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A NONCOMMUTATIVE WEIGHT-DEPENDENT GENERALIZATION OF THE BINOMIAL THEOREM

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Dedicated to Christian Krattenthaler on the occasion of his 60th birthday

Abstract. A weight-dependent generalization of the binomial theorem for noncommuting variables is presented. This result extends the well-known binomial theorem for q-commuting variables by a generic weight function depending on two integers. For two special cases of the weight function, in both cases restricting it to depend only on a single integer, the noncommutative binomial theorem involves an expansion involving complete symmetric functions, and elementary symmetric functions, respectively. Another special case concerns the weight function to be a suitably chosen elliptic (i.e., doubly-periodic meromorphic) function, in which case an elliptic generalization of the binomial theorem is obtained. The latter is utilized to quickly recover Frenkel and Turaev's elliptic hypergeometric ${}_{10}V_9$ summation formula, an identity fundamental to the theory of elliptic hypergeometric series. Further specializations yield noncommutative binomial theorems of basic hypergeometric type.

1. Introduction

For an indeterminate q, let $\mathbb{C}_q[x,y]$ be the associative unital algebra over $\mathbb{C}$ generated by x and y, satisfying the relation

$$yx = qxy. \tag{1.1}$$

$\mathbb{C}_q[x,y]$ can be regarded as a q-deformation of the commutative algebra $\mathbb{C}[x,y]$. The variables x, y forming $\mathbb{C}_q[x,y]$ are referred to as *q-commuting* variables.

The following binomial theorem for q-commuting variables is well known and usually attributed to M.-P. Schützenberger [17]. However, for the case of x and y being $n \times n$ square matrices with complex entries and q (then necessarily) being a root of unity (else (1.1) cannot be satisfied), a proof, which extends verbatim to the general case, was already given in 1950 by the Scottish mathematician H.S.A. Potter [13]. For an excellent account of the history of (1.1) in the context of matrix theory, see [7].

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Proposition 1. *The following identity is valid in* $\mathbb{C}_q[x,y]$*:*

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}. \tag{1.2}$$

Here,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \tag{1.3}$$

is the *q-binomial coefficient*, defined for nonnegative integers n and k with $n \geq k$, where, for an indeterminate a, the *q-shifted factorial* is defined as

$$(a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1}), \qquad \text{for} \quad k = 0,1,2,\ldots. \tag{1.4}$$

Proposition 1 plays an important role in the theory of quantum groups and q-special functions, see [8, 11].

While the author's initial goal was to generalize Proposition 1 to the "elliptic case", investigations led to the discovery of a yet more general result, a noncommutative binomial theorem involving a generic weight function that depends on two integers.

This paper is organized as follows. In Section 2 a noncommutative algebra is introduced for a generic weight function depending on two integers. The three commutation relations which define this algebra are responsible for the validity of the noncommutative binomial theorem. The chosen weight function uniquely determines the corresponding binomial coefficients. These appear as coefficients in the expansion of the noncommutative binomial theorem in Theorem 1, which is the main result of this paper. This result can also be very nicely combinatorially interpreted in terms of weighted lattice paths. By multiplying two instances of the binomial theorem and suitably taking coefficients, a convolution formula for the weight-dependent binomial coefficients is deduced, while two other convolution formulae are derived by means of the combinatorics of weighted lattice paths. Section 3 focuses on two specific choices of the weight function where (as is well-known) the binomial coefficients become symmetric functions, namely complete symmetric functions and elementary symmetric functions, respectively. However, the noncommutative binomial theorems involving the complete and elementary symmetric functions in (3.1)/(3.2) and (3.4)/(3.5) already appear to be new (which is quite surprising, given its simplicity). The situation is particularly interesting in Section 4 where an *elliptic* (i.e., doubly-periodic meromorphic) weight function is considered. In this case the noncommutative algebra can be very elegantly described in terms of shifts on two of the variables. The four variables forming this algebra are referred to as *elliptic-commuting* variables, while the coefficients appearing in the binomial expansion of these variables are *elliptic binomial coefficients* (which in fact are even totally elliptic functions). The convolution formula for the latter turns out to be a variant of Frenkel and Turaev's elliptic hypergeometric ${}_{10}V_9$ summation formula, an identity fundamental to the theory of elliptic hypergeometric series. While this suggests that the noncommutative elliptic binomial theorem should be useful in the theory of elliptic hypergeometric series, related elliptic special functions, and elliptic quantum groups (see [6, Sec. 11], [19], and [4], respectively), it is possible that the more general weight-dependent result in Theorem 1, or at least its symmetric function specializations in (3.1)/(3.2) and (3.4)/(3.5), will be similarly useful, e.g., in symmetric function theory in general or, speculatively, even in the construction of quantum groups involving symmetric functions. A further challenge of combinatorial nature consists in finding possible *weight-dependent* noncommutative extensions of MacMahon's Master Theorem (which would maybe generalize the results of [10]). Already working out an *elliptic* extension of MacMahon's Master Theorem would be an exciting achievement. In Appendix A the whole set-up of Section 2 is extended by introducing an additional weight function, again depending on two integers. This leads to a further extension of the noncommutative algebra and corresponding binomial theorem by which one in principle is able to consider more general cases (which however is not pursued further in this paper). Finally, in Appendix B, particularly attractive basic hypergeometric specializations of the elliptic case are considered and made explicit. (This section may serve as a teaser. Some readers, who are familiar with basic hypergeometric series, may enjoy looking at Subsections B.2 and B.3 first, and verify, say, the $n = 2, 3$ cases of the binomial theorems in (B.8) and (B.12) by hand.)

Acknowledgements. I would like to thank Tom Koornwinder for private discussions on the problem of finding an elliptic extension of the binomial theorem for q-commuting variables. These discussions took place during the workshop on "Elliptic integrable systems, isomonodromy problems, and hypergeometric functions" at the Max Planck Institute for Mathematics in Bonn, July 21–25, 2008. I would further like to thank Tom for his continued interest and encouragement. I would also like to thank Johann Cigler for fruitful discussions on the noncommutative binomial theorem. Finally, I would like to thank Volker Strehl for his interest and for suggesting to add the alternative combinatorial interpretation (2.6) to the discussion of the weight-dependent binomial coefficients in Subsection 2.2.

The main results of this paper were presented at the "Discrete Systems and Special Functions" workshop at the Isaac Newton Institute for Mathematical Sciences in Cambridge, June 29 – July 3, 2009. (The elliptic case was already presented at several occasions before, the first time at a seminar at Nagoya University on September 3, 2008.) I am indebted to the organizers of both of these meetings (Yu.I. Manin, M. Noumi, E.M. Rains, H. Rosengren, V.P. Spiridonov, and P. Clarkson, R. Halburd, M. Noumi, A.O. Daalhuis, respectively) for inviting me to these workshops which have been highly stimulating.

2. Weight-dependent commutation relations and binomial theorem

2.1. **A noncommutative algebra.** Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive and nonnegative integers, respectively. We will work in the following noncommutative algebra. (A slight extension of this algebra is considered in Appendix A.)

Definition 1. For a doubly-indexed sequence of indeterminates $(w(s,t))_{s,t\in\mathbb{N}}$ let $\mathbb{C}_w[x,y]$ be the associative unital algebra over $\mathbb{C}$ generated by x, y and the $w(s,t)$, with $s,t \in \mathbb{N}$, satisfying the following three relations:

$$yx = w(1,1)\,xy, \tag{2.1a}$$

$$x\,w(s,t) = w(s+1,t)\,x, \tag{2.1b}$$

$$y\,w(s,t) = w(s,t+1)\,y, \tag{2.1c}$$

for all $(s,t) \in \mathbb{N}^2$.

We refer to the $w(s,t)$, and more generally, products (and even polynomials) of the $w(s,t)$, as "weights" (in consideration of the combinatorial interpretation in Subsection 2.2). Notice that, for $w(s,t) = q$ for all $s,t \in \mathbb{N}$, where q is some indeterminate, $\mathbb{C}_w[x,y]$ reduces to $\mathbb{C}_q[x,y]$, the algebra of two q-commuting variables considered in the Introduction.

The relations in (2.1) for generic weights $w(s,t)$ are indeed well-defined. Define the *canonical form* of an element in $\mathbb{C}_w[x,y]$ to be of the form

$$\sum_{k,l\ge 0} c_{k,l} x^k y^l,$$

where $c_{k,l} \neq 0$ for finitely many k,l. Here the coefficients $c_{k,l}$ are elements of $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{N}}]$, the polynomial ring over $\mathbb{C}$ of the indeterminates $(w(s,t))_{s,t\in\mathbb{N}}$. Since, according to (2.1b) and (2.1c), x and y act as independent shift operators on the components of the weights, it is straightforward to verify that the canonical form of an arbitrary expression in $\mathbb{C}_w[x,y]$ is unique. (In the terminology of [1], all elements of $\mathbb{C}_w[x,y]$ are reduction-unique.) This follows by induction (on the minimal number of commutation relations needed to bring an expression into canonical form) and observing that the simple expression $yx\,w(s,t)$ reduces to a unique canonical form regardless in which order (e.g., y and x, or x and $w(s,t)$, are swapped first, etc.) the commutation relations are applied for this purpose.

2.2. **Weight-dependent binomial coefficients.** As before, we consider a doubly-indexed sequence of indeterminate weights $(w(s,t))_{s,t\in\mathbb{N}}$. For $s \in \mathbb{N}$ and $t \in \mathbb{N}_0$, we write

$$W(s,t) := \prod_{j=1}^{t} w(s,j) \tag{2.2a}$$

(the empty product, which occurs when $t=0$, being defined as 1) for brevity. Note that for $s,t \in \mathbb{N}$ we have

$$w(s,t) = \frac{W(s,t)}{W(s,t-1)}. \tag{2.2b}$$

To distinguish, we refer to the $W(s,t)$ as *big* weights, and to the $w(s,t)$ as *small* weights.

Let the *weight-dependent binomial coefficients* be defined by

$${}_w\!\begin{bmatrix}0\\0\end{bmatrix} = 1, \qquad {}_w\!\begin{bmatrix}n\\k\end{bmatrix} = 0, \qquad \text{for } n \in \mathbb{N}_0, \text{ and } k \in -\mathbb{N} \text{ or } k > n, \tag{2.3a}$$

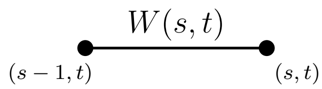
and

$${}_w\!\begin{bmatrix}n+1\\k\end{bmatrix} = {}_w\!\begin{bmatrix}n\\k\end{bmatrix} + {}_w\!\begin{bmatrix}n\\k-1\end{bmatrix} W(k,n+1-k), \qquad \text{for } n,k \in \mathbb{N}_0. \tag{2.3b}$$

The more general *double* weight-dependent binomial coefficients involving two generic weight functions are defined in Equation (A.2). To avoid possible misconception, it should be stressed that the weight-dependent binomial coefficients in (2.3) in general do *not* satisfy the symmetry ${}_w\!\left[{n \atop k}\right] = {}_w\!\left[{n \atop n-k}\right]$. (In case they are symmetric, a second recursion from (2.3b) can be immediately deduced, from which together with (2.3b) a closed product formula for the weight-dependent binomial coefficients can be derived.) It follows from (2.3) immediately by induction that

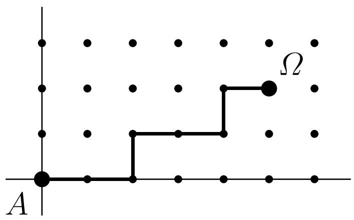
$$ {}_w\!\begin{bmatrix} n \\ 0 \end{bmatrix} = {}_w\!\begin{bmatrix} n \\ n \end{bmatrix} = 1 \qquad \text{for } n \in \mathbb{N}_0. \tag{2.4}$$

The big weights $W(s,t)$ and the weight-dependent binomial coefficients ${}_w\!\left[{n \atop k}\right]$ have an elegant combinatorial interpretation in terms of *weighted lattice paths*. Consider lattice paths in the planar integer lattice consisting of positively directed unit vertical and horizontal steps. Such paths can be "enumerated" with respect to a generic weight function w. In particular, assign the big weight $W(s,t)$ to each horizontal step $(s-1,t) \to (s,t)$,



and (for the moment) assign weight 1 to each vertical step $(s,t-1) \to (s,t)$. (We will consider the more general case of having an additional weight function v defined on the vertical steps in Appendix A.) Further, define the weight $\omega(P)$ of a path P to be the product of the weights of all its steps. For instance, the following path $P_0(A \to \Omega)$ from $A = (0,0)$ to $\Omega = (5,2)$

P_0 :



has weight

$$\omega(P_0) = 1 \cdot 1 \cdot W(3,1) \cdot W(4,1) \cdot W(5,2) = w(3,1)w(4,1)w(5,1)w(5,2).$$

(Equivalently, this corresponds to picking up the weights $w(s,t)$ for each of the points $(s,t-1)$ strictly below the path, for $s,t \geq 1$). We will come back to this specific example shortly after the proof of Theorem 1.

Given two points $A, \Omega \in \mathbb{N}_0^2$, let $\mathcal{P}(A \to \Omega)$ be the set of all paths from A to Ω. Further, define

$$\omega(\mathcal{P}(A \to \Omega)) := \sum_{P \in \mathcal{P}(A \to \Omega)} \omega(P)$$

to be the *generating function* with respect to the weight function ω of all paths from A to Ω. Now it is clear that

$$\omega(\mathcal{P}((0,0) \to (k, n-k))) = {}_w\!\begin{bmatrix} n \\ k \end{bmatrix}. \tag{2.5}$$

Indeed, the path generating function satisfies the same relations as the binomial coefficients in (2.3). The initial conditions (2.3a) being clear, the validity of the recursion (2.3b) stems from the fact that the final step of a path ending in $(k, n+1-k)$ is either *vertical* or *horizontal*.

If one thinks of lattice paths consisting of East and North steps as 0-1-sequences (0 corresponding to an East step and 1 to a North step), then the relation (2.5) can also be interpreted as a weighted enumeration of 0-1-sequences of length n with exactly k occurrences of 0, with respect to a weighted inversion statistic that keeps track not only of the position of the respective inversion in the sequence but also of the number of 0's and 1's that have already appeared before in the respective subsequence. Specifically, under this interpretation, Equation (2.5) now reads

$$\sum_{\substack{\text{0-1-sequences } \alpha \text{ of length } n\\ \text{with } k \text{ occurrences of } 0}} \prod_{\text{inversions } (s,t) \text{ of } \alpha} W(s,t) = {}_w\begin{bmatrix} n \\ k \end{bmatrix}, \tag{2.6}$$

where (s,t) is an *inversion* of $\alpha = (\alpha_1 \dots, \alpha_n)$ (for $\alpha_1, \dots, \alpha_n \in \{0,1\}$ with k occurrences of 0 and $n-k$ occurrences of 1) if and only if $\alpha_{s+t} = 0$ and the subsequence $(\alpha_1, \dots, \alpha_{s+t-1})$ contains $s-1$ occurrences of 0 and t occurrences of 1.

It should be clear that the relations (2.3), (2.5) and (2.6) are equivalent. So one could also just define the weight-dependent binomial coefficients by (2.5) or (2.6), and then show that they satisfy the recursion (2.3). (The combinatorial interpretations (2.5) and (2.6) are certainly nice and show how the weight-dependent binomial coefficients naturally appear, while the Pascal triangle relation (2.3) describes how they are recursively computed.)

In [16], lattice paths in $\mathbb{Z}^2$ were enumerated with respect to the specific *elliptic* weight function $w(s,t) = w_{a,b;q,p}(s,t)$ as defined in (4.7), giving as generating functions the (closed form) elliptic binomial coefficients $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{a,b;q,p}$ in (4.4). It was exactly this result which inspired the search for the algebra $\mathbb{C}_w[x,y]$ and the binomial theorem in Theorem 1. We will have a closer look at the elliptic case in Section 4.

2.3. **A noncommutative binomial theorem.** We have the following elegant result.

Theorem 1 (Weight-dependent binomial theorem)**.** *Let $n \in \mathbb{N}_0$. Then the following identity is valid in $\mathbb{C}_w[x,y]$:*

$$(x+y)^n = \sum_{k=0}^{n} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}. \tag{2.7}$$

Proof. We proceed by induction on n. For $n=0$ the formula is trivial. Now let $n \geq 0$ (n being fixed) and assume that we have already shown the formula for all nonnegative integers $\leq n$. We need to show

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} {}_w\begin{bmatrix} n+1 \\ k \end{bmatrix} x^k y^{n+1-k}. \tag{2.8}$$

By the recursion formula (2.3b), the right-hand side is

$$\sum_{k=0}^{n+1} {}_w\!\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n+1-k} + \sum_{k=0}^{n+1} {}_w\!\begin{bmatrix} n \\ k-1 \end{bmatrix} W(k, n+1-k)\, x^k y^{n+1-k}$$
$$= \sum_{k=0}^{n} {}_w\!\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k} y + \sum_{k=0}^{n} {}_w\!\begin{bmatrix} n \\ k \end{bmatrix} W(k+1, n-k)\, x^{k+1} y^{n-k},$$

where the range of summation in each of the sums was delimited due to (2.3a). It remains to be shown that

$$W(k+1, n-k)\, x^{k+1} y^{n-k} = x^k y^{n-k}\, x. \tag{2.9}$$

In particular, using (2.1b) (and induction) we have

$$W(k+1, n-k)\, x^{k+1} y^{n-k} = x^k W(1, n-k)\, x y^{n-k},$$

so (2.9) is shown as soon as we establish

$$W(1, n-k)\, x y^{n-k} = y^{n-k} x. \tag{2.10}$$

The left-hand side is

$$\left(\prod_{j=1}^{n-k} w(1,j)\right) x y y^{n-k-1} = \left(\prod_{j=2}^{n-k} w(1,j)\right) y x y^{n-k-1} = y \left(\prod_{j=1}^{n-k-1} w(1,j)\right) x y^{n-k-1}$$
$$= y\, W(1, n-k-1)\, x y^{n-k-1}, \tag{2.11}$$

where we have first used (2.1a) and then (2.1c). The identity (2.10) follows now from (2.11) immediately by induction on $n - k \geq 0$. $\square$

Coming back to the interpretation of the weight-dependent binomial coefficients as generating functions for weighted lattice paths (see Subsection 2.2), the expansion (2.7) in Theorem 1 itself has an accordingly nice interpretation. The identification of expressions (or "words" consisting of concatenated symbols) in $\mathbb{C}_w[x, y]$ and lattice paths in $\mathbb{Z}^2$ works locally (variable by variable, or step by step) as follows:

$$x \quad \longleftrightarrow \quad \text{horizontal step,}$$
$$y \quad \longleftrightarrow \quad \text{vertical step.}$$

Under this correspondence, "xy" means that a horizontal step is followed by a vertical step, while "yx" means that a vertical step is followed by a horizontal step (we read from left to right). The relations of the algebra $\mathbb{C}_w[x, y]$ in Definition 1 exactly take into account the changes of the respective weights when consecutive horizontal and vertical steps are being interchanged. For instance, the specific path P_0 considered in Subsection 2.2 corresponds to the algebraic expression

$$x\,x\,y\,x\,x\,y\,x = x\,x\,w(1,1)\,x\,y\,x\,w(1,1)\,x\,y = w(3,1)w(5,2)\,x^3\,w(1,1)\,x\,y\,x\,y$$
$$= w(3,1)w(5,2)w(4,1)\,x^4\,w(1,1)\,x\,y^2 = w(3,1)w(5,2)w(4,1)w(5,1)\,x^5 y^2 = w(P_0)\,x^5 y^2,$$

where the left coefficient (of the double monomial $x^5 y^2$) in the canonical form of the algebraic expression is the weight of the path. Concluding, the left-hand side of (2.7), i.e., $(x+y)^n$, translates into paths of length n having for each step a choice of going in horizontal or vertical positive direction, while the right-hand side of (2.7) refines the counting according to the number of horizontal steps k (for $0 \le k \le n$).

2.4. **Convolution formulae.** We are ready to apply the weight-dependent binomial theorem in Theorem 1 to derive weight-dependent extensions of the well-known (Vandermonde) convolution of binomial coefficients,

$$\binom{n+m}{k} = \sum_{j=0}^{\min(k,n)} \binom{n}{j}\binom{m}{k-j}. \tag{2.12}$$

The following corollaries, although being derived with the help of noncommuting variables, themselves concern identities of *commuting* elements.

Corollary 1 (First weight-dependent binomial convolution formula)**.** *Let n, m, and k be nonnegative integers. For the binomial coefficients in* (2.3)*, defined by the doubly-indexed sequence of indeterminate weights $(w(s,t))_{s,t\in\mathbb{N}}$, we have the following formal identity in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{N}}]$:*

$${}_w\!\begin{bmatrix} n+m \\ k \end{bmatrix} = \sum_{j=0}^{\min(k,n)} {}_w\!\begin{bmatrix} n \\ j \end{bmatrix} \left(x^j y^{n-j}\, {}_w\!\begin{bmatrix} m \\ k-j \end{bmatrix} y^{j-n} x^{-j} \right) \prod_{i=1}^{k-j} W(i+j, n-j). \tag{2.13}$$

The above identity in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{N}}]$ is *formal* as it contains the expression

$$x^j y^{n-j}\, {}_w\!\begin{bmatrix} m \\ k-j \end{bmatrix} y^{j-n} x^{-j} \tag{2.14}$$

(which evaluates to an expression in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{N}}]$) in the summand, where $x, y \notin \mathbb{C}[(w(s,t))_{s,t\in\mathbb{N}}]$. The x and y are understood to be shift operators as defined in (2.1b) and (2.1c). Formally, $x^j y^{n-j}$ has to commute with ${}_w\!\left[\begin{smallmatrix} m \\ k-j \end{smallmatrix}\right]$ which will involve various shifts of the weight functions implicitly appearing in the w-binomial coefficient. Afterwards $x^j y^{n-j}$ will cancel with its formal inverse $y^{j-n} x^{-j}$.

First we state a useful lemma.

Lemma 1. *The following identity holds in $\mathbb{C}_w[x,y]$:*

$$y^k x^l = \left(\prod_{i=1}^{l} W(i,k) \right) x^l y^k, \qquad \textit{for } k, l \in \mathbb{N}_0. \tag{2.15}$$

Proof. The $k = 0$ or $l = 0$ cases are trivial. For $k, l \ge 1$, the identity (2.15) follows straightforwardly by double induction starting with the $k = l = 1$ case which is (2.1a). To prove the validity of (2.15) for $k = 1$ and $l > 1$, assume that $yx^s = \left(\prod_{i=1}^{s} W(i,1)\right) x^s y$ has already been shown for $1 \le s < l$. Then

$$\begin{aligned} yx^l &= w(1,1)\, xyx^{l-1} = W(1,1)\, x \left(\prod_{i=1}^{l-1} W(i,1) \right) x^{l-1} y \\ &= \left(W(1,1) \prod_{i=1}^{l-1} W(i+1,1) \right) x^l y = \left(\prod_{i=1}^{l} W(i,1) \right) x^l y, \end{aligned}$$

as to be shown. Now, for fixed $k, l \geq 1$, assume that $y^t x^l = \big(\prod_{i=1}^{l} W(i,t)\big) x^l y^t$ has already been shown for $1 \leq t < k$. Then

$$\begin{aligned} y^k x^l &= y\bigg(\prod_{i=1}^{l} W(i,k-1)\bigg) x^l y^{k-1} = y\bigg(\prod_{i=1}^{l}\prod_{j=1}^{k-1} w(i,j)\bigg) x^l y^{k-1} \\ &= \bigg(\prod_{i=1}^{l}\prod_{j=1}^{k-1} w(i,j+1)\bigg) y x^l y^{k-1} = \bigg(\prod_{i=1}^{l}\prod_{j=1}^{k} w(i,j)\bigg) x^l y^k = \bigg(\prod_{i=1}^{l} W(i,k)\bigg) x^l y^k, \end{aligned}$$

which establishes the lemma. ☐

Proof of Corollary 1. Working in $\mathbb{C}_w[x,y]$, we expand $(x+y)^{n+m}$ in two different ways and suitably extracts coefficients. On the one hand,

$$(x+y)^{n+m} = \sum_{k=0}^{n+m} {}_w\!\begin{bmatrix} n+m \\ k \end{bmatrix} x^k y^{n+m-k}. \tag{2.16}$$

On the other hand,

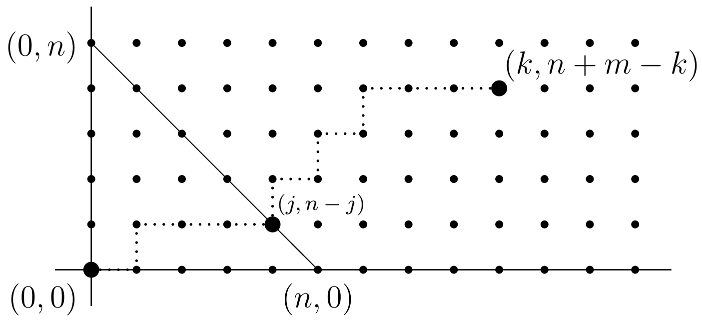
$$\begin{aligned} (x+y)^{n+m} &= (x+y)^n\,(x+y)^m \\ &= \sum_{j=0}^{n}\sum_{l=0}^{m} {}_w\!\begin{bmatrix} n \\ j \end{bmatrix} x^j y^{n-j}\, {}_w\!\begin{bmatrix} m \\ l \end{bmatrix} x^l y^{m-l} \\ &= \sum_{j=0}^{n}\sum_{l=0}^{m} {}_w\!\begin{bmatrix} n \\ j \end{bmatrix} x^j y^{n-j}\, {}_w\!\begin{bmatrix} m \\ l \end{bmatrix} y^{j-n} x^{-j}\, x^j y^{n-j} x^l y^{m-l}. \end{aligned} \tag{2.17}$$

Now use Lemma 1 to apply

$$x^j y^{n-j} x^l y^{m-l} = \bigg(\prod_{i=1}^{l} W(i+j, n-j)\bigg) x^{j+l} y^{n+m-j-l}, \quad \text{for } n \geq j,$$

and extract and equate (left) coefficients of $x^k y^{n+m-k}$ in (2.16) and (2.17). We thus immediately establish the convolution formula (2.13). ☐

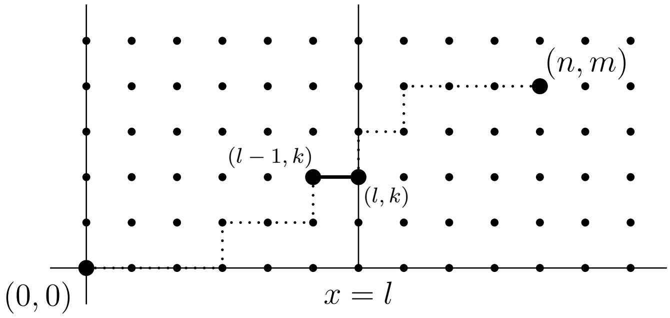
In terms of interpreting the weight-dependent binomial coefficients as generating functions for weighted lattice paths (see Eq. (2.5)), the identity (2.13) translates into a convolution of paths with respect to a *diagonal*,



where the corresponding generating function identity is immediately seen to be

$$\omega\big(\mathcal{P}((0,0)\to(k,n+m-k))\big) \\ = \sum_{j=0}^{\min(k,n)} \omega\big(\mathcal{P}((0,0)\to(j,n-j))\big)\,\omega\big(\mathcal{P}((j,n-j)\to(k,n+m-k))\big). \quad (2.18)$$

We can also consider convolution with respect to a *vertical line*,



which corresponds to the identity

$$\omega\big(\mathcal{P}((0,0)\to(n,m))\big) \\ = \sum_{k=0}^{m} \omega\big(\mathcal{P}((0,0)\to(l-1,k))\big)\,W(l,k)\,\omega\big(\mathcal{P}((l,k)\to(n,m))\big). \quad (2.19)$$

Here, l is fixed ($1\le l\le n$), while the nonnegative integer k is uniquely determined by the height of the path when it reaches the vertical line $x=l$ first.

In terms of our weights w, we have the following result.

Corollary 2 (Second weight-dependent binomial convolution formula)**.** *Let n, m, and k be nonnegative integers with $1\le l\le n$. For the binomial coefficients in* (2.3)*, defined by the doubly-indexed sequence of indeterminate weights $(w(s,t))_{s,t\in\mathbb{N}}$, we have the following formal identity in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{N}}]$:*

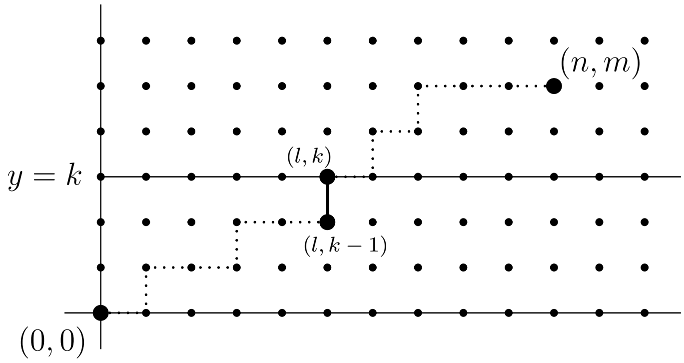
$${}_w\!\begin{bmatrix} n+m \\ n \end{bmatrix} = \sum_{k=0}^{m} {}_w\!\begin{bmatrix} k+l-1 \\ l-1 \end{bmatrix} \left(x^l y^k \, {}_w\!\begin{bmatrix} n+m-l-k \\ n-l \end{bmatrix} y^{-k} x^{-l} \right) \prod_{i=0}^{n-l} W(i+l,k). \quad (2.20)$$

Proof. Translating the generating function identity (2.19) into an identity in $\mathbb{C}_w[x,y]$ with weight function w, we immediately obtain

$${}_w\!\begin{bmatrix} n+m \\ n \end{bmatrix} x^n y^m = \sum_{k=0}^{m} {}_w\!\begin{bmatrix} k+l-1 \\ l-1 \end{bmatrix} W(l,k)\, x^l y^k \, {}_w\!\begin{bmatrix} n+m-l-k \\ n-l \end{bmatrix} x^{n-l} y^{m-k}.$$

The further analysis is now similar to the proof of Corollary 1. The y's have to be moved to the far right, then the x's, hereby creating weights and shifts, while in the end the left coefficients of $x^n y^m$ have to be extracted and equated to establish (2.20). □

Finally, we can also consider convolution with respect to a *horizontal line*,



which corresponds to the identity

$$\omega\big(\mathcal{P}((0,0)\to(n,m))\big)=\sum_{l=0}^{n}\omega\big(\mathcal{P}((0,0)\to(l,k-1))\big)\,\omega\big(\mathcal{P}((l,k)\to(n,m))\big). \tag{2.21}$$

Here, k is fixed ($1\le k\le m$), while the nonnegative integer l is the uniquely determined abscissa of the path when it reaches the horizontal line $y=k$ first.

In terms of our weights w, we have the following result.

Corollary 3 (Third weight-dependent binomial convolution formula)**.** *Let n, m, and k be nonnegative integers with $1\le k\le m$. For the binomial coefficients in (2.3), defined by the doubly-indexed sequence of indeterminate weights $(w(s,t))_{s,t\in\mathbb{N}}$, we have the following formal identity in $\mathbb{C}[(w(s,t))_{s,t\in\mathbb{N}}]$:*

$$ {}_w\!\begin{bmatrix}n+m\\ n\end{bmatrix}=\sum_{l=0}^{n}{}_w\!\begin{bmatrix}l+k-1\\ l\end{bmatrix}\left(x^ly^k\,{}_w\!\begin{bmatrix}n+m-l-k\\ n-l\end{bmatrix}y^{-k}x^{-l}\right)\prod_{i=1}^{n-l}W(i+l,k). \tag{2.22}$$

Proof. Translating the generating function identity (2.21) into an identity in $\mathbb{C}_w[x,y]$ with weight function w, we immediately obtain

$$ {}_w\!\begin{bmatrix}n+m\\ n\end{bmatrix}x^ny^m=\sum_{l=0}^{n}{}_w\!\begin{bmatrix}l+k-1\\ l\end{bmatrix}x^ly^{k-1}y\,{}_w\!\begin{bmatrix}n+m-l-k\\ n-l\end{bmatrix}x^{n-l}y^{m-k}.$$

The convolution formula (2.22) follows immediately after application of Lemma 1 (where we have $k\ge 1$, thus do not have to distinguish cases), moving first the y's, then the x's, to the right and finally extracting and equating the left coefficients of x^ny^m. ☐

3. Symmetric functions

Here we take a closer look at two important specializations of the weights $w(s,t)$, both involving symmetric functions. (See [12] for a classical textbook on symmetric function theory).

3.1. **Complete symmetric functions.** The first choice is $w(s,t)=a_t/a_{t-1}$. In this case we have (essentially corresponding to the h-labeling of lattice paths in [15, Sec. 4.5])

$$ {}_w\!\begin{bmatrix}n\\ k\end{bmatrix}=h_k(a_0,a_1,\dots,a_{n-k})\,a_0^{-k},$$

where $h_k(a_0, a_1, \dots, a_m)$ is the *complete symmetric function* of order k, defined by

$$h_0(a_0, a_1, \dots, a_m) := 1$$

and

$$h_k(a_0, a_1, \dots, a_m) := \sum_{0 \le j_1 \le j_2 \le \cdots \le j_k \le m} a_{j_1} a_{j_2} \cdots a_{j_k}, \qquad \text{for} \quad k > 0.$$

Indeed, the complete symmetric functions satisfy the recursion

$$h_k(a_0, a_1, \dots, a_{m+1}) = h_k(a_0, a_1, \dots, a_m) + a_{m+1} h_{k-1}(a_0, a_1, \dots, a_{m+1}),$$

which readily follows from specializing the recursion (2.3) for the weight-dependent binomial coefficients.

The relations of the algebra $\mathbb{C}_w[x, y]$ in Definition 1 now reduce to

$$yx = \frac{a_1}{a_0} xy, \tag{3.1a}$$

$$x \frac{a_t}{a_{t-1}} = \frac{a_t}{a_{t-1}} x, \tag{3.1b}$$

$$y \frac{a_t}{a_{t-1}} = \frac{a_{t+1}}{a_t} y, \tag{3.1c}$$

for all $t \in \mathbb{N}$.

The noncommutative binomial theorem in Theorem 1 now becomes

$$(x + y)^n = \sum_{k=0}^{n} h_k(a_0, a_1, \dots, a_{n-k})\, a_0^{-k}\, x^k y^{n-k}, \tag{3.2}$$

which, despite its simplicity, appears to be new.

From Corollary 1 and (3.1b)/(3.1c), the convolution

$$h_k(a_0, a_1, \dots, a_{n+m-k}) = \sum_{j=0}^{\min(k,n)} h_j(a_0, \dots, a_{n-j})\, h_{k-j}(a_{n-j}, \dots, a_{n+m-k}) \tag{3.3a}$$

is obtained. On the other hand, Corollaries 2 and 3 reduce to the identities

$$h_n(a_0, a_1, \dots, a_m) = \sum_{k=0}^{m} h_{l-1}(a_0, \dots, a_k)\, a_k\, h_{n-l}(a_k, \dots, a_m), \text{ for a fixed } 1 \le l \le n, \tag{3.3b}$$

and

$$h_n(a_0, a_1, \dots, a_m) = \sum_{l=0}^{n} h_l(a_0, \dots, a_{k-1})\, h_{n-l}(a_k, \dots, a_m), \text{ for a fixed } 1 \le k \le m, \tag{3.3c}$$

respectively. The identity in (3.3c) is the special case of a well-known convolution formula for Schur functions [12, p. 72, Eq. (5.10)], for which the indexing partitions are reduced to at most one row. The other two identities, (3.3a) and (3.3b), are most likely already known as well, although the author has not been able to find them explicitly in the literature. (It is in any case not very difficult to establish them directly by combinatorial arguments.)

3.2. **Elementary symmetric functions.** Another choice is $w(s,t) = a_{s+t}/a_{s+t-1}$. In this case we have (essentially corresponding to the e-labeling of lattice paths in [15, Sec. 4.5])

$$ {}_w\begin{bmatrix} n \\ k \end{bmatrix} = \frac{e_k(a_1,\dots,a_n)}{a_1\cdots a_k}, $$

where $e_k(a_1,\dots,a_n)$ is the *elementary symmetric function* of order k, defined by

$$ e_0(a_1,\dots,a_n) := 1 $$

and

$$ e_k(a_1,\dots,a_n) := \sum_{1\le j_1<j_2<\cdots<j_k\le n} a_{j_1}a_{j_2}\cdots a_{j_k}, \qquad \text{for} \quad k>0. $$

Indeed, the elementary symmetric functions satisfy the recursion

$$ e_k(a_1,\dots,a_{n+1}) = e_k(a_1,\dots,a_n) + a_{n+1}e_{k-1}(a_1,\dots,a_n), $$

which again readily follows from specializing (2.3).

The relations of the algebra $\mathbb{C}_w[x,y]$ in Definition 1 now reduce to

$$ yx = \frac{a_2}{a_1}xy, \tag{3.4a} $$

$$ x\frac{a_{s+t}}{a_{s+t-1}} = \frac{a_{s+t+1}}{a_{s+t}}x, \tag{3.4b} $$

$$ y\frac{a_{s+t}}{a_{s+t-1}} = \frac{a_{s+t+1}}{a_{s+t}}y, \tag{3.4c} $$

for all $s,t\in\mathbb{N}$.

The noncommutative binomial theorem in Theorem 1 now becomes

$$ (x+y)^n = \sum_{k=0}^{n} \frac{e_k(a_1,\dots,a_n)}{a_1\cdots a_k}\, x^k y^{n-k}, \tag{3.5} $$

which, despite its simplicity, appears to be new.

The convolutions in Corollaries 1, 2, and 3, respectively, give

$$ e_k(a_1,a_2,\dots,a_{n+m}) = \sum_{j=0}^{\min(k,n)} e_j(a_1,\dots,a_n)\, e_{k-j}(a_{n+1},\dots,a_{n+m}), \tag{3.6a} $$

$$ e_n(a_1,a_2,\dots,a_{n+m}) = \sum_{k=0}^{m} e_{l-1}(a_1,\dots,a_{l+k-1})\, a_{l+k}\, e_{n-l}(a_{l+k+1},\dots,a_{n+m}), \qquad \text{for a fixed } 1\le l\le n, \tag{3.6b} $$

and

$$e_n(a_1,a_2,\dots,a_{n+m}) = \sum_{l=0}^{n} e_l(a_1,\dots,a_{l+k-1})\, e_{n-l}(a_{l+k+1},\dots,a_{n+m}),$$
$$\text{for a fixed } 1 \le k \le m. \tag{3.6c}$$

The identity in (3.6a) is another special case (compare with (3.3c)) of the convolution formula for Schur functions in [12, p. 72, Eq. (5.10)], for which the indexing partitions are now reduced to at most one *column*. The other two identities, (3.6b) and (3.6c), are most likely already known as well, although the author has not been able to find them explicitly in the literature. (Again, it is not very difficult to establish them directly by combinatorial arguments.)

4. Elliptic hypergeometric series

In this section we concentrate on the so-called *elliptic* case. It was this case which, for the author, served as a motivation to look out for generalizations of the q-commuting variables (1.1).

We explain some important notions from the theory of elliptic hypergeometric series which we will use here (see also [6, Ch. 11]).

Let the modified Jacobi theta function with argument x and nome p be defined by

$$\theta(x) = \theta(x;p) := (x;p)_\infty (p/x;p)_\infty\,, \qquad \theta(x_1,\dots,x_m) = \prod_{k=1}^{m} \theta(x_k),$$

where $x, x_1, \dots, x_m \neq 0$, $|p| < 1$, and $(x;p)_\infty = \prod_{k=0}^{\infty}(1 - xp^k)$.

These functions satisfy the simple properties

$$\theta(x) = -x\,\theta(1/x), \tag{4.1a}$$

$$\theta(px) = -\frac{1}{x}\,\theta(x), \tag{4.1b}$$

and the three-term addition formula (cf. [22, p. 451, Example 5])

$$\theta(xy, x/y, uv, u/v) - \theta(xv, x/v, uy, u/y) = \frac{u}{y}\,\theta(yv, y/v, xu, x/u). \tag{4.2}$$

The relation (4.2) is not obvious but crucial for the theory of elliptic hypergeometric series. (Inductive proofs of summation formulae usually involve functional equations or recursions which are established by means of the theta addition formula.) A proof of (4.2) (and more general results) is given in [14, Eq. (3.4) being a special case of Lemma 3.3].

Now define the *theta shifted factorial* (or *q,p-shifted factorial*) by

$$(a;q,p)_n = \begin{cases} \prod_{k=0}^{n-1} \theta(aq^k), & n = 1,2,\dots, \\ 1, & n = 0, \\ 1/\prod_{k=0}^{-n-1} \theta(aq^{n+k}), & n = -1,-2,\dots. \end{cases}$$

For compact notation, we write

$$(a_1,a_2,\dots,a_m;q,p)_n = \prod_{k=1}^{m} (a_k;q,p)_n.$$

Notice that $\theta(x;0) = 1 - x$ and, hence, $(a;q,0)_n = (a;q)_n$ is a *q-shifted factorial* in base q.

Observe that

$$(pa;q,p)_n = (-1)^n a^{-n} q^{-\binom{n}{2}} (a;q,p)_n, \tag{4.3}$$

which follows from (4.1b). A list of other useful identities for manipulating the q,p-shifted factorials is given in [6, Sec. 11.2].

By definition, a function $g(u)$ is *elliptic* if it is a doubly-periodic meromorphic function of the complex variable u.

Without loss of generality, by the theory of theta functions, we may assume that

$$g(u) = \frac{\theta(a_1 q^u, a_2 q^u, \dots, a_s q^u; p)}{\theta(b_1 q^u, b_2 q^u, \dots, b_s q^u; p)}\, z$$

(i.e., an abelian function of some degree s), for a constant z and some $a_1, a_2, \dots, a_s$, $b_1, \dots, b_s$, and p, q with $|p| < 1$, where the *elliptic balancing condition* (cf. [18]), namely

$$a_1 a_2 \cdots a_s = b_1 b_2 \cdots b_s,$$

holds. If we write $q = e^{2\pi i\sigma}$, $p = e^{2\pi i\tau}$, with complex σ, τ, then $g(u)$ is indeed periodic in u with periods σ^{-1} and $\tau\sigma^{-1}$. Keeping this notation for p and q, denote the *field of elliptic functions* over $\mathbb{C}$ of the complex variable u, meromorphic in u with the two periods σ^{-1} and $\tau\sigma^{-1}$ by $\mathbb{E}_{q^u;q,p}$.

More generally, denote the *field of totally elliptic multivariate functions* over $\mathbb{C}$ of the complex variables $u_1, \dots, u_n$, meromorphic in each variable with equal periods, σ^{-1} and $\tau\sigma^{-1}$, of double periodicity, by $\mathbb{E}_{q^{u_1},\dots,q^{u_n};q,p}$.

After these prerequisites, we are ready to turn to our elliptic generalization of the q-binomial coefficient. (The corresponding elliptic weight function will come out automatically.) For indeterminates a, b, complex numbers q, p (with $|p| < 1$), and nonnegative integers n, k, define the *elliptic binomial coefficient* as follows (this is exactly the expression for $\omega(\mathcal{P}((0,0) \to (k, n-k)))$ in [16, Th. 2.1]):

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b; q, p)_{n-k}}{(q, aq, bq^{1+2k}, aq/b; q, p)_{n-k}}. \tag{4.4}$$

Note that this definition of the elliptic binomial coefficient (which reduces to the usual q-binomial coefficient after taking the limits $p \to 0$, $a \to 0$, and $b \to 0$, in this order) is different from the much simpler one given in [6, Eq. (11.2.61)], the latter being a straightforward theta shifted factorial extension of the q-binomial coefficient but actually *not* being elliptic. In fact, as pointed out in [16], it is not difficult to see that the expression in (4.4) is *totally elliptic*, i.e., elliptic in each of $\log_q a$, $\log_q b$, k, and n (viewed as complex parameters), with equal periods of double periodicity, which fully justifies the notion "elliptic". In particular, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{a,b;q,p} \in \mathbb{E}_{a,b,q^n,q^k;q,p}$.

It is immediate from the definition of (4.4) that (for integers n, k) we have

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n \\ n \end{bmatrix}_{a,b;q,p} = 1, \tag{4.5a}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} = 0, \qquad \text{whenever} \quad k = -1, -2, \dots, \quad \text{or} \quad k > n. \tag{4.5b}$$

Furthermore, using the theta addition formula in (4.2) one can verify the following recursion formula for the elliptic binomial coefficients:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{a,b;q,p} = \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} + \begin{bmatrix} n \\ k-1 \end{bmatrix}_{a,b;q,p} W_{a,b;q,p}(k, n+1-k), \tag{4.5c}$$

for nonnegative integers n and k, where

$$W_{a,b;q,p}(s,t) := \frac{\theta(aq^{s+2t}, bq^{2s}, bq^{2s-1}, aq^{1-s}/b, aq^{-s}/b)}{\theta(aq^s, bq^{2s+t}, bq^{2s+t-1}, aq^{1+t-s}/b, aq^{t-s}/b)} q^t. \tag{4.6}$$

Clearly, $W_{a,b;q,p}(s,0) = 1$, for all s. If we let $p \to 0$, $a \to 0$, then $b \to 0$ (in this order), the relations in (4.5) reduce to

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1,$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{n+1-k},$$

for positive integers n and k with $n \geq k$, which is a well-known recursion for the q-binomial coefficients.

According to (2.2b) we have for the small weights

$$w_{a,b;q,p}(s,t) := \frac{W_{a,b;q,p}(s,t)}{W_{a,b;q,p}(s,t-1)} = \frac{\theta(aq^{s+2t}, bq^{2s+t-2}, aq^{t-s-1}/b)}{\theta(aq^{s+2t-2}, bq^{2s+t}, aq^{t-s+1}/b)} q, \tag{4.7}$$

for $s, t \in \mathbb{N}$.

We refer to $w_{a,b;q,p}(s,t)$ (and to $W_{a,b;q,p}(s,t)$) as an *elliptic weight function.* Recall that in [16] lattice paths in the integer lattice $\mathbb{Z}^2$ were enumerated with respect to precisely this weight function. A similar weight function was subsequently used by A. Borodin, V. Gorin and E.M. Rains in [2, Sec. 10] (see in particular the expression obtained for $\frac{w(i,j+1)}{w(i,j)}$ on p. 780 of that paper) in the context of weighted lozenge tilings.

4.1. **An elliptic binomial theorem.** For the elliptic case, the commutation relations from Definition 1 are particularly elegant and can be formulated as follows. Recall (see the prerequisites we just covered in between Equations (4.3) and (4.4)) that $\mathbb{E}_{a,b;q,p}$ denotes the field of totally elliptic functions over $\mathbb{C}$, in the complex variables $\log_q a$ and $\log_q b$, with equal periods σ^{-1}, $\tau\sigma^{-1}$ (where $q = e^{2\pi i\sigma}$, $p = e^{2\pi i\tau}$, $\sigma, \tau \in \mathbb{C}$), of double periodicity.

Definition 2. For four noncommuting variables x, y, a, b, where a and b commute with each other, and two complex numbers q, p with $|p| < 1$, let $\mathbb{C}_{a,b;q,p}[x,y]$ denote the associative unital algebra over $\mathbb{C}$, generated by x, y and the set of all totally elliptic functions $\mathbb{E}_{a,b;q,p}$, satisfying the following three relations:

$$yx = \frac{\theta(aq^3, bq, a/bq; p)}{\theta(aq, bq^3, aq/b; p)} qxy, \tag{4.8a}$$

$$xf(a,b) = f(aq,bq^2)x, \tag{4.8b}$$

$$yf(a,b) = f(aq^2,bq)y, \tag{4.8c}$$

for all $f \in \mathbb{E}_{a,b;q,p}$.

We refer to the variables x, y, a, b forming $\mathbb{C}_{a,b;q,p}[x,y]$ as *elliptic-commuting* variables.

Notice that, in comparison with (2.1) the pair of positive integers (s,t) does not appear explicitly in the commutation relations (4.8). It is easy to verify that the actions of x, respectively y, on a weight $w(s,t)$ exactly correspond to shifts of the parameters a and b as described in (4.8b) and (4.8c).

The algebra $\mathbb{C}_{a,b;q,p}[x,y]$ reduces to $\mathbb{C}_q[x,y]$ if we formally let $p \to 0$, $a \to 0$, then $b \to 0$ (in this order), while (having eliminated the nome p) relaxing the condition of ellipticity.

As in (2.1), the relations in (4.8) are well-defined as any expression in $\mathbb{C}_{a,b;q,p}[x,y]$ can be put in a unique canonical form regardless in which order the commutation relations are applied for this purpose.

The generic weight-dependent noncommutative binomial theorem in Theorem 1 reduces now to the following identity.

Theorem 2 (Elliptic binomial theorem)**.** *Let $n \in \mathbb{N}_0$. Then*

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} x^k y^{n-k} \tag{4.9}$$

holds in $\mathbb{C}_{a,b;q,p}[x,y]$.

Remark 1. As Erik Koelink has kindly pointed out, a result very similar to Theorem 2 has been proved in [9, Eq. (3.5)], as an identity in the elliptic $U(2)$ quantum group (or, equivalently, the $\mathfrak{h}$-Hopf algebroid $\mathcal{F}_R(U(2))$). Nevertheless, although both results involve a "binomial" expansion of noncommuting variables, the correspondence between the two results is not entirely clear. It is possible, however, that such a correspondence would be easier to make out for another (yet to be established) version of elliptic binomial theorem in the framework of the more general situation in Appendix A with two weight functions v and w (where v and W contribute about the same number of factors).

4.2. **Frenkel and Turaev's ${}_{10}V_9$ summation.** *Elliptic hypergeometric series* are series $\sum_{k\ge 0} c_k$ where $c_0 = 1$ and $g(k) = c_{k+1}/c_k$ is an elliptic function of k, with k considered as a complex variable.

Elliptic hypergeometric series first appeared as elliptic solutions of the Yang–Baxter equation (or elliptic $6j$-symbols) in work by E. Date, M. Jimbo, A. Kuniba, T. Miwa, and M. Okado [3] in 1987, and a decade later by E. Frenkel and V. Turaev [5]. The latter authors were the first to find summation and transformation formulae satisfied by elliptic hypergeometric series. In particular, by exploiting the symmetries of the elliptic $6j$-symbols they derived the (now-called) ${}_{12}V_{11}$ transformation. By specializing this result they obtained the (now-called) ${}_{10}V_9$ summation (see also [6, Eq. (11.4.1)]), an identity which is fundamental to the theory of elliptic hypergeometric series.

Proposition 2 (Frenkel and Turaev's ${}_{10}V_9$ summation). *Let $n \in \mathbb{N}_0$ and $a, b, c, d, e, q, p \in \mathbb{C}$ with $|p| < 1$. Then we have*

$$\sum_{k=0}^{n} \frac{\theta(aq^{2k};p)}{\theta(a;p)} \frac{(a,b,c,d,e,q^{-n};q,p)_k}{(q,aq/b,aq/c,aq/d,aq/e,aq^{n+1};q,p)_k} q^k = \frac{(aq,aq/bc,aq/bd,aq/cd;q,p)_n}{(aq/b,aq/c,aq/d,aq/bcd;q,p)_n}, \tag{4.10}$$

where $a^2q^{n+1} = bcde$.

For $p = 0$, the ${}_{10}V_9$ summation reduces to *Jackson's ${}_8\phi_7$ summation* [6, Eq. (II.22)]. Interestingly, the ${}_{10}V_9$ stands at the "*bottom*" of the hierarchy of identities that are direct elliptic extensions of any of the classical basic hypergeometric series identities listed in Appendices II and III of Gasper and Raman's book [6]. We point out that, while one cannot take confluent limits in an elliptic hypergeometric series, identities of lower order can be obtained by suitably specializing the parameters. In particular, the specialization $e = aq/d$ in (4.10) leads to an ${}_8V_7$ summation. The systematic study of elliptic hypergeometric series commenced at about the turn of the millennium, after further pioneering work of V.P. Spiridonov and A.S. Zhedanov [20], and of S.O. Warnaar [21].

By the elliptic specialization of the convolution formula in Corollary 1, we recover Frenkel and Turaev's [5] ${}_{10}V_9$ summation in the following form (where the requirement of n and m being nonnegative integers can be removed by repeated analytic continuation).

Corollary 4. *Let $n, m, k \in \mathbb{N}_0$ and $a, b, q, p \in \mathbb{C}$ with $|p| < 1$. Then we have*

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_{a,b;q,p} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p} \begin{bmatrix} m \\ k-j \end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p} \prod_{i=1}^{k-j} W_{a,b;q,p}(i+j,n-j), \tag{4.11}$$

where the elliptic binomial coefficients and the weight function $W_{a,b;q,p}$ are defined in (4.5) *and* (4.6).

To see the correspondence with Proposition 2, replace the summation index k in Equation (4.10) by j and substitute the 6-tuple of parameters (a, b, c, d, e, n) appearing in Equation (4.10) by $(bq^{-n}/a, q^{-n}/a, bq^{1+n+m}, bq^{-n-m+k}/a, q^{-n}, k)$. (This substitution is reversible if q^{-n} and q^{-m} are treated as complex variables. This is fine, as the terminating parameter has changed from n to k.) The resulting summation can be written, after some elementary manipulations of theta shifted factorials, exactly in the form of Equation (4.11).

Interestingly, Corollaries 2 and 3 also yield essentially the same result, namely variants of Frenkel and Turaev's ${}_{10}V_9$ summation. In particular, from Corollary 2 we obtain the identity

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_{a,b;q,p} = \sum_{k=0}^{m} \begin{bmatrix} k+l-1 \\ l-1 \end{bmatrix}_{a,b;q,p} \begin{bmatrix} n+m-l-k \\ n-l \end{bmatrix}_{aq^{l+2k},bq^{2l+k};q,p} \prod_{i=0}^{n-l} w_{a,b;q,p}(i+l,k)$$

(which is the $(a,b,c,d,e,n) \mapsto (aq^l, bq^l, aq^{1+n+m}, aq^{-n}/b, q^l, m)$ case of Equation (4.10)), where the requirement of n and l being nonnegative integers can be removed by repeated analytic continuation.

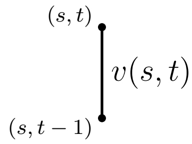
On the other hand, from Corollary 3 we obtain

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_{a,b;q,p} = \sum_{l=0}^{n} \begin{bmatrix} l+k-1 \\ l \end{bmatrix}_{a,b;q,p} \begin{bmatrix} n+m-l-k \\ n-l \end{bmatrix}_{aq^{l+2k}, bq^{2l+k};q,p} \prod_{i=1}^{n-l} w_{a,b;q,p}(i+l,k)$$

(which is the ($k \mapsto l$, then) $(a,b,c,d,e,n) \mapsto (bq^k, aq^k, bq^{1+n+m}, bq^{-m}/a, q^k, n)$ case of Equation (4.10)), where again the requirement of m and k being nonnegative integers can be removed by repeated analytic continuation.

Appendix A. A generalization involving an additional weight function

A substantial amount of the analysis of Section 2 can be readily generalized to the situation where one not only has weights $w(s,t)$ attributed to the horizontal steps but also additional indeterminate weights $v(s,t)$ on the vertical steps. More precisely, the weight of a vertical step in the (first quadrant of the) integer lattice $\mathbb{Z}^2$ from $(s,t-1)$ to (s,t) shall be $v(s,t)$.



For instance, the path P_0 from Subsection 2.2 now has the weight

$$\begin{aligned} \omega(P_0) &= 1 \cdot 1 \cdot v(2,1) \cdot W(3,1) \cdot W(4,1) \cdot v(4,2) \cdot W(5,2) \\ &= v(2,1)w(3,1)w(4,1)v(4,2)w(5,1)w(5,2). \end{aligned}$$

Keeping the other notions from Section 2, let us describe how the results look like in this generalized setting.

First we have the following extension of the noncommutative algebra $\mathbb{C}_w[x;y]$:

Definition 3. For two doubly-indexed sequences of indeterminates $(v(s,t))_{s,t\in\mathbb{N}}$ and $(w(s,t))_{s,t\in\mathbb{N}}$, let $\mathbb{C}_{v,w}[x,y]$ be the associative unital algebra over $\mathbb{C}$ generated by x,y and the $v(s,t)$ and $w(s,t)$, $s,t \in \mathbb{N}$, satisfying the following five relations:

$$yx = w(1,1)\, xy, \tag{A.1a}$$

$$x\, v(s,t) = v(s+1,t)\, x, \tag{A.1b}$$

$$x\, w(s,t) = w(s+1,t)\, x, \tag{A.1c}$$

$$y\, v(s,t) = v(s,t+1)\, y, \tag{A.1d}$$

$$y\, w(s,t) = w(s,t+1)\, y, \tag{A.1e}$$

for all $(s,t) \in \mathbb{N}^2$.

As in Subsection 2.2, we define the big weight $W(s,t)$ to be the product $\prod_{j=1}^{t} w(s,j)$ of the small w-weights.

Let the *double weight-dependent binomial coefficients* be defined by

$$ {}_{v,w}\begin{bmatrix}0\\0\end{bmatrix}=1, \qquad {}_{v,w}\begin{bmatrix}n\\k\end{bmatrix}=0 \qquad \text{for } n\in\mathbb{N}_0, \text{ and } k\in-\mathbb{N} \text{ or } k>n, \tag{A.2a} $$

and

$$ {}_{v,w}\begin{bmatrix}n+1\\k\end{bmatrix} = {}_{v,w}\begin{bmatrix}n\\k\end{bmatrix} v(k,n+1-k) + {}_{v,w}\begin{bmatrix}n\\k-1\end{bmatrix} W(k,n+1-k) \quad \text{for } n,k\in\mathbb{N}_0. \tag{A.2b} $$

It is obvious that the double weight-dependent binomial coefficients ${}_{v,w}\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]$ have again a nice combinatorial interpretation in terms of weighted lattice paths. The generating function $\omega_{v,w}$ with respect to the weights v and w of all paths from $(0,0)$ to $(k,n-k)$ is clearly

$$ \omega_{v,w}(\mathcal{P}((0,0)\to(k,n-k))) = {}_{v,w}\begin{bmatrix}n\\k\end{bmatrix}. \tag{A.3} $$

The noncommutative binomial theorem in Theorem 1 extends to the following.

Theorem 3 (Double weight-dependent binomial theorem)**.** *Let $n\in\mathbb{N}_0$. Then*

$$ (x+v(0,1)\,y)^n = \sum_{k=0}^{n} {}_{v,w}\begin{bmatrix}n\\k\end{bmatrix} x^k y^{n-k}. \tag{A.4} $$

holds in $\mathbb{C}_{v,w}[x,y]$.

The proof of this theorem is a simple extension of the proof of Theorem 1.

The double weight-dependent binomial coefficients ${}_{v,w}\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]$ have the advantage that they also cover various (generalizations of) important sequences. In particular, the (q-)*Stirling numbers of the first kind* arise when $v(s,t)=1-s-t$ (respectively $v(s,t)=(q^{s+t-1}-1)/(1-q)$) and $w(s,t)=1$, for all $s,t\in\mathbb{Z}$, whereas the (q-)*Stirling numbers of the second kind* arise when $v(s,t)=s$ (respectively $v(s,t)=(1-q^s)/(1-q)$) and $w(s,t)=1$, for all $s,t\in\mathbb{Z}$.

Notice that the algebra $\mathbb{C}_{v,w}[x,y]$ is not very interesting when $w(s,t)=1$ for all $s,t\in\mathbb{Z}$. It is certainly worthwhile to look for nontrivial (and "nice") applications of Theorem 3 for suitable choices of the weight functions v and w (where none of them is the identity function). This is not pursued further here, the focus being laid on symmetric functions, elliptic hypergeometric series and some basic hypergeometric specializations.

Appendix B. Basic hypergeometric specializations

Here some particularly attractive specializations of the elliptic weights $w_{a,b;q,p}(s,t)$ from Section 4 are considered. The corresponding binomial coefficients and associated commutation relations are given explicitly, while the summations that are obtained by convolution are identified.

For some standard terminology related to basic hypergeometric series, in particular the terms *balanced*, *well-poised*, *very-well-poised*, and the definition of an ${}_r\phi_s$ *basic hypergeometric series*, the reader is referred to the classical textbook [6].

B.1. **The balanced very-well-poised case.** If we specialize the elliptic weight function in (4.7) by letting $p \to 0$, we obtain the weights

$$w_{a,b;q}(s,t) = \frac{(1-aq^{s+2t})(1-bq^{2s+t-2})(1-aq^{t-s-1}/b)}{(1-aq^{s+2t-2})(1-bq^{2s+t})(1-aq^{t-s+1}/b)}q, \tag{B.1a}$$

the associated big weights being

$$W_{a,b;q}(s,t) = \frac{(1-aq^{s+2t})(1-bq^{2s})(1-bq^{2s-1})(1-aq^{1-s}/b)(1-aq^{-s}/b)}{(1-aq^{s})(1-bq^{2s+t})(1-bq^{2s+t-1})(1-aq^{1+t-s}/b)(1-aq^{t-s}/b)}q^t. \tag{B.1b}$$

The corresponding binomial coefficients are

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q} = \frac{(q^{1+k}, aq^{1+k}, bq^{1+k}, aq^{1-k}/b; q)_{n-k}}{(q, aq, bq^{1+2k}, aq/b; q)_{n-k}}, \tag{B.2}$$

where we are using the suggestive compact notation $(a_1, \dots, a_m; q)_j = \prod_{l=1}^m (a_l; q)_j$ for products of q-shifted factorials.

Now, in the unital algebra $\mathbb{C}_{a,b;q}[x,y]$ over $\mathbb{C}$ defined by the five commutation relations

$$yx = \frac{(1-aq^3)(1-bq)(1-a/bq)}{(1-aq)(1-bq^3)(1-aq/b)}qxy, \tag{B.3a}$$

$$xa = qax, \tag{B.3b}$$

$$xb = q^2bx, \tag{B.3c}$$

$$ya = q^2ay, \tag{B.3d}$$

$$yb = qby, \tag{B.3e}$$

the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q} x^k y^{n-k} \tag{B.4}$$

holds. Convolution yields *Jackson's balanced very-well-poised terminating* ${}_8\phi_7$ *summation* [6, Appendix (II.22)] (which of course is the $p \to 0$ case of Frenkel and Turaev's ${}_{10}V_9$ summation, see Corollary 4 and the two identities appearing thereafter).

B.2. **The balanced case.** If in (B.1a) we let $a \to 0$, we obtain the weights

$$w_{0,b;q}(s,t) = \frac{(1-bq^{2s+t-2})}{(1-bq^{2s+t})}q, \tag{B.5a}$$

the associated big weights being

$$W_{0,b;q}(s,t) = \frac{(1-bq^{2s})(1-bq^{2s-1})}{(1-bq^{2s+t})(1-bq^{2s+t-1})}q^t. \tag{B.5b}$$

The corresponding binomial coefficients are

$$\begin{bmatrix} n \\ k \end{bmatrix}_{0,b;q} = \frac{(q^{1+k}, bq^{1+k}; q)_{n-k}}{(q, bq^{1+2k}; q)_{n-k}}. \tag{B.6}$$

Now, in the unital algebra $\mathbb{C}_{0,b;q}[x,y]$ over $\mathbb{C}$ defined by the three commutation relations

$$yx = \frac{(1-bq)}{(1-bq^3)}qxy, \tag{B.7a}$$

$$xb = q^2bx, \tag{B.7b}$$

$$yb = qby, \tag{B.7c}$$

the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{0,b;q} x^k y^{n-k} \tag{B.8}$$

holds. Convolution yields the *balanced q-Pfaff–Saalschütz summation* [6, Appendix (II.12)], which is a summation for a balanced terminating ${}_3\phi_2$ series. It is clear that reducing the weight in (B.5a) yet further by letting $b \to 0$, one arrives at the standard q-weight connected to q-commuting variables. In this case, convolution gives the *q-Chu–Vandermonde summation* [6, Appendix (II.6)/(II.7)], a summation for a terminating ${}_2\phi_1$ series.

B.3. **The very-well-poised case.** If in (B.1a) we let $b \to 0$, we obtain the weights

$$w_{a,0;q}(s,t) = \frac{(1-aq^{s+2t})}{(1-aq^{s+2t-2})}q^{-1}, \tag{B.9a}$$

the associated big weights being

$$W_{a,0;q}(s,t) = \frac{(1-aq^{s+2t})}{(1-aq^{s})}q^{-t}. \tag{B.9b}$$

The corresponding binomial coefficients are

$$\begin{bmatrix} n \\ k \end{bmatrix}_{a,0;q} = \frac{(q^{1+k}, aq^{1+k}; q)_{n-k}}{(q, aq; q)_{n-k}} q^{k(k-n)}. \tag{B.10}$$

Now, in the unital algebra $\mathbb{C}_{a,0;q}[x,y]$ over $\mathbb{C}$ defined by the three commutation relations

$$yx = \frac{(1-aq^3)}{(1-aq)}q^{-1}xy, \tag{B.11a}$$

$$xa = qax, \tag{B.11b}$$

$$ya = q^2ay, \tag{B.11c}$$

the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{a,0;q} x^k y^{n-k} \tag{B.12}$$

holds. Convolution yields the *very-well-poised terminating ${}_6\phi_5$ summation* [6, Appendix (II.21)]. It is clear that the further $a \to \infty$ limit of (B.9a) leads again to the classical case of q-commuting variables (whereas $a \to 0$ leads to the same with q replaced by q^{-1}).

References

[1] G.M. Bergman, “The diamond lemma for ring theory”, *Adv. Math.* **29** (1978), 178–218.

[2] A. Borodin, V. Gorin, E.M. Rains, “q-Distributions on boxed plane partitions”, *Selecta Math. (N.S.)* **16** (2010), no. 4, 731–789.

[3] E. Date, M. Jimbo, A. Kuniba, T. Miwa, M. Okado, “Exactly solvable SOS models: local height probabilities and theta function identities”, *Nuclear Phys. B* **290** (1978), 231–273.

[4] P. Etingof and O. Schiffmann, “Lectures on the dynamical Yang–Baxter equations”, in A. Pressley (ed.), Quantum groups and Lie theory, pp. 89–129, Cambridge University Press, Cambridge, 2001.

[5] I.B. Frenkel and V.G. Turaev, “Elliptic solutions of the Yang–Baxter equation and modular hypergeometric functions”, in V.I. Arnold et al. (eds.), *The Arnold–Gelfand Mathematical Seminars*, pp. 171–204, Birkhäuser, Boston, 1997.

[6] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, second edition, Encyclopedia of Mathematics and Its Applications **96**, Cambridge University Press, Cambridge, 2004.

[7] O. Holtz, V. Mehrmann, H. Schneider, “Potter, Wielandt, and Drazin on the matrix equation $AB = \omega BA$: new answers to old questions”, *Amer. Math. Monthly* **111** (2004), no. 8, 655–667.

[8] E. Koelink, “Eight lectures on quantum groups and q-special functions”, *Rev. Colombiana Mat.* **30** (1996), no. 2, 93–180.

[9] E. Koelink, Y. van Norden and H. Rosengren “U(2) quantum group and elliptic hypergeometric series”, *Comm. Math. Phys.* **245** (2004), no. 3, 519–537.

[10] M. Konvalinka and I. Pak, “Non-commutative extensions of the MacMahon Master Theorem”, *Adv. Math.* **216** (2007), no. 1, 29–61.

[11] T.H. Koornwinder, “Special functions and q-commuting variables”, in *Special functions, q-series and related topics* (Toronto, ON, 1995), pp. 131–166, *Fields Inst. Commun.* **14**, Amer. Math. Soc., Providence, RI, 1997.

[12] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Clarendon Press, Oxford, 1995.

[13] H.S.A. Potter, “On the latent roots of quasi-commutative matrices”, *Amer. Math. Monthly* **57** (1950), 321–322.

[14] H. Rosengren and M.J. Schlosser, “Elliptic determinant evaluations and the Macdonald identities for affine root systems”, *Compos. Math.* 142 (2006), no. 4, 937–961.

[15] B.E. Sagan, *The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions*, second edition, Graduate Texts in Mathematics **203**, Springer–Verlag, New York, 2001.

[16] M.J. Schlosser, “Elliptic enumeration of nonintersecting lattice paths”, *J. Combin. Theory Ser. A* **114** (3) (2007), 505–521.

[17] M.-P. Schützenberger, “Une interprétation de certaines solutions de l’équation fonctionnelle: $F(x + y) = F(x)F(y)$”. *C. R. Acad. Sci. Paris* **236** (1953), 352–353.

[18] V.P. Spiridonov, “Theta hypergeometric series”, in V.A. Malyshev and A.M. Vershik (eds.), *Asymptotic Combinatorics with Applications to Mathematical Physics*, pp. 307–327, Kluwer Acad. Publ., Dordrecht, 2002.

[19] V.P. Spiridonov, “Essays on the theory of elliptic hypergeometric functions”, Russian Math. Surveys **63** (2008), 405–472.

[20] V.P. Spiridonov and A.S. Zhedanov, “Spectral transformation chains and some new biorthogonal rational functions”, *Comm. Math. Phys.* **10** (2000), 49–83.

[21] S.O. Warnaar, “Summation and transformation formulas for elliptic hypergeometric series”, *Constr. Approx.* **18** (2002), 479–502.

[22] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, 4th edition, Cambridge University Press, Cambridge, 1962.

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