} \\ &= \sum_{\mu_1 > \dots > \mu_l \geq 0} \det \left(a_{\lambda_i,\mu_j} \right)_{1 \leq i,j \leq l} P_\mu^{\mathcal{G}}(\boldsymbol{x}), \end{aligned}$$

where μ runs over all strict partitions of length $l-1$ or l. Similarly, if $n+l$ is odd, then we have

$$P_\lambda^{\mathcal{F}}(\boldsymbol{x}) = \frac{1}{\Delta(\boldsymbol{x})} \sum_{\alpha \in \mathbb{N}^{l+1}} a_{\lambda_1,\alpha_1} \dots a_{\lambda_l,\alpha_l} a_{0,\alpha_{l+1}} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & V_\alpha^{\mathcal{G}}(\boldsymbol{x}) \\ -{}^t V_\alpha^{\mathcal{G}}(\boldsymbol{x}) & O \end{pmatrix}.$$

Since $a_{0,l} = 0$ for $l > 0$, we see that

$$P_\lambda^{\mathcal{F}}(\boldsymbol{x}) = \sum_{\mu_1 > \dots > \mu_l > 0} \det \left(a_{\lambda_i,\mu_j} \right)_{1 \leq i,j \leq l} P_\mu^{\mathcal{G}}(\boldsymbol{x}),$$

where μ runs over all strict partitions of length l. For a strict partition μ of length $l-1$ or l, we put

$$a_{\lambda,\mu} = \det \left(a_{\lambda_i,\mu_j} \right)_{1 \leq i,j \leq l},$$

where $\mu_l = 0$ if $l(\mu) = l-1$. We prove that $a_{\lambda,\lambda} \neq 0$ and $a_{\lambda,\mu} = 0$ unless $\lambda \supset \mu$. Since the matrix $\left(a_{\lambda_i,\lambda_j} \right)_{1 \leq i,j \leq l}$ is upper-triangular, we have $a_{\lambda,\lambda} = \prod_{i=1}^{l} a_{\lambda_i,\lambda_i} \neq 0$. If there is an index k such that $\lambda_k < \mu_k$, then we have $\lambda_i \leq \lambda_k < \mu_k \leq \mu_j$ and $a_{\lambda_i,\mu_j} = 0$ for $i \geq k$ and $j \leq k$ and thus $a_{\lambda,\mu} = 0$. Hence we obtain the desired result. □

Corollary 5.2. *The generalized P-functions $\{P_\lambda^{\mathcal{F}}(\boldsymbol{x}) : \lambda \in \mathcal{S}^{(n)}\}$ associated with a fixed sequence $\mathcal{F}$ form a basis of $\Gamma^{(n)}$.*

5.2. **Pieri-type rule.** Let $q_r(\boldsymbol{x}) = Q_{(r)}(\boldsymbol{x})$ be the Schur Q-function corresponding to a one-row partition (r), and set $q_0(\boldsymbol{x}) = 1$. It is known (see [12, III.(8.5)]) that $\Gamma^{(n)}$ is generated by $q_r(\boldsymbol{x})$ $(r \geq 1)$. Thus the algebra structure of $\Gamma^{(n)}$ is governed by the multiplication rule for $q_r(\boldsymbol{x})$s.

Theorem 5.3. *Let $\mathcal{F} = \{f_d\}_{d=0}^{\infty}$ be an admissible sequence. We define formal power series $b_r^s(z)$ by the relation*

$$f_r(u) \cdot \frac{1+uz}{1-uz} = \sum_{s=0}^{\infty} b_r^s(z) f_s(u). \tag{5.1}$$

We define the modified Pieri coefficients $c_{\mu,r}^{\lambda}$ by the relation

$$P_\mu^{\mathcal{F}}(\boldsymbol{x}) \cdot q_r(\boldsymbol{x}) = \sum_{\lambda} c_{\mu,r}^{\lambda} P_\lambda^{\mathcal{F}}(\boldsymbol{x}), \tag{5.2}$$

where the summation is taken over all strict partitions. Then the generating function of modified Pieri coefficients is given by

$$\sum_{r=0}^{\infty} c^{\lambda}_{\mu,r} z^r = \begin{cases} \det B^{\lambda}_{\mu}, & \text{if } n + l(\mu) \text{ is even and } l(\lambda) = l(\mu), \\ \det B^{\lambda^0}_{\mu}, & \text{if } n + l(\mu) \text{ is even and } l(\lambda) = l(\mu) - 1, \\ \det B^{\lambda}_{\mu^0}, & \text{if } n + l(\mu) \text{ is even and } l(\lambda) = l(\mu) + 1, \\ \det B^{\lambda^0}_{\mu^0}, & \text{if } n + l(\mu) \text{ is odd and } l(\lambda) = l(\mu), \\ \det B^{\lambda}_{\mu^0}, & \text{if } n + l(\mu) \text{ is odd and } l(\lambda) = l(\mu) + 1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.3}$$

where $B^{\alpha}_{\beta} = \left(b^{\alpha_i}_{\beta_j}(z) \right)_{1 \le i,j \le r}$ *for* $\alpha = (\alpha_1, \dots, \alpha_r)$ *and* $\beta = (\beta_1, \dots, \beta_r)$.

If we put

$$c^{\lambda}_{\mu}(z) = \sum_{r=0}^{\infty} c^{\lambda}_{\mu,r} z^r,$$

then it follows from (3.3) and (5.2) that

$$P^{\mathcal{F}}_{\mu}(z) \cdot \Pi_z(\boldsymbol{x}) = \sum_{\lambda} c^{\lambda}_{\mu}(z) P^{\mathcal{F}}_{\lambda}(z). \tag{5.4}$$

In order to prove the above theorem, we derive a Nimmo-type expression for the product $P^{\mathcal{F}}_{\mu}(\boldsymbol{x}) \cdot \Pi_z(\boldsymbol{x})$ in terms of a Pfaffian.

Lemma 5.4. *For a sequence* $\alpha = (\alpha_1, \dots, \alpha_r)$ *of nonnegative integers, we put*

$$W_{\alpha}(\boldsymbol{x}) = \left(f_{\alpha_j}(x_i) \cdot \frac{1 + x_i z}{1 - x_i z} \right)_{1 \le i \le n, 1 \le j \le r}.$$

Then, for a strict partition μ *of length* m, *we have*

$$P^{\mathcal{F}}_{\mu}(\boldsymbol{x}) \cdot \Pi_z(\boldsymbol{x}) = \begin{cases} \dfrac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & W_{\mu}(\boldsymbol{x}) & W_{(0)}(\boldsymbol{x}) & \mathbf{1} \\ -{}^t W_{\mu}(\boldsymbol{x}) & 0 & 0 & 0 \\ -{}^t W_{(0)}(\boldsymbol{x}) & 0 & 0 & 1 \\ -{}^t \mathbf{1} & 0 & -1 & 0 \end{pmatrix}, & \text{if } n + m \text{ is even}, \\ \dfrac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & W_{\mu^0}(\boldsymbol{x}) \\ -{}^t W_{\mu^0}(\boldsymbol{x}) & 0 \end{pmatrix}, & \text{if } n + m \text{ is odd}. \end{cases}$$

Proof. The method of the proof is the same as in the proof of Theorem 2.3, so we will only give a sketch of the proof.

Since $\Pi_z(\boldsymbol{x}) = \prod_{i=1}^{n} (1 + x_i z)/(1 - x_i z)$ is invariant under the symmetric group S_n, it follows from Corollary 2.5 that

$$P_{\mu}(\boldsymbol{x}) \cdot \Pi_z(\boldsymbol{x}) = \sum_{w \in S_n / S_{n-m}} w \left(\prod_{i=1}^{m} f_{\mu_i}(x_i) \prod_{i=1}^{n} \frac{1 + x_i z}{1 - x_i z} \prod_{\substack{1 \le i < j \le n \\ i \le m}} \frac{x_i + x_j}{x_i - x_j} \right)$$

$$= \frac{(-1)^{\binom{n}{2}+\binom{n-m}{2}}}{\Delta_n(\boldsymbol{x})} \times \sum_{w \in S_n/S_{n-m}} \operatorname{sgn}(w) w \left(\prod_{i=1}^{m} \left(f_{\mu_i}(x_i) \frac{1+x_i z}{1-x_i z} \right) \prod_{i=m+1}^{n} \frac{1+x_i z}{1-x_i z} \prod_{m+1 \le i < j \le n} \frac{x_j - x_i}{x_j + x_i} \right).$$

By using (A.5) with $p = 2$, $y_1 = z$, $y_2 = 0$, or $p = 1$, $y_1 = z$, we obtain

$$P_\mu(\boldsymbol{x}) \cdot \Pi_z(\boldsymbol{x}) = \frac{(-1)^{\binom{n}{2}+\binom{n-m}{2}}}{\Delta(\boldsymbol{x})} \sum_{w \in S_n/S_m \times S_{n-m}} \operatorname{sgn}(w) w \Big(\det W_\mu(\boldsymbol{x}_{[m]}) \operatorname{Pf} X \Big),$$

where the skew-symmetric matrix X is given by

$$X = \begin{cases} \begin{pmatrix} A(\boldsymbol{x}_{[n]\setminus[m]}) & W_{(0)}(\boldsymbol{x}_{[n]\setminus[m]}) & \mathbf{1} \\ -{}^t W_{(0)}(\boldsymbol{x}_{[n]\setminus[m]}) & 0 & 1 \\ -{}^t \mathbf{1} & -1 & 0 \end{pmatrix}, & \text{if } n+m \text{ is even}, \\ \begin{pmatrix} A(\boldsymbol{x}_{[n]\setminus[m]}) & W_{(0)}(\boldsymbol{x}_{[n]\setminus[m]}) \\ -{}^t W_{(0)}(\boldsymbol{x}_{[n]\setminus[m]}) & 0 \end{pmatrix}, & \text{if } n+m \text{ is odd}. \end{cases}$$

Now we can use a Pfaffian analogue of Laplace expansion (Proposition A.5) to complete the proof. □

Proof of Theorem 5.3. The argument is similar to that in the proof of Proposition 5.1.

First we consider the case where $n+m$ is even. In this case, by using the multilinearity and the expansion along the last row/column, we have

$$\operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & W_\mu(\boldsymbol{x}) & W_{(0)}(\boldsymbol{x}) & \mathbf{1} \\ -{}^t W_\mu(\boldsymbol{x}) & O & O & O \\ -{}^t W_{(0)}(\boldsymbol{x}) & O & 0 & 1 \\ -{}^t \mathbf{1} & O & -1 & 0 \end{pmatrix}$$
$$= \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & W_{\mu^0}(\boldsymbol{x}) & \mathbf{1} \\ -{}^t W_{\mu^0}(\boldsymbol{x}) & O & O \\ -{}^t \mathbf{1} & O & 0 \end{pmatrix} + (-1)^{n+m} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & W_\mu(\boldsymbol{x}) \\ -{}^t W_\mu(\boldsymbol{x}) & 0 \end{pmatrix}.$$

Since $W_{(r)}(\boldsymbol{x}) = \sum_{s=0}^{\infty} b_r^s(z) V_{(s)}(\boldsymbol{x})$ by (5.1), we can use the multilinear and alternating property to obtain

$$\operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & W_{\mu^0}(\boldsymbol{x}) & \mathbf{1} \\ -{}^t W_{\mu^0}(\boldsymbol{x}) & O & O \\ -{}^t \mathbf{1} & O & 0 \end{pmatrix}$$
$$= \sum_{\alpha \in \mathbb{N}^{m+1}} \prod_{i=1}^{m+1} b_{\mu_i}^{\alpha_i}(z) \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & V_\alpha(\boldsymbol{x}) & \mathbf{1} \\ -{}^t V_\alpha(\boldsymbol{x}) & O & O \\ -{}^t \mathbf{1} & O & 0 \end{pmatrix}$$
$$= \sum_{\lambda_1 > \cdots > \lambda_{m+1} \ge 0} \det \left(b_{\mu_j}^{\lambda_i}(z) \right)_{1 \le i,j \le m+1} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & V_\lambda(\boldsymbol{x}) & \mathbf{1} \\ -{}^t V_\lambda(\boldsymbol{x}) & O & O \\ -{}^t \mathbf{1} & O & 0 \end{pmatrix}.$$

If $\lambda_{m+1} = 0$, then the last column of $V_\lambda(\boldsymbol{x})$ coincides with $\mathbf{1}$, so the corresponding Pfaffian vanishes. Hence we have

$$\frac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & W_{\mu^0}(\boldsymbol{x}) & \mathbf{1} \\ -{}^t W_{\mu^0}(\boldsymbol{x}) & O & O \\ -{}^t\mathbf{1} & O & 0 \end{pmatrix} = \sum_{\lambda} \det B^{\lambda}_{\mu^0} P_\lambda(\boldsymbol{x}),$$

where λ runs over all strict partitions of length $m+1$. Similarly we have

$$\frac{1}{\Delta(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & W_{\mu}(\boldsymbol{x}) \\ -{}^t W_{\mu}(\boldsymbol{x}) & 0 \end{pmatrix} = \sum_{\lambda} \det B^{\lambda^*}_{\mu} P_\lambda(\boldsymbol{x}),$$

where λ runs over all strict partitions of length $m-1$ or m, and $\lambda^* = \lambda^0$ or λ.

The case where $n+m$ is odd can be treated in a similar manner. □

If $\mathcal{F}$ is constant-term free, then we have a simpler formula.

Corollary 5.5. *If $\mathcal{F}$ is constant-term free, then we have*

$$c^{\lambda}_{\mu}(z) = \begin{cases} \det B^{\lambda}_{\mu}, & \textit{if } l(\lambda) = l(\mu), \\ \det B^{\lambda}_{\mu^0}, & \textit{if } l(\lambda) = l(\mu)+1, \\ 0, & \textit{otherwise,} \end{cases}$$

under the same notation as in Theorem 5.3.

Proof. By substituting $t = 0$ in (5.1) and using the assumption $f_d(0) = \delta_{d,0}$, we see that $b^0_r(z) = \delta_{r,0}$. Hence we have

$$\det B^{\lambda^0}_{\mu^0} = \det B^{\lambda}_{\mu}, \quad \det B^{\lambda^0}_{\mu} = 0,$$

and we obtain the corollary. □

6. Applications to factorial P- and Q-functions

In this section, we focus on Ivanov's factorial P- and Q-functions.

6.1. **Factorial P- and Q-functions.**

Recall the definition of Ivanov's factorial P- and Q-functions. Let $\boldsymbol{a} = (a_0, a_1, \dots)$ be parameters, called *factorial parameters*. We define the factorial monomial $(u|\boldsymbol{a})^d$ by putting

$$(u|\boldsymbol{a})^d = \prod_{i=0}^{d-1} (u - a_i).$$

Then the *factorial P-function* $P_\lambda(\boldsymbol{x}|\boldsymbol{a})$ is defined to be the generalized P-function $P^{\mathcal{F}}_\lambda(\boldsymbol{x})$ associated with $\mathcal{F} = \{(u|\boldsymbol{a})^d\}_{d=0}^{\infty}$ (see Definition 1.2). The *factorial Q-function* $Q_\lambda(\boldsymbol{x}|\boldsymbol{a})$ is defined by

$$Q_\lambda(\boldsymbol{x}|\boldsymbol{a}) = 2^{l(\lambda)} P_\lambda(\boldsymbol{x}|\boldsymbol{a}).$$

The factorial Q-function $Q_\lambda(\boldsymbol{x}|\boldsymbol{a})$ is also the generalized P-function $P^{\mathcal{F}'}_\lambda(\boldsymbol{x})$ associated with the sequence $\mathcal{F}' = \{f'_d\}_{d=0}^{\infty}$ given by

$$f'_d(u) = \begin{cases} 1, & \text{if } d = 0, \\ 2(u|\boldsymbol{a})^d, & \text{if } d \geq 1. \end{cases}$$

Note that the sequence $\mathcal{F} = \{(u|\boldsymbol{a})^d\}_{d=0}^{\infty}$ is constant-term free if and only if $a_0 = 0$.

Since the factorial P- and Q-functions are special cases of our generalized P-functions, we can recover some formulas in [6] and [7] from the results of Section 2. For example, we recover [7, Theorem 9.1] without assumption $a_0 = 0$:

$$Q_\lambda(\boldsymbol{x}|\boldsymbol{a}) = \operatorname{Pf}\left(Q_{(\lambda_i,\lambda_j)}(\boldsymbol{x}|\boldsymbol{a})\right)_{1 \le i,j \le r},$$

where $r = l(\lambda)$ if $l(\lambda)$ is even and $l(\lambda)+1$ if $l(\lambda)$ is odd, and $\lambda_{l(\lambda)+1} = 0$. (We use the convention (2.8).)

Next we compute explicitly the dual of $\mathcal{F} = \{(t|\boldsymbol{a})^d\}_{d=0}^{\infty}$ introduced in Section 3.

Lemma 6.1. *Let $\widehat{\mathcal{F}} = \{\widehat{f}_d\}_{d=0}^{\infty}$ be the dual of $\mathcal{F} = \{(u|\boldsymbol{a})^d\}_{d=0}^{\infty}$. Then we have*

$$\widehat{f}_d(v) = \begin{cases} \dfrac{1+a_0 v}{1-a_0 v}, & \text{if } d = 0, \\ \dfrac{2v^d}{\prod_{i=0}^{d}(1-a_i v)}, & \text{if } d \ge 1. \end{cases}$$

Proof. Put $f_d(u) = (u|\boldsymbol{a})^d$. The sequence $\{\widehat{f}_d\}_{d=0}^{\infty}$ is uniquely determined by the relation (3.1). Since $f_0 = 1$, we see that $\widehat{f}_0(v)$ is determined by substituting $u = a_0$ in (3.1), and we obtain $\widehat{f}_0(v) = (1+a_0 v)/(1-a_0 v)$. Let $r > 0$. Since $f_k(a_r) = 0$ for $k > r$ and $f_r(a_r) \neq 0$, we see that $\widehat{f}_r(t)$ is determined inductively by the relation

$$\sum_{k=0}^{r} \widehat{f}_k(v) f_k(a_r) = \frac{1+a_r v}{1-a_r v}.$$

Hence it is enough to show

$$\frac{1+a_r v}{1-a_r v} = \frac{1+a_0 v}{1-a_0 v} + \sum_{k=1}^{r} \frac{2v^r}{\prod_{i=0}^{k}(1-a_i v)} \prod_{j=0}^{k-1}(a_r - a_j). \tag{6.1}$$

By using

$$\frac{1+a_r v}{1-a_r v} - \frac{1-a_0 v}{1-a_0 v} = \frac{2v(a_r-a_0)}{(1-a_0 v)(1-a_r v)},$$

and cancelling the common factor $2z(a_r - a_0)/(1-a_0 z)$, we see that (6.1) is equivalent to

$$\frac{1}{1-a_r v} = \sum_{k=1}^{r} \frac{v^{k-1}}{\prod_{i=1}^{k}(1-a_i v)} \prod_{j=1}^{k-1}(a_r - a_j). \tag{6.2}$$

We proceed by induction on r to prove (6.2). The case $r = 1$ is trivial. If $r > 1$, then by the induction hypothesis with factorial parameters $(a_2, \dots, a_r)$, we have

$$\frac{1}{1-a_r v} = \sum_{k=2}^{r} \frac{v^{k-2}}{\prod_{i=2}^{k}(1-a_i v)} \prod_{j=2}^{k-1}(a_r - a_j).$$

Hence we have

$$\sum_{k=1}^{r} \frac{v^{k-1}}{\prod_{i=1}^{k}(1-a_i v)} \prod_{j=1}^{k-1}(a_r - a_i) = \frac{1}{1-a_1 v} + \frac{v(a_r - a_1)}{1-a_1 v} \sum_{k=2}^{r} \frac{v^{k-2}}{\prod_{i=2}^{k}(1-a_i v)} \prod_{j=2}^{k-1}(a_r - a_j)$$

$$= \frac{1}{1-a_1 v} + \frac{v(a_r - a_1)}{1-a_1 v} \cdot \frac{1}{1-a_r v}$$
$$= \frac{1}{1-a_r v}.$$

This completes the proof. $\square$

We denote by $\widehat{P}_\lambda(\boldsymbol{x}|\boldsymbol{a})$ the dual P-function $\widehat{P}^{\mathcal{F}}_\lambda(\boldsymbol{x})$ associated with $\mathcal{F} = \{(u|\boldsymbol{a})^d\}_{d=0}^\infty$. If the first parameter a_0 is equal to 0, then we can recover Korotkikh's dual P-function (see [8, Definition 2] and (3.8)) given by

$$\widehat{P}_\lambda(\boldsymbol{x}|\boldsymbol{a}) = \frac{2^l}{(n-l)!} \sum_{w \in S_n} w \left(\prod_{i=1}^{l} \frac{x_i^{\lambda_i}}{\prod_{k=0}^{\lambda_i}(1-a_k x_i)} \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{x_i + x_j}{x_i - x_j} \right)$$

and the Cauchy-type identity (see [8, Theorem 8])

$$\sum_\lambda P_\lambda(\boldsymbol{x}|\boldsymbol{a}) \widehat{P}_\lambda(\boldsymbol{x}|\boldsymbol{a}) = \prod_{i,j=1}^{n} \frac{1+x_i y_j}{1-x_i y_j}.$$

6.2. **Factorial skew P-functions.** For two strict partitions λ and μ and a positive integer p, we denote by $P_{\lambda/\mu,p}(\boldsymbol{x}|\boldsymbol{a})$ the generalized skew P-functions $P^{\mathcal{F}}_{\lambda/\mu,p}(\boldsymbol{x})$ associated with $\mathcal{F} = \{(u|\boldsymbol{a})^d\}_{d=0}^\infty$, and call it the *factorial skew P-function* (see Definition 4.1). Since $(u|\boldsymbol{a})^r$ depends only on $a_0, a_1, \dots, a_{r-1}$, it follows from the definition (1.4) that $P_{(r)}(\boldsymbol{x}|\boldsymbol{a})$ also depends only on $a_0, a_1, \dots, a_{r-1}$. So we write $P_{(r)}(\boldsymbol{x}|a_0, a_1, \dots, a_{r-1})$ for $P_{(r)}(\boldsymbol{x}|\boldsymbol{a})$.

Proposition 6.2. *We define $R_{r/k}(\boldsymbol{x}|\boldsymbol{a})$ by the relation*

$$P_{(r)}(\boldsymbol{x}, y|\boldsymbol{a}) = \sum_{k=0}^{\infty} R_{r/k}(\boldsymbol{x}|\boldsymbol{a})(y|\boldsymbol{a})^k.$$

Then we have we have

$$R_{r/k}(\boldsymbol{x}|\boldsymbol{a}) = \begin{cases} P_{(r)}(\boldsymbol{x}|-a_0, a_1, \dots, a_{r-1}), & \text{if } k = 0, \\ P_{(r-k)}(\boldsymbol{x}|0, a_{k+1}, \dots, a_{r-1}), & \text{if } 1 \le k \le r-1, \\ 1, & \text{if } k = r, \\ 0, & \text{if } k > r. \end{cases} \tag{6.3}$$

In particular, if $a_0 = 0$, then we have

$$R_{r/k}(\boldsymbol{x}|\boldsymbol{a}) = P_{(r-k)}(\boldsymbol{x}|0, a_{k+1}, \dots, a_{r-1}). \tag{6.4}$$

If $\boldsymbol{x} = (x)$ consists of a single variable, then $P_{(r)}(x|\boldsymbol{a}) = (x|\boldsymbol{a})^r$. Hence we obtain

Corollary 6.3. *If $\boldsymbol{x} = (x)$ consists of a single variable, then we have*

$$R_{r/k}(x|\boldsymbol{a}) = \begin{cases} (x+a_0)\prod_{i=1}^{r-1}(x-a_i), & \text{if } k = 0, \\ x\prod_{i=k+1}^{r-1}(x-a_i), & \text{if } 1 \le k \le r-1, \\ 1, & \text{if } k = r, \\ 0, & \text{if } k > r. \end{cases}$$

In the proof of Proposition 6.2, we need the following relations for elementary symmetric polynomials $e_r(\boldsymbol{x})$.

Lemma 6.4. (1) *If $k > 0$ and $l > 0$, then we have*

$$\sum_{m=1}^{r-1} e_{m-k}(x_1, \dots, x_m) e_{r-m-l}(x_{m+2}, \dots, x_r) = e_{r-k-l}(x_1, \dots, x_r).$$

(2) *If $l > 0$, then we have*

$$2\sum_{m=1}^{r-1} e_m(x_1, \dots, x_m) e_{r-m-l}(x_{m+2}, \dots, x_r) + e_{r-l}(-x_1, x_2, \dots, x_r) = e_{r-l}(x_1, x_2, \dots, x_r).$$

Proof. For $1 \le a < b \le r$, we put $[a, b] = \{a, a+1, \dots, b\}$ and denote by $\binom{[a,b]}{p}$ the set of p-element subsets of $[a, b]$. If we put $x_I = \prod_{i \in I} x_i$ for $I \subset [a, b]$, then we have

$$e_p(x_a, \dots, x_b) = \sum_{I \in \binom{[a,b]}{p}} x_I.$$

(1) We define a map

$$\varphi : \bigsqcup_{m=1}^{r-1} \binom{[1, m]}{m-k} \times \binom{[m+2, r]}{r-m-l} \to \binom{[1, r]}{r-k-l}$$

by $\varphi(I, J) = I \sqcup J$. Given $K \in \binom{[1,r]}{r-k-l}$, let $m+1$ be the $(k+1)$st smallest element in the $(k+l)$-element subset $[r] \setminus K$ and put $I = K \cap [1, m]$ and $J = K \cap [m+2, r]$. Then the correspondence $K \mapsto (I, J)$ gives the inverse map of φ, and we obtain the desired identity.

(2) Since we have

$$e_{r-l}(-x_1, x_2, \dots, x_r) = -\sum_{\substack{I \in \binom{[1,r]}{r-l} \\ 1 \in I}} x_I + \sum_{\substack{I \in \binom{[1,r]}{r-l} \\ 1 \notin I}} x_I,$$

it is enough to show that

$$\sum_{m=1}^{r-1} e_m(x_1, \dots, x_m) e_{r-m-l}(x_{m+2}, \dots, x_r) = \sum_{\substack{K \in \binom{[1,r]}{r-l} \\ 1 \in K}} x_K.$$

We define a map

$$\psi : \bigsqcup_{m=1}^{r-1} \binom{[m+2, r]}{r-m-l} \to \left\{ K \in \binom{[1, r]}{r-l} : 1 \in K \right\}$$

by $\psi(J) = [1, m] \sqcup J$. Given $K \in \binom{[1,r]}{r-l}$ with $1 \in K$, let m be the maximum integer m satisfying $[1, m] \subset K$, and put $J = K \setminus [1, m]$. Then the correspondence $K \mapsto J$ gives the inverse map of ψ, and we obtain the desired identity. $\square$

Now we prove Proposition 6.2.

Proof of Proposition 6.2. We need to show that

$$\begin{aligned} P_{(r)}(\boldsymbol{x}, y|a_0, \dots, a_{r-1}) = P_{(r)}(\boldsymbol{x}| - a_0, a_1, \dots, a_{r-1}) \\ + \sum_{k=1}^{r-1} P_{(r-k)}(\boldsymbol{x}|0, a_{k+1}, \dots, a_{r-1})(y|a_0, \dots, a_{k-1})^k \\ + (y|a_0, \dots, a_{r-1})^r. \end{aligned} \tag{6.5}$$

We compare the coefficients of $P_{(k)}(\boldsymbol{x})y^l$ in the expansions of both sides, where $P_{(k)}(\boldsymbol{x})$ is the Schur P-function. We denote by $a_{k,l}$ and $b_{k,l}$ the coefficients of $P_{(k)}(\boldsymbol{x})y^l$ on the left- and right-hand sides, respectively.

Plugging $(u|\boldsymbol{a})^r = \sum_{m=0}^{r}(-1)^{r-m}e_{r-m}(a_0, \dots, a_{r-1})u^m$ into the definition (1.4) and using the multilinearity of Pfaffians, we have

$$P_{(r)}(\boldsymbol{x}|\boldsymbol{a}) = \begin{cases} \sum_{m=1}^{r}(-1)^{r-m}e_{r-m}(a_0, \dots, a_{r-1})P_{(m)}(\boldsymbol{x}), & \text{if } n \text{ is even,} \\ \sum_{m=0}^{r}(-1)^{r-m}e_{r-m}(a_0, \dots, a_{r-1})P_{(m)}(\boldsymbol{x}), & \text{if } n \text{ is odd.} \end{cases} \tag{6.6}$$

Since $Q_{(r)}(\boldsymbol{x}) = 2P_{(r)}(\boldsymbol{x})$, it follows from (3.3) that

$$1 + 2\sum_{r=1}^{\infty} P_{(r)}(\boldsymbol{x}, y)z^r = \prod_{i=1}^{n} \frac{1 + x_i z}{1 - x_i z} \cdot \frac{1 + yz}{1 - yz} = \left(1 + 2\sum_{r=1}^{\infty} P_{(r)}(\boldsymbol{x})z^r\right)\left(1 + 2\sum_{r=1}^{\infty} y^r z^r\right).$$

Equating coefficients of z^r, we get

$$P_{(r)}(\boldsymbol{x}, y) = P_{(r)}(\boldsymbol{x}) + 2\sum_{h=1}^{r-1} P_{(r-h)}(\boldsymbol{x})y^h + y^r. \tag{6.7}$$

Using (6.6) and (6.7), we have

$$\begin{aligned} &P_{(r)}(\boldsymbol{x}, y|\boldsymbol{a}) \\ &= \begin{cases} e_r(a_0, \dots, a_{r-1}) \\ \quad + \sum_{m=1}^{r}(-1)^{r-m}e_{r-m}(a_0, \dots, a_{r-1})\left(P_{(m)}(\boldsymbol{x}) + 2\sum_{l=1}^{m-1} P_{(m-l)}(\boldsymbol{x})y^l + y^m\right), \\ \hfill \text{if } n \text{ is even,} \\ \sum_{m=1}^{r}(-1)^{r-m}e_{r-m}(a_0, \dots, a_{r-1})\left(P_{(m)}(\boldsymbol{x}) + 2\sum_{l=1}^{m-1} P_{(m-l)}(\boldsymbol{x})y^l + y^m\right), \\ \hfill \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Hence the coefficient $a_{k,l}$ of $P_{(k)}(\boldsymbol{x})y^l$ on the left-hand side of (6.5) is given by

$$a_{k,l} = \begin{cases} e_r(a_0, \dots, a_{r-1}), & \text{if } k = 0,\ l = 0 \text{ and } n \text{ is even,} \\ 0, & \text{if } k = 0,\ l = 0 \text{ and } n \text{ is odd,} \\ (-1)^{r-l}e_{r-l}(a_0, \dots, a_{r-1}), & \text{if } k = 0 \text{ and } l > 0, \\ 2(-1)^{r-k-l}e_{r-k-l}(a_0, \dots, a_{r-1}), & \text{if } k > 0. \end{cases}$$

In a similar manner, we can compute the coefficient $b_{k,l}$ on the right-hand side of (6.5) and see that

(a) if $k = l = 0$, then

$$b_{0,0} = \begin{cases} e_r(a_0, \dots, a_{r-1}), & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}, \end{cases}$$

(b) if $k = 0$ and $l > 0$, then

$$b_{0,l} = (-1)^{r-l} e_{r-l}(a_0, \dots, a_{r-1}),$$

(c) if $k > 0$ and $l = 0$, then

$$b_{k,0} = (-1)^{r-k} \left(\begin{array}{l} e_{r-k}(-a_0, a_1, \dots, a_{r-1}) \\ + \displaystyle\sum_{m=1}^{r-1} e_{r-m-k}(0, a_{m+1}, \dots, a_{r-1}) e_m(a_0, \dots, a_{m-1}) \end{array} \right),$$

(d) if $k > 0$ and $l > 0$, then

$$b_{k,l} = (-1)^{r-k-l} \sum_{m=1}^{r-1} e_{r-m-k}(0, a_{m+1}, \dots, a_{r-1}) e_{m-l}(a_0, \dots, a_{m-1}).$$

Now, by using Lemma 6.4, we see that $a_{k,l} = b_{k,l}$ and obtain (6.5). □

By an argument similar to the one in the proof of Proposition 4.6, we can derive a determinant formula for $P_{\lambda/\mu,p}(x|\boldsymbol{a})$ for a single variable x.

Theorem 6.5. *Let $\boldsymbol{a} = (a_0, a_1, \dots)$ be factorial parameters. For two strict partitions λ of length l and μ of length m, the factorial skew P-function $P_{\lambda/\mu,p}(x|\boldsymbol{a})$ in a single variable x is given as follows:*

(1) *We have $P_{\lambda/\mu,p}(x|\boldsymbol{a}) = 0$ unless $\lambda \supset \mu$ and $m = l$ or $l - 1$.*
(2) *If $\lambda \supset \mu$ and $m = l$ or $l - 1$, then we have*

$$P_{\lambda/\mu,p}(x|\boldsymbol{a}) = \det \left(R_{\lambda_i/\mu_j}(x|\boldsymbol{a}) \right)_{1 \le i,j \le l}.$$

Proof. It follows from Proposition 4.4 that $P_{\lambda/\mu,p}(x|\boldsymbol{a}) = 0$ unless $\lambda \supset \mu$. By an argument similar to the one in the proof of Proposition 4.5, we can show:

(a) if $l \equiv p$ and $m \equiv p \bmod 2$, then

$$P_{\lambda/\mu,p}(x|\boldsymbol{a}) = \begin{cases} \det \left(R_{\lambda_i/\mu_j}(x|\boldsymbol{a}) \right)_{1 \le i,j \le l}, & \text{if } l = m, \\ 0, & \text{otherwise}, \end{cases}$$

(b) if $l \equiv p$ and $m \not\equiv p \bmod 2$, or if $l \not\equiv p$ and $m \equiv p \bmod 2$, then

$$P_{\lambda/\mu,p}(x|\boldsymbol{a}) = \begin{cases} \det \left(R_{\lambda_i/\mu_j}(x|\boldsymbol{a}) \right)_{1 \le i,j \le l}, & \text{if } l = m + 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mu_l = 0$.

It remains to consider the case where $l \not\equiv p$ and $m \not\equiv p \bmod 2$. By the definition (4.3) and the multilinearity of Pfaffians we have

$$\begin{aligned} P_{\lambda/\mu,p}(x|\boldsymbol{a}) &= \operatorname{Pf}\begin{pmatrix} O_l & T_\lambda(\boldsymbol{x}) & T_{\lambda/0}(\boldsymbol{x}) & M_{\lambda/\mu}(\boldsymbol{x}) \\ -{}^tT_\lambda(\boldsymbol{x}) & 0 & 1 & O \\ -{}^tT_{\lambda/0}(\boldsymbol{x}) & -1 & 0 & O \\ -{}^tM_{\lambda/\mu}(\boldsymbol{x}) & O & O & O_m \end{pmatrix} \\ &= \operatorname{Pf}\begin{pmatrix} O_l & T_\lambda(x) & T_{\lambda/0}(x) & M_{\lambda/\mu}(x) \\ -{}^tT_\lambda(x) & 0 & 0 & O \\ -{}^tT_{\lambda/0}(x) & 0 & 0 & O \\ -{}^tM_{\lambda/\mu}(x) & O & O & O_m \end{pmatrix} \\ &\qquad + \operatorname{Pf}\begin{pmatrix} O_l & T_\lambda(x) & 0 & M_{\lambda/\mu}(x) \\ -{}^tT_\lambda(x) & 0 & 1 & O \\ 0 & -1 & 0 & O \\ -{}^tM_{\lambda/\mu} & O & O & O_m \end{pmatrix}, \end{aligned}$$

where $T_\lambda(x)$ and $T_{\lambda/0}(x)$ are the column vectors with ith entry $P_{(\lambda_i)}(x|\boldsymbol{a})$ and $R_{\lambda_i/0}(x|\boldsymbol{a})$, respectively. Since $P_{(r)}(x|\boldsymbol{a}) = (x-a_0)\prod_{i=1}^{r-1}(x-a_i)$ and $R_{r/0}(x|\boldsymbol{a}) = (x+a_0)\prod_{i=1}^{r-1}(x-a_i)$ by Corollary 6.3, we see that $T_\lambda(x)$ and $T_{\lambda/0}(x)$ are linearly independent. Hence the first Pfaffian vanishes. By expanding the second Pfaffian along the $(l+2)$nd row/column, we have

$$P_{\lambda/\mu,p}(x) = \operatorname{Pf}\begin{pmatrix} O_l & M_{\lambda/\mu}(x) \\ -{}^tM_{\lambda/\mu}(x) & O_m \end{pmatrix} = \begin{cases} \det\left(R_{\lambda_i/\mu_j}(x)\right)_{1\le i,j\le l}, & \text{if } l = m, \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof. □

We can use this theorem to provide a lattice path proof of the tableau description of factorial P- and Q-functions given in [1, Theorem 2.1] and [7, Theorem 4.3] (the case where $a_0 = 0$).

6.3. **Modified Pieri coefficients.** In this subsection, we give a combinatorial description of the modified Pieri coefficients for factorial P-functions. Recall that the skew shifted diagram $S(\lambda/\mu) = S(\lambda) \setminus S(\mu)$ is called a *border strip* if it is connected and contains no 2×2 block of cells.

Theorem 6.6. *Let $\boldsymbol{a} = (a_0, a_1, \dots)$ be factorial parameters. We define the modified Pieri coefficient $c^\lambda_{\mu,r}(\boldsymbol{a})$ by the relation*

$$P_\mu(\boldsymbol{x}|\boldsymbol{a}) \cdot q_r(\boldsymbol{x}) = \sum_\lambda c^\lambda_{\mu,r}(\boldsymbol{a}) P_\lambda(\boldsymbol{x}|\boldsymbol{a}), \tag{6.8}$$

where $q_r(\boldsymbol{x}) = Q_{(r)}(\boldsymbol{x})$ is Schur's Q-function, and λ runs over all strict partitions. For two strict partitions λ and μ, we consider the generating function of modified Pieri coefficients

$$c^\lambda_\mu(z|\boldsymbol{a}) = \sum_{r=0}^\infty c^\lambda_{\mu,r}(\boldsymbol{a}) z^r.$$

Then we have:

(1) *If the skew shifted diagram $S(\lambda/\mu)$ contains a 2×2 block of cells, then we have $c^\lambda_\mu(z|\boldsymbol{a}) = 0$.*

(2) *Suppose that $S(\lambda/\mu)$ contains no 2×2 block of cells. Let*

$$S(\lambda/\mu) = \bigsqcup_{i=1}^{r} S((\lambda_{m(i)}, \dots, \lambda_{M(i)})/(\mu_{m(i)}, \dots, \mu_{M(i)}))$$

be the decomposition of $S(\lambda/\mu)$ into a disjoint union of border strips, where $m(1) \le M(1) < m(2) \le M(2) < \cdots < m(r) \le M(r)$. Then we have

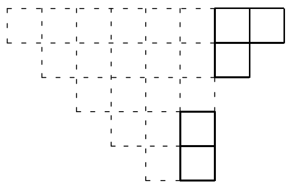
$$c^\lambda_\mu(z|\boldsymbol{a}) = \prod_{k \in K} \frac{1 + a_{\mu_k} z}{1 - a_{\mu_k} z} \prod_{i=1}^{r} \frac{2 z^{\lambda_{m(i)} - \mu_{M(i)}}}{\prod_{j=\mu_{M(i)}}^{\lambda_{m(i)}} (1 - a_j z)}, \tag{6.9}$$

where

$$K = \begin{cases} \{k : 1 \le k \le l(\mu),\ \lambda_k = \mu_k\}, & \text{if } n + l(\mu) \text{ is even,} \\ \{k : 1 \le k \le l(\mu) + 1,\ \lambda_k = \mu_k\}, & \text{if } n + l(\mu) \text{ is odd.} \end{cases}$$

(3) *In particular, the modified Pieri coefficient $c^\lambda_{\mu,r}(\boldsymbol{a})$ is a polynomial in the factorial parameters $a_0, a_1, \dots$ with nonnegative integer coefficients.*

Example 6.7. Let $\lambda = (8, 6, 4, 3, 2)$ and $\mu = (6, 5, 4, 2, 1)$. Then the skew shifted diagram $S(\lambda/\mu)$ is decomposed into a disjoint union of border strips as follows:

$S(\lambda/\mu) =$ 

$$= S((8, 6)/(6, 5)) \sqcup S((3, 2)/(2, 1)).$$

Since we have

$$K = \begin{cases} \{3\}, & \text{if } n \text{ is even,} \\ \{3, 6\}, & \text{if } n \text{ is odd,} \end{cases}$$

we obtain

$$c^{(8,6,4,3,2)}_{(6,5,4,2,1)}(z|\boldsymbol{a}) = \begin{cases} \dfrac{1 + a_4 z}{1 - a_4 z} \cdot \dfrac{2z^{8-5}}{\prod_{i=5}^{8}(1 - a_i z)} \cdot \dfrac{2z^{3-1}}{\prod_{i=1}^{3}(1 - a_i z)}, & \text{if } n \text{ is even,} \\ \dfrac{1 + a_4 z}{1 - a_4 z} \cdot \dfrac{1 + a_0 z}{1 - a_0 z} \cdot \dfrac{2z^{8-5}}{\prod_{i=5}^{8}(1 - a_i z)} \cdot \dfrac{2z^{3-1}}{\prod_{i=1}^{3}(1 - a_i z)}, & \text{if } n \text{ is odd.} \end{cases}$$

Setting $a_0 = a_1 = \cdots = 0$, we recover the Pieri rule for Schur P-functions.

Corollary 6.8 (Morris [15, Theorem 1])**.** *For a strict partition μ and a nonnegative integer k, we have*

$$P_\mu(\boldsymbol{x}) \cdot q_r(\boldsymbol{x}) = \sum_{\lambda} 2^{a(\lambda,\mu)} P_\lambda(\boldsymbol{x}),$$

where λ runs over all strict partitions such that $|\lambda| - |\mu| = r$ and $S(\lambda/\mu)$ contains no 2×2 block, and where $a(\lambda, \mu)$ is the number of connected components of $S(\lambda/\mu)$.

Now we use Theorem 5.3 to give a proof of Theorem 6.6. Let $b_r^s(z|\boldsymbol{a})$ be the coefficient in the expansion

$$(u|\boldsymbol{a})^r \cdot \frac{1+uz}{1-uz} = \sum_{s \geq 0} b_r^s(z|\boldsymbol{a})(u|\boldsymbol{a})^s. \tag{6.10}$$

The following lemma gives an explicit formula for $b_r^s(z|\boldsymbol{a})$.

Lemma 6.9. *For two nonnegative integers r and s, we have*

$$b_r^s(z|\boldsymbol{a}) = \begin{cases} \dfrac{1+a_r z}{1-a_r z}, & \text{if } s = r, \\ \dfrac{2z^{s-r}}{\prod_{j=r}^{s}(1-a_j z)}, & \text{if } s > r, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We need to prove

$$(u|\boldsymbol{a})^r \frac{1+uz}{1-uz} = \frac{1+a_r u}{1-a_r u}(u|\boldsymbol{a})^r + \sum_{s=r+1}^{\infty} \frac{2z^{s-r}}{\prod_{j=r}^{s}(1-a_j z)}(u|\boldsymbol{a})^s. \tag{6.11}$$

By dividing both sides of (6.11) by $\prod_{i=0}^{r-1}(u-a_i)$ and then shifting the indices of factorial parameters, we may assume $r = 0$. The latter case follows from Lemma 6.1. $\square$

We prove Theorem 6.6 by computing the determinant given in (5.3) of Theorem 5.3.

Proof of Theorem 6.6. By (5.3), we see that a nonzero $c_\mu^\lambda(z|\boldsymbol{a})$ is equal to the determinant whose (i,j) entry is equal to $b_{\mu_j}^{\lambda_i} = b_{\mu_j}^{\lambda_i}(z|\boldsymbol{a})$.

Claim 6.10. *We have $c_\mu^\lambda(z|\boldsymbol{a}) = 0$ unless $S(\lambda) \supset S(\mu)$.*

Proof. If there exists an index k such that $\lambda_k < \mu_k$, then we have $\lambda_i < \lambda_k < \mu_k < \mu_j$ for $i \geq k$ and $j \leq k$. By Lemma 6.9, we have $b_{\mu_j}^{\lambda_i} = 0$ for $i \geq k$ and $j \leq k$, thus $c_\mu^\lambda = 0$. $\square$

In what follows we assume that $S(\lambda) \supset S(\mu)$. In this case, by Theorem 5.3, we have

$$c_\mu^\lambda(z|\boldsymbol{a}) = \begin{cases} \det B_\mu^\lambda, & \text{if } n + l(\mu) \text{ is even and } l(\lambda) = l(\mu), \\ \det B_{\mu^0}^{\lambda^0}, & \text{if } n + l(\mu) \text{ is odd and } l(\lambda) = l(\mu), \\ \det B_{\mu^0}^{\lambda}, & \text{if } l(\lambda) = l(\mu) + 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $n + l(\mu)$ is odd and $l(\lambda) = l(\mu) = l$, then $b_{\mu_j}^0 = 0$ for $1 \leq j \leq l$ by Lemma 6.9. By expanding the determinant along the last row, we have

$$\det B_{\mu^0}^{\lambda^0} = b_0^0 \cdot \det B_\mu^\lambda.$$

Hence it is enough to compute the determinant $\det B_{\mu^*}^\lambda$, where $\mu^* = \mu$ or μ^0. By abuse of notation, we write simply B_μ^λ for B_μ^λ and $B_{\mu^0}^\lambda$ in the following.

First we assume that $S(\lambda/\mu)$ is not connected. In this case, there exists an index k such that $\lambda_{k+1} < \mu_k$ or $\lambda_k = \mu_k$.

Claim 6.11. *Suppose that $S(\lambda) \supset S(\mu)$.*

(1) *If there exists an index k such that $\lambda_{k+1} < \mu_k$, then we have*

$$\det B_\mu^\lambda = \det B_{\mu'}^{\lambda'} \cdot \det B_{\mu''}^{\lambda''},$$

where

$$\begin{aligned} ' &= (\lambda_1, \dots, \lambda_k), & \mu' &= (\mu_1, \dots, \mu_k), \\ \lambda'' &= (\lambda_{k+1}, \dots, \lambda_l), & \mu'' &= (\mu_{k+1}, \dots, \mu_l). \end{aligned}$$

(2) *If there exists an index k such that $\lambda_k = \mu_k > 0$, then we have*

$$\det B_\mu^\lambda = \det B_{\mu'}^{\lambda'} \cdot b_{\mu_k}^{\lambda_k} \cdot \det B_{\mu''}^{\lambda''},$$

where

$$\begin{aligned} ' &= (\lambda_1, \dots, \lambda_{k-1}), & \mu' &= (\mu_1, \dots, \mu_{k-1}), \\ \lambda'' &= (\lambda_{k+1}, \dots, \lambda_l), & \mu'' &= (\mu_{k+1}, \dots, \mu_l). \end{aligned}$$

Proof. (1) If $i \geq k+1$ and $j \leq k$, then we have $\lambda_i \leq \lambda_{k+1} < \mu_k \leq \mu_j$ and $b_{\mu_j}^{\lambda_i} = 0$, thus

$$\det B_\mu^\lambda = \det \begin{pmatrix} B_{\mu'}^{\lambda'} & * \\ O & B_{\mu''}^{\lambda''} \end{pmatrix} = \det B_{\mu'}^{\lambda'} \cdot \det B_{\mu''}^{\lambda''}.$$

(2) By a similar consideration, we have

$$\det B_\mu^\lambda = \det \begin{pmatrix} B_{\mu'}^{\lambda'} & * & * \\ O & b_{\mu_k}^{\lambda_k} & * \\ O & 0 & B_{\mu''}^{\lambda''} \end{pmatrix} = \det B_{\mu'}^{\lambda'} \cdot b_{\mu_k}^{\lambda_k} \cdot \det B_{\mu''}^{\lambda''}. \qquad \square$$

Now we consider the case where $S(\lambda/\mu)$ is connected.

Claim 6.12. *Suppose that $S(\lambda) \supset S(\mu)$ and $S(\lambda/\mu)$ is connected. If $S(\lambda/\mu)$ contains a 2×2 block of cells, then we have* $\det B_\mu^\lambda = 0$.

Proof. If $S(\lambda/\mu)$ contains a 2×2 square, then there exists an index k such that $\lambda_{k+1} > \mu_k$. We take the smallest such index k. Then we have

$$\lambda_1 > \mu_1 = \lambda_2 > \mu_2 = \lambda_3 > \cdots > \mu_{k-1} = \lambda_k > \lambda_{k+1} > \mu_k.$$

(Since $S(\lambda/\mu)$ is connected, we have $\lambda_{i+1} \geq \mu_i$ if $\lambda_{i+1} > 0$.) It follows from Lemma 6.9 that, if $t > s > r$, then

$$b_r^t = b_r^s \cdot \frac{z^{t-s}}{(1 - a_{s+1}z) \cdots (1 - a_t z)}. \tag{6.12}$$

We proceed by induction on k. If $k = 1$, then the first row of B_μ^λ is a scalar multiple of the second row by (6.12), and $\det B_\mu^\lambda = 0$. If $k > 1$, then by subtracting the $(k+1)$st row multiplied by $z^{\lambda_k - \lambda_{k+1}}/(1 - a_{\lambda_{k+1}+1}z) \cdots (1 - a_{\lambda_k}z)$ from the kth row in $\det B_\mu^\lambda$, and then expanding the resulting determinant along the kth row, we have

$$\det B_\mu^\lambda = (-1)^{k+(k-1)} b_{\mu_{k-1}}^{\lambda_k} \det B_{\mu'}^{\lambda'},$$

where λ' (respectively μ') is the strict partition obtained from λ (respectively μ) by removing λ_k (respectively μ_{k-1}), i.e.,

$$\lambda' = (\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_l), \quad \mu' = (\mu_1, \dots, \mu_{k-2}, \mu_k, \dots, \mu_l).$$

Since $\det B^{\lambda'}_{\mu'} = 0$ by the induction hypothesis, we obtain $\det B^{\lambda}_{\mu} = 0$. □

Now it remains to compute $\det B^{\lambda}_{\mu}$ when $S(\lambda/\mu)$ is a border strip.

Claim 6.13. *If $S(\lambda/\mu)$ is a border strip, i.e.,*

$$\lambda_1 > \mu_1 = \lambda_2 > \mu_2 = \lambda_3 > \cdots > \mu_{l-1} = \lambda_l > \mu_l,$$

then we have

$$\det B^{\lambda}_{\mu} = \frac{2 z^{\lambda_1 - \mu_l}}{\prod_{i=\mu_l}^{\lambda_1} (1 - a_i z)}.$$

Proof. We proceed by induction on l. A direct computation verifies the cases $l = 1$ and $l = 2$. So we assume $l \geq 3$. By Lemma 6.9, we have

$$\begin{aligned} b^{\lambda_i}_{\mu_l} &= \frac{z^{\mu_{l-1} - \mu_l}}{\prod_{j=\mu_l}^{\mu_{l-1}-1} (1 - a_j z)} \cdot b^{\lambda_i}_{\mu_{l-1}} \quad (1 \leq i \leq l-1), \\ b^{\lambda_l}_{\mu_l} &= \frac{z^{\mu_{l-1} - \mu_l}}{\prod_{j=\mu_l}^{\mu_{l-1}-1} (1 - a_j z)} \cdot \frac{2}{1 - a_{\lambda_l}} \cdot b^{\lambda_l}_{\mu_{l-1}}. \end{aligned}$$

Factor out the common term $z^{\mu_{l-1}-\mu_l} / \prod_{j=\mu_l}^{\mu_{l-1}-1} (1 - a_j z)$ from the last column of $\det B^{\lambda}_{\mu}$, and then subtract the lth column from the $(l-1)$st column. Since we have

$$b^{\lambda_l}_{\mu_{l-1}} - \frac{2}{1 - a_{\lambda_l}} \cdot b^{\lambda_l}_{\mu_{l-1}} = -1,$$

we expand the resulting determinant along the $(l-1)$st column to see

$$\det B^{\lambda}_{\mu} = \frac{z^{\mu_{l-1} - \mu_l}}{\prod_{j=\mu_l}^{\mu_{l-1}-1} (1 - a_j z)} \cdot \det B^{\lambda'}_{\mu'},$$

where $\lambda' = (\lambda_1, \dots, \lambda_{l-1})$ and $\mu' = (\mu_1, \dots, \mu_{l-1})$. Using the induction hypothesis, we obtain Claim 6.13. □

Combining the above claims together completes the proof of the theorem. □

Based on Theorem 6.6 (3) and some experimental evidence, we propose the following conjecture.

Conjecture 6.14. *We define $f^{\lambda}_{\mu,\nu}(\boldsymbol{a})$ by the formula*

$$P_\mu(\boldsymbol{x}|\boldsymbol{a}) P_\nu(\boldsymbol{x}) = \sum_{\lambda} f^{\lambda}_{\mu,\nu}(\boldsymbol{a}) P_\lambda(\boldsymbol{x}|\boldsymbol{a}), \tag{6.13}$$

where λ runs over all strict partitions. Then the coefficient $f^{\lambda}_{\mu,\nu}(\boldsymbol{a})$ is a polynomial in $\boldsymbol{a} = (a_0, a_1, \dots)$ with nonnegative integer coefficients. More generally, if we expand the product of factorial P-functions corresponding to different factorial parameters $\boldsymbol{a} = (a_0, a_1, \dots)$ and $-\boldsymbol{b} = (-b_0, -b_1, \dots)$ as a linear combination of $P_\lambda(\boldsymbol{x}|\boldsymbol{a})$s,

$$P_\mu(\boldsymbol{x}|\boldsymbol{a}) P_\nu(\boldsymbol{x}|-\boldsymbol{b}) = \sum_{\lambda} f^{\lambda}_{\mu,\nu}(\boldsymbol{a}, \boldsymbol{b}) P_\lambda(\boldsymbol{x}|\boldsymbol{a}), \tag{6.14}$$

then the coefficient $f^{\lambda}_{\mu,\nu}(\boldsymbol{a}, \boldsymbol{b})$ is a polynomial in $\boldsymbol{a}$ and $\boldsymbol{b}$ with nonnegative integer coefficients.

Cho and Ikeda [1, Theorem 4.6] gave a combinatorial formula for the Pieri-type coefficients $f^{\lambda}_{\mu,(r)}(\boldsymbol{a},-\boldsymbol{a})$, which implies that $f^{\lambda}_{\mu,(r)}(\boldsymbol{a},-\boldsymbol{a})$ is a polynomial in $a_i \pm a_j$ with $i > j$ with nonnegative integer coefficients.

Remark 6.15. Let $s_\lambda(\boldsymbol{x}|\boldsymbol{a})$ be the factorial Schur function with factorial parameters $\boldsymbol{a}$, and expand

$$s_\mu(\boldsymbol{x}|\boldsymbol{a})s_\nu(\boldsymbol{x}|-\boldsymbol{b}) = \sum_\lambda m^\lambda_{\mu,\nu}(\boldsymbol{a},\boldsymbol{b}) s_\lambda(\boldsymbol{x}|\boldsymbol{a}).$$

Then Molev and Sagan [14, Theorem 3.1] gave a combinatorial formula for the coefficient $m^\lambda_{\mu,\nu}(\boldsymbol{a},\boldsymbol{b})$, which implies that $m^\lambda_{\mu,\nu}(\boldsymbol{a},\boldsymbol{b})$ is a polynomial in $\boldsymbol{a}$ and $\boldsymbol{b}$ with nonnegative integer coefficients.

7. P-functions associated with classical root systems

In this section, we show that the Hall–Littlewood functions at $t=-1$ associated with the classical root systems can be written as generalized P-functions associated with certain polynomial sequences.

7.1. **Hall–Littlewood function associated with root systems.** Macdonald [13] generalized the definition of Hall–Littlewood functions to any root system. Let Φ be a root system in a Euclidean vector space V and fix a positive system Φ^+. We denote by Λ and Λ^+ the weight lattice and the set of dominant weights, respectively. Let $K = \mathbb{Q}(t)$ be the rational function field in an indeterminate t, and let $K[\Lambda]$ be the group algebra of Λ with basis $\{e^\lambda : \lambda \in \Lambda\}$ over K. Let W be the Weyl group of Φ. Then the Hall–Littlewood functions associated with the root system Φ are defined as follows.

Definition 7.1. The Hall–Littlewood function $\mathbb{P}^\Phi_\lambda \in K[\Lambda]$ corresponding to a dominant weight $\lambda \in \Lambda^+$ is defined by

$$\mathbb{P}^\Phi_\lambda = \frac{1}{W_\lambda(t)} \sum_{w \in W} w \left(e^\lambda \prod_{\alpha \in \Phi^+} \frac{1 - te^{-\alpha}}{1 - e^{-\alpha}} \right), \tag{7.1}$$

where $W_\lambda(t) = \sum_{w \in W_\lambda} t^{l(w)}$ is the Poincaré polynomial of W_λ, with $W_\lambda = \{w \in W : w\lambda = \lambda\}$ being the stabilizer of λ in W.

In this section, we consider the root systems of types $X_n = B_n$, C_n and D_n. Let V be the n-dimensional Euclidean vector space with orthonormal basis $\varepsilon_1, \dots, \varepsilon_n$. We put $x_i = e^{\varepsilon_i}$ for $1 \le i \le n$ and write $\mathbb{P}^\Phi_\lambda = \mathbb{P}^\Phi_\lambda(\boldsymbol{x};t)$. Let $\Phi(X_n) \subset V$ be the root system of type X_n with the positive system $\Phi^+(X_n)$ given by

$$\begin{aligned}
\Phi^+(B_n) &= \{\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\} \cup \{\varepsilon_i : 1 \le i \le n\}, \\
\Phi^+(C_n) &= \{\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\} \cup \{2\varepsilon_i : 1 \le i \le n\}, \\
\Phi^+(D_n) &= \{\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\}.
\end{aligned}$$

Then the set $\Lambda^+(X_n)$ of dominant weights is given by

$$\Lambda^+(B_n) = \left\{ \sum_{i=1}^n \lambda_i \varepsilon_i : (\lambda_i)_{i=1}^n \in \mathbb{Z}^n \cup (\mathbb{Z}+1/2)^n,\ \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0 \right\},$$

$$\Lambda^+(C_n) = \left\{ \sum_{i=1}^n \lambda_i \varepsilon_i : (\lambda_i)_{i=1}^n \in \mathbb{Z}^n, \ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \right\},$$

$$\Lambda^+(D_n) = \left\{ \sum_{i=1}^n \lambda_i \varepsilon_i : (\lambda_i)_{i=1}^n \in \mathbb{Z}^n \cup (\mathbb{Z}+1/2)^n, \ \lambda_1 \geq \lambda_2 \geq \dots \lambda_{n-1} \geq |\lambda_n| \right\},$$

where $\mathbb{Z}+1/2 = \{r+1/2 : r \in \mathbb{Z}\}$. We identify a partition λ of length $l \leq n$ with a dominant weight $\lambda_1\varepsilon_1 + \cdots + \lambda_l \varepsilon_l \in \Lambda^+(X_n)$.

We note that a reciprocal Laurent polynomial $g(z) = \sum_{i=-d}^d a_i z^i$ with $a_i = a_{-i}$ and $a_d \neq 0$ can be written as $g(z) = f(z+z^{-1})$ for some polynomial of degree d. We use the notation $\boldsymbol{x} + \boldsymbol{x}^{-1} = (x_1 + x_1^{-1}, \dots, x_n + x_n^{-1})$. The following is the main theorem of this section.

Theorem 7.2. (1) *Let $\mathcal{F}^B = \{f_d^B\}_{d=0}^\infty$ be the sequence of polynomials defined by*

$$f_0^B = 1, \quad f_d^B(x+x^{-1}) = (x^d - x^{-d}) \frac{x^{1/2} + x^{-1/2}}{x^{1/2} - x^{-1/2}} \quad (d \geq 1).$$

For a strict partition λ of length $l \leq n$, we have

$$\mathbb{P}_\lambda^{\Phi(B_n)}(\boldsymbol{x}; -1) = P_\lambda^{\mathcal{F}^B}(\boldsymbol{x} + \boldsymbol{x}^{-1}). \tag{7.2}$$

(2) *Let $\mathcal{F}^C = \{f_d^C\}_{d=0}^\infty$ be the sequence of polynomials defined by*

$$f_0^C = 1, \quad f_d^C(x+x^{-1}) = (x^d - x^{-d}) \frac{x + x^{-1}}{x - x^{-1}} \quad (d \geq 1).$$

For a strict partition λ of length $l \leq n$, we have

$$\mathbb{P}_\lambda^{\Phi(C_n)}(\boldsymbol{x}; -1) = P_\lambda^{\mathcal{F}^C}(\boldsymbol{x} + \boldsymbol{x}^{-1}). \tag{7.3}$$

(3) *Let $\mathcal{F}^D = \{f_d^D\}_{d=0}^\infty$ be the sequence of polynomials defined by*

$$f_0^D = 1, \quad f_d^D(x+x^{-1}) = x^d + x^{-d} \quad (d \geq 1).$$

For a strict partition λ of length $l < n$, we have

$$\mathbb{P}_\lambda^{\Phi(D_n)}(\boldsymbol{x}; -1) = P_\lambda^{\mathcal{F}^D}(\boldsymbol{x} + \boldsymbol{x}^{-1}), \tag{7.4}$$

and, for a strict partition λ of length n, we have

$$\mathbb{P}_\lambda^{\Phi(D_n)}(\boldsymbol{x}; -1) + \mathbb{P}_{\lambda'}^{\Phi(D_n)}(\boldsymbol{x}; -1) = P_\lambda^{\mathcal{F}^D}(\boldsymbol{x} + \boldsymbol{x}^{-1}), \tag{7.5}$$

where $\lambda' = \lambda_1\varepsilon_1 + \cdots + \lambda_{n-1}\varepsilon_{n-1} - \lambda_n \varepsilon_n$.

The first few terms of the sequences $\mathcal{F}^B$, $\mathcal{F}^C$ and $\mathcal{F}^D$ are

$$\begin{aligned} &f_1^B(u) = u+2, && f_2^B(u) = u^2+2u, && f_3^B(u) = u^3+2u^2-u-2, \\ &f_1^C(u) = u, && f_2^C(u) = u^2, && f_3^C(u) = u^3-u, \\ &f_1^D(u) = u, && f_2^D(u) = u^2-2, && f_3^D(u) = u^3-3u. \end{aligned}$$

See Lemma 7.7 for the generating functions of $f_d^X(x+x^{-1})$.

Definition 7.3. Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be indeterminates and λ a strict partition of length $l \le n$. We define symmetric Laurent polynomials $P^X_\lambda(\boldsymbol{x})$ and $Q^X_\lambda(\boldsymbol{x})$ of type X_n, where $X \in \{B, C, D\}$, by putting

$$P^X_\lambda(\boldsymbol{x}) = P^{\mathcal{F}^X}_\lambda(\boldsymbol{x} + \boldsymbol{x}^{-1}), \quad Q^X_\lambda(\boldsymbol{x}) = 2^l P^{\mathcal{F}^X}_\lambda(\boldsymbol{x} + \boldsymbol{x}^{-1}),$$

where $\mathcal{F}^X$ is the polynomial sequence given in Theorem 7.2. We call $Q^B_\lambda(\boldsymbol{x})$, $Q^C_\lambda(\boldsymbol{x})$ and $Q^D_\lambda(\boldsymbol{x})$ the *odd orthogonal Q-function*, *symplectic Q-function* and *even orthogonal Q-function*, respectively.

Note that $Q^X_\lambda(\boldsymbol{x})$ is obtained as the generalized P-function $P^{\mathcal{G}}_\lambda(\boldsymbol{x}+\boldsymbol{x}^{-1})$ associated with the sequence $\mathcal{G}^X = \{g^X_d\}_{d=0}^\infty$ given by

$$g^X_d(u) = \begin{cases} 1, & \text{if } d = 0, \\ 2f^X(u), & \text{if } d \ge 1. \end{cases}$$

In order to prove Theorem 7.2, we recall the structure of the Weyl groups of type B_n, C_n and D_n. Let T_n be the abelian group of order 2^n generated by $t_1, \dots, t_n$ subject to the relations $t_i^2 = 1$ $(1 \le i \le n)$ and $t_i t_j = t_j t_i$ $(1 \le i, j \le n)$, and $W_n = T_n \rtimes S_n$ the semidirect product of T_n with the symmetric group S_n, where S_n acts on T_n by permuting $t_1, \dots, t_n$. Put $T'_n = \{t_1^{u_1} \dots t_n^{u_n} : \sum_{i=1}^n u_i \equiv 0 \bmod 2\}$ and $W'_n = T'_n \rtimes S_n$ the semidirect product of T'_n with S_n. Then W_n is isomorphic to the Weyl groups of type B_n and C_n, and W'_n is isomorphic to the Weyl group of type D_n. The natural action of S_n on V and $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is extended to W_n by

$$t_i \varepsilon_k = \begin{cases} -\varepsilon_i, & \text{if } k = i, \\ \varepsilon_k, & \text{if } k \ne i, \end{cases} \qquad t_i x_k = \begin{cases} x_i^{-1}, & \text{if } k = i, \\ x_k, & \text{if } k \ne i. \end{cases}$$

If λ is a strict partition of length l, then the stabilizer W_λ of $\lambda_1\varepsilon_1 + \dots + \lambda_l \varepsilon_l$ is isomorphic to W_{n-l} for types B_n and C_n, and to W'_{n-l} for type D_n.

Lemma 7.4. *For a strict partition λ of length l, we have*

$$\mathbb{P}^{\Phi(B_n)}_\lambda(\boldsymbol{x}; t) = \sum_{w \in W_n/W_{n-l}} w \left(\prod_{i=1}^l x_i^{\lambda_i} \prod_{i=1}^l \frac{1 - t x_i^{-1}}{1 - x_i^{-1}} \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{(1 - t x_i^{-1} x_j)(1 - t x_i^{-1} x_j^{-1})}{(1 - x_i^{-1} x_j)(1 - x_i^{-1} x_j^{-1})} \right),$$

$$\mathbb{P}^{\Phi(C_n)}_\lambda(\boldsymbol{x}; t) = \sum_{w \in W_n/W_{n-l}} w \left(\prod_{i=1}^l x_i^{\lambda_i} \prod_{i=1}^l \frac{1 - t x_i^{-2}}{1 - x_i^{-2}} \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{(1 - t x_i^{-1} x_j)(1 - t x_i^{-1} x_j^{-1})}{(1 - x_i^{-1} x_j)(1 - x_i^{-1} x_j^{-1})} \right),$$

$$\mathbb{P}^{\Phi(D_n)}_\lambda(\boldsymbol{x}; t) = \sum_{w \in W'_n/W'_{n-l}} w \left(\prod_{i=1}^l x_i^{\lambda_i} \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{(1 - t x_i^{-1} x_j)(1 - t x_i^{-1} x_j^{-1})}{(1 - x_i^{-1} x_j)(1 - x_i^{-1} x_j^{-1})} \right).$$

Proof. For a general root system Φ with Weyl group W, we have (see [10, Theorem 2.8])

$$\sum_{w \in W} w \left(\prod_{\alpha \in \Phi^+} \frac{1 - t e^{-\alpha}}{1 - e^{-\alpha}} \right) = \sum_{w \in W} t^{l(w)}.$$

We can use the same argument as the one in the proof of Lemma 2.4 to prove this lemma. □

Proof of Theorem 7.2. (1) We can take $\{wu : w \in S_n/S_{n-l}, u \in T_l\}$ as a complete set of coset representatives of W_n/W_{n-l}, where $T_l = \langle t_1, \dots, t_l \rangle$. Since the product

$$\prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{(1 + x_i^{-1} x_j)(1 + x_i^{-1} x_j^{-1})}{(1 - x_i^{-1} x_j)(1 - x_i^{-1} x_j^{-1})}$$

is invariant under T_l, we see that

$$\sum_{w \in T_l} w \left(\prod_{i=1}^{l} x_i^{\lambda_i} \frac{1 + x_i^{-1}}{1 - x_i^{-1}} \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{(1 + x_i^{-1} x_j)(1 + x_i^{-1} x_j^{-1})}{(1 - x_i^{-1} x_j)(1 - x_i^{-1} x_j^{-1})} \right)$$
$$= \prod_{i=1}^{l} \left(x_i^{\lambda_i} \frac{1 + x_i^{-1}}{1 - x_i^{-1}} + x_i^{-\lambda_i} \frac{1 + x_i^{1}}{1 - x_i^{1}} \right) \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{(1 + x_i^{-1} x_j)(1 + x_i^{-1} x_j^{-1})}{(1 - x_i^{-1} x_j)(1 - x_i^{-1} x_j^{-1})}.$$

By using

$$\frac{(1 + x_i^{-1} x_j)(1 + x_i^{-1} x_j^{-1})}{(1 - x_i^{-1} x_j)(1 - x_i^{-1} x_j^{-1})} = \frac{(x_i + x_i^{-1}) + (x_j + x_j^{-1})}{(x_i + x_i^{-1}) - (x_j + x_j^{-1})},$$

we have

$$\mathbb{P}_\lambda^{\Phi(B_n)}(\boldsymbol{x}; -1)$$
$$= \sum_{w \in S_n/S_{n-l}} w \left(\prod_{i=1}^{l} \left(x_i^{\lambda_i} - x_i^{-\lambda_i} \right) \frac{x_i^{1/2} + x_i^{-1/2}}{x_i^{1/2} - x_i^{-1/2}} \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{(x_i + x_i^{-1}) + (x_j + x_j^{-1})}{(x_i + x_i^{-1}) - (x_j + x_j^{-1})} \right).$$

Comparing this with (2.5), we obtain (7.2).

(2) can be shown in the same manner as (1).

(3) Suppose that $l < n$. Since $W_n = W'_n \sqcup W'_n t_n$, $W_{n-l} = W'_{n-l} \sqcup W'_{n-l} t_n$ and

$$\prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{(1 + x_i^{-1} x_j)(1 + x_i^{-1} x_j^{-1})}{(1 - x_i^{-1} x_j)(1 - x_i^{-1} x_j^{-1})}$$

is invariant under t_n, we see that

$$\mathbb{P}_\lambda^{\Phi(D_n)}(\boldsymbol{x}; -1) = \sum_{w \in W_n/W_{n-l}} w \left(\prod_{i=1}^{l} x_i^{\lambda_i} \prod_{\substack{1 \le i < j \le n \\ i \le l}} \frac{(x_i + x_i^{-1}) + (x_j + x_j^{-1})}{(x_i + x_i^{-1}) - (x_j + x_j^{-1})} \right).$$

If $l = n$, then the stabilizer W_λ is trivial and we have

$$\begin{aligned}
&\mathbb{P}_\lambda^{\Phi(D_n)}(\boldsymbol{x};-1) + \mathbb{P}_{\lambda'}^{\Phi(D_n)}(\boldsymbol{x};-1) \\
&\qquad = \sum_{w \in W_n'} w \left(\prod_{i=1}^{n-1} x_i^{\lambda_i} x_n^{\lambda_n} \prod_{1 \le i < j \le n} \frac{(1 + x_i^{-1} x_j)(1 + x_i^{-1} x_j^{-1})}{(1 - x_i^{-1} x_j)(1 - x_i^{-1} x_j^{-1})} \right) \\
&\qquad\qquad + \sum_{w \in W_n'} w \left(\prod_{i=1}^{n-1} x_i^{\lambda_i} x_n^{-\lambda_n} \prod_{1 \le i < j \le n} \frac{(1 + x_i^{-1} x_j)(1 + x_i^{-1} x_j^{-1})}{(1 - x_i^{-1} x_j)(1 - x_i^{-1} x_j^{-1})} \right) \\
&\qquad = \sum_{w \in W_n} w \left(\prod_{i=1}^{n} x_i^{\lambda_i} \prod_{1 \le i < j \le n} \frac{(x_i + x_i^{-1}) + (x_j + x_j^{-1})}{(x_i + x_i^{-1}) - (x_j + x_j^{-1})} \right).
\end{aligned}$$

The rest of the proof is the same as in the proof of (1). □

7.2. **Generating functions.** Since the Q-functions $Q_\lambda^X(\boldsymbol{x})$ of type X are special cases of generalized P-functions, we have a Schur-type Pfaffian formula.

Proposition 7.5. *For a strict partition λ of length l, we have*

$$Q_\lambda^X(\boldsymbol{x}) = \operatorname{Pf}\left(Q_{(\lambda_i,\lambda_j)}^X(\boldsymbol{x}) \right)_{1 \le i,j \le r},$$

where $r = l$ or $l+1$ according to whether l is even or odd, and where we use the convention (2.8).

Hence, in order to obtain $Q_\lambda^X(\boldsymbol{x})$ for a general strict partition λ, we need to know $Q_{(r)}^X(\boldsymbol{x})$ and $Q_{(r,s)}^X(\boldsymbol{x})$. We compute the generating functions for them. In order to state formulas, we introduce formal power series $\varphi^X(z)$ and $\psi^X(z)$ by putting

$$\varphi^X(z) = \begin{cases} \dfrac{(1+z)^2}{1+z^2}, & \text{if } X = B, \\ 1, & \text{if } X = C, \\ \dfrac{1-z^2}{1+z^2}, & \text{if } X = D, \end{cases} \qquad \psi^X(z) = \begin{cases} \dfrac{2z}{1+z^2}, & \text{if } X = B, \\ 0, & \text{if } X = C, \\ -\dfrac{2z^2}{1+z^2}, & \text{if } X = D. \end{cases}$$

Furthermore, we put

$$\widetilde{\Pi}_z(\boldsymbol{x}) = \prod_{i=1}^n \frac{(1 + x_i z)(1 + x_i^{-1} z)}{(1 - x_i z)(1 - x_i^{-1} z)}.$$

Then we have the following formulas.

Proposition 7.6. (1) *The generating function for $Q_{(r)}^X(\boldsymbol{x})$ is given by*

$$\sum_{r=0}^\infty Q_{(r)}^X(\boldsymbol{x}) z^r = \varphi^X(z) \widetilde{\Pi}_z(\boldsymbol{x}) - (-1)^n \psi^X(z).$$

(2) *The generating function for $Q_{(r,s)}^X(\boldsymbol{x})$ is given by*

$$\sum_{r,s \ge 0} Q_{(r,s)}^X(\boldsymbol{x}) z^r w^s = \frac{(z-w)(1-zw)}{(z+w)(1+zw)} \cdot \varphi^X(z) \varphi^X(w) \left(\widetilde{\Pi}_z(\boldsymbol{x}) \widetilde{\Pi}_w(\boldsymbol{x}) - 1 \right)$$

$$+ (-1)^n \left(\varphi^X(z)\psi^X(w)\widetilde{\Pi}_z(\boldsymbol{x}) - \varphi^X(w)\psi^X(z)\widetilde{\Pi}_w(\boldsymbol{x}) \right)$$
$$+ \psi^X(z) - \psi^X(w).$$

By a straightforward case-by-case computation, we can show the following lemma.

Lemma 7.7. *We have*

$$1 + 2\sum_{r=1}^{\infty} f_r^X(x + x^{-1}) z^r = \varphi^X(z) \cdot \frac{(1+xz)(1+x^{-1}z)}{(1-xz)(1-x^{-1}z)} + \psi^X(z).$$

Proof of Proposition 7.6. By Theorem 7.2 and (1.4), we see that the Q-function $Q_\lambda^X(\boldsymbol{x})$ corresponding to a strict partition λ of length l is expressed as

$$Q_\lambda^X(\boldsymbol{x}) = \frac{1}{\widetilde{\Delta}(\boldsymbol{x})} \operatorname{Pf} \begin{pmatrix} \widetilde{A}(\boldsymbol{x}) & \widetilde{V}_{\lambda^*}^X(\boldsymbol{x}) \\ -{}^t\widetilde{V}_{\lambda^*}^X(\boldsymbol{x}) & O \end{pmatrix},$$

where $\lambda^* = \lambda$ or λ^0 according to whether $n + l$ is even or odd, and

$$\widetilde{A}(\boldsymbol{x}) = \left(\frac{(x_j + x_j^{-1}) - (x_i + x_i^{-1})}{(x_j + x_j^{-1}) + (x_i + x_i^{-1})} \right)_{1 \le i,j \le n} = \left(\frac{(x_i - x_j)(1 - x_i x_j)}{(x_i + x_j)(1 + x_i x_j)} \right)_{1 \le i,j \le n},$$

$$\widetilde{\Delta}(\boldsymbol{x}) = \prod_{1 \le i < j \le n} \frac{(x_j + x_j^{-1}) - (x_i + x_i^{-1})}{(x_j + x_j^{-1}) + (x_i + x_i^{-1})} = \prod_{1 \le i < j \le n} \frac{(x_i - x_j)(1 - x_i x_j)}{(x_i + x_j)(1 + x_i x_j)},$$

$$\widetilde{V}^X_{(\alpha_1,\dots,\alpha_r)}(\boldsymbol{x}) = \left(\chi(\alpha_j) f^X_{\alpha_j}(x_i + x_i^{-1}) \right)_{1 \le i \le n, 1 \le j \le r}, \quad \chi(d) = \begin{cases} 1, & \text{if } d = 0, \\ 2, & \text{if } d \ge 1. \end{cases}$$

Now, by an argument similar to the one in the proof of Proposition 3.2, we can establish this proposition by using the Pfaffian evaluations in Proposition A.3 together with Lemma 7.7 and the relation $\varphi^X(z) - \psi^X(z) = 1$. The details are left to the readers. $\square$

Appendix A. Pfaffian formulas

In this appendix, we collect several useful Pfaffian identities.

A.1. **Pfaffians.** Recall the definition and some properties of Pfaffians (see [5] for an exposition). Let $X = \left(x_{ij} \right)_{1 \le i,j \le 2m}$ be a skew-symmetric matrix of order $2m$. The *Pfaffian* of X, denoted by $\operatorname{Pf}(X)$, is defined by

$$\operatorname{Pf}(X) = \sum_{\sigma \in F_{2m}} \operatorname{sgn}(\sigma) \prod_{i=1}^{m} x_{\sigma(2i-1),\sigma(2i)}, \tag{A.1}$$

where F_{2m} is the set of permutations $\sigma \in S_{2m}$ satisfying $\sigma(1) < \sigma(3) < \cdots < \sigma(2m-1)$ and $\sigma(2i-1) < \sigma(2i)$ for $1 \le i \le m$.

Pfaffians are multilinear and alternating in the following sense. Let $X = \left(x_{ij} \right)_{1 \le i,j \le n}$ be an even-sized skew-symmetric matrix and fix a row/column index k. If the entries of the kth row and kth column of X are written as $x_{i,j} = \alpha x'_{i,j} + \beta x''_{i,j}$ for $i = k$ or $j = k$, then

$$\operatorname{Pf} X = \alpha \operatorname{Pf} X' + \beta \operatorname{Pf} X'',$$

where X' (respectively X'') is the skew-symmetric matrix obtained from X by replacing the entries x_{ij} for $i = k$ or $j = k$ with x'_{ij} (respectively x''_{ij}). Moreover, for a permutation $\sigma \in S_n$, we have

$$\operatorname{Pf}\big(x_{\sigma(i),\sigma(j)}\big)_{1 \le i,j \le n} = \operatorname{sgn}(\sigma) \operatorname{Pf}\big(x_{i,j}\big)_{1 \le i,j \le n}.$$

It follows that, if Y is the skew-symmetric matrix obtained from X by adding the kth row multiplied by a scalar α to the lth row and then adding the kth column multiplied by α to the lth column, then we have $\operatorname{Pf} Y = \operatorname{Pf} X$.

We use the following notations for submatrices. For a positive integer n, we put $[n] = \{1, 2, \dots, n\}$. Given a subset $I \subset [n]$, we put $\Sigma(I) = \sum_{i \in I} i$. For an $M \times N$ matrix $X = \big(x_{i,j}\big)_{1 \le i \le M, 1 \le j \le N}$ and subsets $I \subset [M]$ and $J \subset [N]$, we denote by $X(I; J)$ the submatrix of X obtained by picking up rows indexed by I and columns indexed by J. If X is a skew-symmetric matrix, then we write $X(I)$ for $X(I; I)$. We use the convention that $\det X(\emptyset; \emptyset) = 1$ and $\operatorname{Pf} X(\emptyset) = 1$.

For an $n \times n$ skew-symmetric matrix $X = \big(x_{i,j}\big)_{1 \le i,j \le n}$, we have the following expansion formula along the kth row/column:

$$\operatorname{Pf} X = \sum_{i=1}^{k-1} (-1)^{k+i-1} x_{i,k} \operatorname{Pf} X([n] \setminus \{i,k\}) + \sum_{i=k+1}^{n} (-1)^{k+i-1} x_{k,i} \operatorname{Pf} X([n] \setminus \{k,i\}). \quad \text{(A.2)}$$

A.2. **Schur's Pfaffian evaluation and its variations.** Recall that

$$A(\boldsymbol{x}) = \left(\frac{x_j - x_i}{x_j + x_i} \right)_{1 \le i,j \le n}, \quad \Delta(\boldsymbol{x}) = \prod_{1 \le i < j \le n} \frac{x_j - x_i}{x_j + x_i}$$

for a sequence $\boldsymbol{x} = (x_1, \dots, x_n)$ of indeterminates. The evaluation of the Pfaffian in (A.3) below originates from [22], and its simple proof can be found in [9]. Equation (A.4) is derived from (A.3) by specializing the last indeterminate to 0.

Proposition A.1. *If n is even, then we have*

$$\operatorname{Pf} A(\boldsymbol{x}) = \Delta(\boldsymbol{x}). \quad \text{(A.3)}$$

If n is odd, then we have

$$\operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & \mathbf{1} \\ -{}^t\mathbf{1} & 0 \end{pmatrix} = \Delta(\boldsymbol{x}), \quad \text{(A.4)}$$

where $\mathbf{1}$ is the all-one column vector.

For two sequences $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_p)$ of indeterminates, we put

$$B(\boldsymbol{x}; \boldsymbol{y}) = \left(\frac{1 + x_i y_j}{1 - x_i y_j} \right)_{1 \le i \le n, 1 \le j \le p}, \quad \Pi(\boldsymbol{x}; \boldsymbol{y}) = \prod_{i=1}^{n} \prod_{j=1}^{p} \frac{1 + x_i y_j}{1 - x_i y_j}.$$

Let $B_z(\boldsymbol{x}) = B(\boldsymbol{x}; (z))$ be the column vector with ith entry $(1 + x_i z)/(1 - x_i z)$ and set $\Pi_z(\boldsymbol{x}) = \Pi(\boldsymbol{x}; (z)) = \prod_{i=1}^{n} (1 + x_i z)/(1 - x_i z)$. Then we have the following variations of Schur's Pfaffian evaluation.

Proposition A.2. (1) *If $n + p$ is even, then we have*

$$\operatorname{Pf} \begin{pmatrix} A(\boldsymbol{x}) & B(\boldsymbol{x}; \boldsymbol{y}) \\ -{}^t B(\boldsymbol{x}; \boldsymbol{y}) & -A(\boldsymbol{y}) \end{pmatrix} = (-1)^{\binom{p}{2}} \Delta(\boldsymbol{x}) \Delta(\boldsymbol{y}) \Pi(\boldsymbol{x}; \boldsymbol{y}). \quad \text{(A.5)}$$

(2) *If n is even, then we have*

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) \\ -{}^tB_z(\boldsymbol{x}) & 0 & 0 \\ -{}^tB_w(\boldsymbol{x}) & 0 & 0 \end{pmatrix} = \Delta(\boldsymbol{x}) \cdot \frac{z-w}{z+w}\bigl(\Pi_z(\boldsymbol{x})\Pi_w(\boldsymbol{x}) - 1\bigr). \tag{A.6}$$

(3) *If n is odd, then we have*

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) & \mathbf{1} \\ -{}^tB_z(\boldsymbol{x}) & 0 & 0 & 0 \\ -{}^tB_w(\boldsymbol{x}) & 0 & 0 & 0 \\ -{}^t\mathbf{1} & 0 & 0 & 0 \end{pmatrix}$$
$$= \Delta(\boldsymbol{x}) \cdot \left\{ \frac{z-w}{z+w}\bigl(\Pi_z(\boldsymbol{x})\Pi_w(\boldsymbol{x}) - 1\bigr) - \Pi_z(\boldsymbol{x}) + \Pi_w(\boldsymbol{x}) \right\}. \tag{A.7}$$

Proof. (1) Apply (A.3) to the indeterminates $(x_1, \dots, x_n, -1/y_1, \dots, -1/y_p)$.

(2) By applying (1) with $p = 2$ and $(y_1, y_2) = (z, w)$, we have

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) \\ -{}^tB_z(\boldsymbol{x}) & 0 & \dfrac{z-w}{z+w} \\ -{}^tB_w(\boldsymbol{x}) & -\dfrac{z-w}{z+w} & 0 \end{pmatrix} = \Delta(\boldsymbol{x})\Pi_z(\boldsymbol{x})\Pi_w(\boldsymbol{x}) \cdot \frac{z-w}{z+w}.$$

By using the multilinearity of Pfaffians, we obtain

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) \\ -{}^tB_z(\boldsymbol{x}) & 0 & \dfrac{z-w}{z+w} \\ -{}^tB_w(\boldsymbol{x}) & -\dfrac{z-w}{z+w} & 0 \end{pmatrix}$$
$$= \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) \\ -{}^tB_z(\boldsymbol{x}) & 0 & 0 \\ -{}^tB_w(\boldsymbol{x}) & 0 & 0 \end{pmatrix} + \frac{z-w}{z+w} \cdot \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & 0 \\ -{}^tB_z(\boldsymbol{x}) & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The last Pfaffian is shown to be equal to $\Delta(\boldsymbol{x})$ by expanding along the last row/column and using (A.3). Thus we obtain (A.6).

(3) Applying (1) with $p = 3$ and $(y_1, y_2, y_3) = (z, w, 0)$, we obtain

$$\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) & -\mathbf{1} \\ -{}^tB_z(\boldsymbol{x}) & 0 & \dfrac{z-w}{z+w} & -1 \\ -{}^tB_w(\boldsymbol{x}) & -\dfrac{z-w}{z+w} & 0 & -1 \\ {}^t\mathbf{1} & 1 & 1 & 0 \end{pmatrix} = \Delta(\boldsymbol{x})\Pi_z(\boldsymbol{x})\Pi_w(\boldsymbol{x}) \cdot \frac{z-w}{z+w} \cdot (-1)^{n+2}.$$

By using the multilinearity we see that

$$
\begin{aligned}
&\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) & -\mathbf{1} \\ -{}^tB_z(\boldsymbol{x}) & 0 & \dfrac{z-w}{z+w} & -1 \\ -{}^tB_w(\boldsymbol{x}) & -\dfrac{z-w}{z+w} & 0 & -1 \\ {}^t\mathbf{1} & 1 & 1 & 0 \end{pmatrix} \\
&= -\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) & \mathbf{1} \\ -{}^tB_z(\boldsymbol{x}) & 0 & 0 & 0 \\ -{}^tB_w(\boldsymbol{x}) & 0 & 0 & 0 \\ -{}^t\mathbf{1} & 0 & 0 & 0 \end{pmatrix} - \frac{z-w}{z+w}\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & 0 & \mathbf{1} \\ -{}^tB_z(\boldsymbol{x}) & 0 & 1 & 0 \\ -{}^tB_w(\boldsymbol{x}) & -1 & 0 & 0 \\ -{}^t\mathbf{1} & 0 & 0 & 0 \end{pmatrix} \\
&\quad - \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) & 0 \\ -{}^tB_z(\boldsymbol{x}) & 0 & \dfrac{z-w}{z+w} & 1 \\ -{}^tB_w(\boldsymbol{x}) & -\dfrac{z-w}{z+w} & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} - \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) & 0 \\ -{}^tB_z(\boldsymbol{x}) & 0 & \dfrac{z-w}{z+w} & 0 \\ -{}^tB_w(\boldsymbol{x}) & -\dfrac{z-w}{z+w} & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
\end{aligned}
$$

The last three Pfaffians can be evaluated by expanding them along a row/column and then using (A.4) and (A.5) with $p = 1$ as follows:

$$
\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & 0 & \mathbf{1} \\ -{}^tB_z(\boldsymbol{x}) & 0 & 1 & 0 \\ -{}^tB_w(\boldsymbol{x}) & -1 & 0 & 0 \\ -{}^t\mathbf{1} & 0 & 0 & 0 \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & \mathbf{1} \\ -{}^t\mathbf{1} & 0 \end{pmatrix} = \Delta(\boldsymbol{x}),
$$

$$
\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) & 0 \\ -{}^tB_z(\boldsymbol{x}) & 0 & \dfrac{z-w}{z+w} & 1 \\ -{}^tB_w(\boldsymbol{x}) & -\dfrac{z-w}{z+w} & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = -\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_w(\boldsymbol{x}) \\ -{}^tB_w(\boldsymbol{x}) & 0 \end{pmatrix} = -\Delta(\boldsymbol{x})\Pi_w(\boldsymbol{x}),
$$

$$
\operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) & B_w(\boldsymbol{x}) & 0 \\ -{}^tB_z(\boldsymbol{x}) & 0 & \dfrac{z-w}{z+w} & 0 \\ -{}^tB_w(\boldsymbol{x}) & -\dfrac{z-w}{z+w} & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} A(\boldsymbol{x}) & B_z(\boldsymbol{x}) \\ -{}^tB_z(\boldsymbol{x}) & 0 \end{pmatrix} = \Delta(\boldsymbol{x})\Pi_z(\boldsymbol{x}).
$$

If these evaluations are combined, the proof of (A.7) is completed. □

The following Pfaffian evaluations are used in Section 7. For $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_p)$, we put

$$
\widetilde{A}(\boldsymbol{x}) = \left(\frac{(x_j + x_j^{-1}) - (x_i + x_i^{-1})}{(x_j + x_j^{-1}) + (x_i + x_i^{-1})} \right)_{1 \le i,j \le n} = \left(\frac{(x_i - x_j)(1 - x_i x_j)}{(x_i + x_j)(1 + x_i x_j)} \right)_{1 \le i,j \le n},
$$

$$
\widetilde{\Delta}(\boldsymbol{x}) = \prod_{1 \le i < j \le n} \frac{(x_j + x_j^{-1}) - (x_i + x_i^{-1})}{(x_j + x_j^{-1}) + (x_i + x_i^{-1})} = \prod_{1 \le i < j \le n} \frac{(x_i - x_j)(1 - x_i x_j)}{(x_i + x_j)(1 + x_i x_j)},
$$

$$\widetilde{B}(\boldsymbol{x};\boldsymbol{y}) = \left(\frac{(1+x_i y_j)(1+x_i^{-1}y_j)}{(1-x_i y_j)(1-x_i^{-1}y_j)} \right)_{1\le i\le n, 1\le j\le p},$$

$$\widetilde{\Pi}(\boldsymbol{x};\boldsymbol{y}) = \prod_{i=1}^{n}\prod_{j=1}^{p} \frac{(1+x_i y_j)(1+x_i^{-1}y_j)}{(1-x_i y_j)(1-x_i^{-1}y_j)}.$$

We write $\widetilde{B}_z(\boldsymbol{x}) = \widetilde{B}(\boldsymbol{x};(z))$ and $\widetilde{\Pi}_z(\boldsymbol{x}) = \widetilde{\Pi}(\boldsymbol{x};(z))$. Then we have the following result.

Proposition A.3. (1) *If n is even, then we have*

$$\operatorname{Pf} \widetilde{A}(\boldsymbol{x}) = \widetilde{\Delta}(\boldsymbol{x}). \tag{A.8}$$

(2) *If n is odd, then we have*

$$\operatorname{Pf} \begin{pmatrix} \widetilde{A}(\boldsymbol{x}) & \mathbf{1} \\ -{}^t\mathbf{1} & 0 \end{pmatrix} = \widetilde{\Delta}(\boldsymbol{x}). \tag{A.9}$$

(3) *If $n+p$ is even, then we have*

$$\operatorname{Pf} \begin{pmatrix} \widetilde{A}(\boldsymbol{x}) & \widetilde{B}(\boldsymbol{x};\boldsymbol{y}) \\ -{}^t\widetilde{B}(\boldsymbol{x};\boldsymbol{y}) & \widetilde{A}(\boldsymbol{y}) \end{pmatrix} = \widetilde{\Delta}(\boldsymbol{x})\widetilde{\Delta}(\boldsymbol{y})\widetilde{\Pi}(\boldsymbol{x};\boldsymbol{y}). \tag{A.10}$$

(4) *If n is even, then we have*

$$\operatorname{Pf} \begin{pmatrix} \widetilde{A}(\boldsymbol{x}) & \widetilde{B}_z(\boldsymbol{x}) & \widetilde{B}_w(\boldsymbol{x}) \\ -{}^t\widetilde{B}_z(\boldsymbol{x}) & 0 & 0 \\ -{}^t\widetilde{B}_w(\boldsymbol{x}) & 0 & 0 \end{pmatrix} = \widetilde{\Delta}(\boldsymbol{x}) \frac{(z-w)(1-zw)}{(z+w)(1+zw)} \big(\widetilde{\Pi}_z(\boldsymbol{x})\widetilde{\Pi}_w(\boldsymbol{x}) - 1 \big). \tag{A.11}$$

(5) *If n is odd, then we have*

$$\operatorname{Pf} \begin{pmatrix} \widetilde{A}(\boldsymbol{x}) & \widetilde{B}_z(\boldsymbol{x}) & \widetilde{B}_w(\boldsymbol{x}) & \mathbf{1} \\ -{}^t\widetilde{B}_z(\boldsymbol{x}) & 0 & 0 & 0 \\ -{}^t\widetilde{B}_w(\boldsymbol{x}) & 0 & 0 & 0 \\ -{}^t\mathbf{1} & 0 & 0 & 0 \end{pmatrix}$$

$$= \widetilde{\Delta}(\boldsymbol{x}) \left\{ \frac{(z-w)(1-zw)}{(z+w)(1+zw)} \big(\widetilde{\Pi}_z(\boldsymbol{x})\widetilde{\Pi}_w(\boldsymbol{x}) - 1 \big) - \widetilde{\Pi}_z(\boldsymbol{x}) + \widetilde{\Pi}_w(\boldsymbol{x}) \right\}. \tag{A.12}$$

Proof. (1) and (2) are obtained by replacing x_i by $x_i + x_i^{-1}$ in (A.3) and (A.4), respectively. (3) is obtained by applying (A.3) with $(x_1+x_1^{-1}, \dots, x_n+x_n^{-1}, -(y_1+y_1^{-1}), \dots, -(y_p+y_p^{-1}))$. (4) and (5) are derived from (3) by an argument similar to the ones in the proof of (A.6) and (A.7), respectively. □

A.3. **Useful formulas for Pfaffians.** The following propositions are Pfaffian analogues of the Sylvester identity, the Laplace expansion formula, and the Cauchy–Binet formula for determinants.

Proposition A.4 ([9, (2.5)])**.** *Let n and m be even integers. If X is an $(n+m)\times(n+m)$ skew-symmetric matrix such that* $\operatorname{Pf} X([n]) \neq 0$, *then we have*

$$\operatorname{Pf} \left(\frac{\operatorname{Pf} X([n] \cup \{n+i, n+j\})}{\operatorname{Pf} X([n])} \right)_{1\le i,j\le m} = \frac{\operatorname{Pf} X}{\operatorname{Pf} X([n])}. \tag{A.13}$$

Proposition A.5 ([19, Corollary 2.4 (1)]). *Let m and n be nonnegative integers with the same parity. If Z is an $m \times m$ skew-symmetric matrix and W an $m \times n$ matrix, then we have*

$$\operatorname{Pf}\begin{pmatrix} Z & W \\ -{}^tW & O_{n,n} \end{pmatrix} = \begin{cases} \sum_I (-1)^{\Sigma(I)+\binom{m}{2}} \operatorname{Pf} Z(I) \det W([m] \setminus I; [n]), & \text{if } m > n, \\ (-1)^{\binom{m}{2}} \det W, & \text{if } m = n, \\ 0, & \text{if } m < n, \end{cases} \tag{A.14}$$

where I runs over all $(m-n)$-element subsets of $[n]$.

Proposition A.6 ([19, Theorem 3.2]). *Let m and n be nonnegative integers with the same parity. Let A and B be $m \times m$ and $n \times n$ skew-symmetric matrices, and let S and T be $m \times l$ and $n \times l$ matrices, respectively. Then we have*

$$\sum_I (-1)^{\binom{\#I}{2}} \operatorname{Pf}\begin{pmatrix} A & S([m];I) \\ -{}^tS([m];I) & O \end{pmatrix} \operatorname{Pf}\begin{pmatrix} B & T([n];I) \\ -{}^tT([n];I) & O \end{pmatrix} = \operatorname{Pf}\begin{pmatrix} A & S{}^tT \\ -T{}^tS & B \end{pmatrix}, \tag{A.15}$$

$$\sum_I \operatorname{Pf}\begin{pmatrix} A & S([m];I) \\ -{}^tS([m];I) & O \end{pmatrix} \operatorname{Pf}\begin{pmatrix} B & T([n];I) \\ -{}^tT([n];I) & O \end{pmatrix} = (-1)^{\binom{n}{2}} \operatorname{Pf}\begin{pmatrix} A & S{}^tT \\ -T{}^tS & -B \end{pmatrix}, \tag{A.16}$$

where I runs over all subsets of $[l]$ with $\#I \equiv m \equiv n \bmod 2$.

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