Cubic realizations of Tamari interval lattices

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Contents

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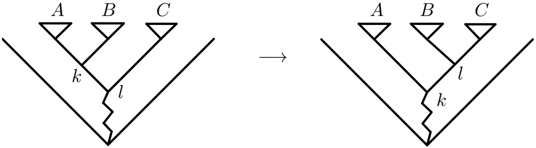
Tamari lattices

Tamari posets [**Tamari**, 1962]:

- ⋆ objects: binary trees with n leaves,
- ⋆ covering relation: right rotation:



- ⋆ partial order relation: $\leqslant_{\mathrm{t}}$.

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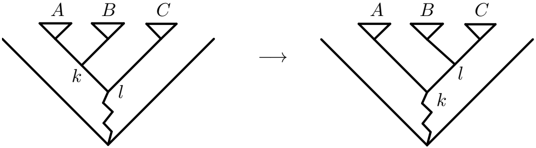

- ⋆ partial order relation: $\leqslant_{\mathrm{t}}$.

Known facts: they are lattices, formula for their number of intervals, admit generalizations (m-Tamari), *etc.*

Tamari interval lattices

Tamari interval posets:

- $\star$ objects: pairs of binary trees $[S, T]$ such that $S \leqslant_{\mathrm{t}} T$,
- $\star$ partial order relation: $\leqslant_{\mathrm{ti}}$:

$$[S, T] \leqslant_{\mathrm{ti}} [S', T'] \iff S \leqslant_{\mathrm{t}} S' \text{ and } T \leqslant_{\mathrm{t}} T'.$$

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Known facts: they are also lattices, their objects are encoded by interval-posets, *etc.*

Work context

Goal: study Tamari interval posets.

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Way: introduce a new encoding of Tamari intervals.

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Results:

- ⋆ simple representation of Tamari intervals,
- ⋆ easy reading of some properties of Tamari intervals,
- ⋆ geometric realization of the lattice.

Contents

Interval-posets

An interval-poset P of size n is a partial order $\lhd$ on the set $\{x_1, \ldots, x_n\}$ such that, for any $i < k$,

(i) if $x_k \lhd x_i$ then for all x_j such that $i < j < k$, one has $x_j \lhd x_i$,

(ii) if $x_i \lhd x_k$ then for all x_j such that $i < j < k$, one has $x_j \lhd x_k$.

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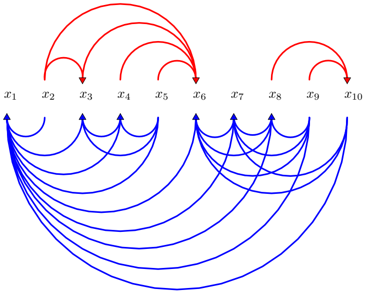

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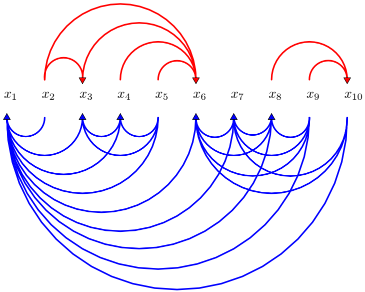

There is a bijection $\rho : \mathcal{IP}_n \to \mathcal{TI}_n$ [**Châtel, Pons**, 2015].

Tamari diagrams

A Tamari diagram is a word $u = u_1 u_2 \ldots u_n$ of integers such that

(i) $0 \leqslant u_i \leqslant n - i$ for all $i \in [n]$;

(ii) $u_{i+j} \leqslant u_i - j$ for all $i \in [n]$ and $0 \leqslant j \leqslant u_i$.

The size of a Tamari diagram is its number of letters [**Palo**, 1986].

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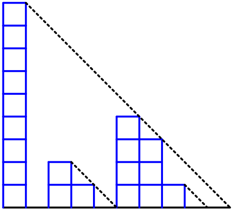

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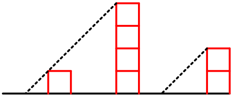

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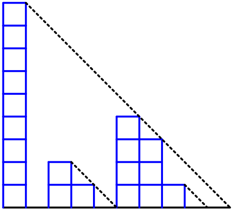

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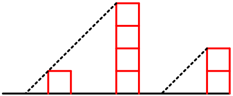

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A word $v = v_1 v_2 \dots v_n$ is a dual Tamari diagram if and only if its reversal is a Tamari diagram.

Compatibility

Let u (resp. v) be a (resp. dual) Tamari diagram of size n.

The diagrams u and v are compatible if $j - i \leqslant u_i$ implies $v_j < j - i$, for all $1 \leqslant i < j \leqslant n$.

In this case, (u, v) is a Tamari interval diagram.

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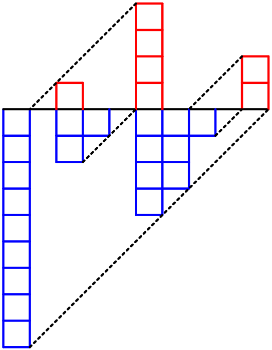

Bijection

Let χ be the map sending a Tamari interval diagram (u, v) of size n to the binary relation $\lhd$ on $\{x_1, \ldots, x_n\}$ where for all $i \in [n]$ and $0 \leqslant l \leqslant u_i$, $x_{i+l} \lhd x_i$, and for all $i \in [n]$ and $0 \leqslant k \leqslant v_i$, $x_{i-k} \lhd x_i$.

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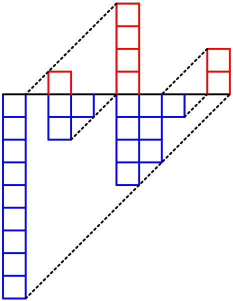

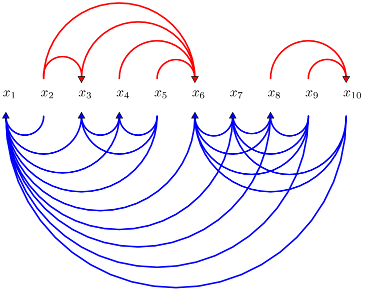

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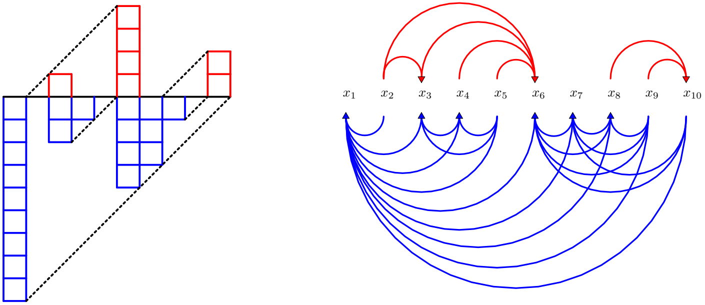

Theorem [C., 2019]

The map χ is a bijection from $\mathcal{TID}_n$ to $\mathcal{IP}_n$.

Cubic coordinates

Let c be a $(n-1)$-tuple with entries in $\mathbb{Z}$. We say that c is a cubic coordinate if the pair (u, v), where u is the word defined by $u_n = 0$ and for all $i \in [n-1]$ by

$$u_i = \max(c_i,\ 0),$$

and v is the word defined by $v_1 = 0$ and for all $2 \leqslant i \leqslant n$ by

$$v_i = |\min(c_{i-1},\ 0)|,$$

is a Tamari interval diagram. The size of a cubic coordinate is its number of entries plus one. The set of cubic coordinates of size n is denoted by $\mathcal{CC}_n$.

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Example

$$\begin{array}{l} v = 0\ 0\ 1\ 0\ 0\ 4\ 0\ 0\ 0\ 2 \\ u = 9\ 0\ 2\ 1\ 0\ 4\ 3\ 1\ 0\ 0 \end{array} \quad \xrightarrow{u_i - v_{i+1}} \quad (9, -1, 2, 1, -4, 4, 3, 1, -2).$$

Some properties

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- ⋆ A cubic coordinate c of size n is synchronized if for all $i \in [n-1]$, $c_i \neq 0$. The set of synchronized cubic coordinates of size n is denoted by $\mathcal{CC}_n^{sync}$. (synchronized Tamari interval, [**Préville-Ratelle, Viennot**, 2017])

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- A Tamari interval diagram (u, v) of size n is new if the following conditions are satisfied:

  (i) $0 \leqslant u_i \leqslant n - i - 1$ for all $i \in [n-1]$;
  (ii) $0 \leqslant v_j \leqslant j - 2$ for all $j \in \{2, \ldots, n\}$;
  (iii) $u_k < l - k - 1$ or $v_l < l - k - 1$ for all $k, l \in [n]$ such that $k + 1 < l$.

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- ⋆ If (u, v) is synchronized then (u, v) is not new.

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Cubic coordinate posets

Let $c, c' \in \mathcal{CC}_n$.

Partial order: $c \leqslant_{\mathrm{cc}} c'$ if and only if $c_i \leqslant c'_i$ for all $i \in [n-1]$.

Covering relation: $c \lessdot c'$ if and only if there is exactly one $i \in [n-1]$ such that $c_i < c'_i$, and if there is a $c'' \in \mathcal{CC}_n$ such that $c \leqslant_{\mathrm{cc}} c'' \leqslant_{\mathrm{cc}} c'$, then either $c = c''$ or $c' = c''$.

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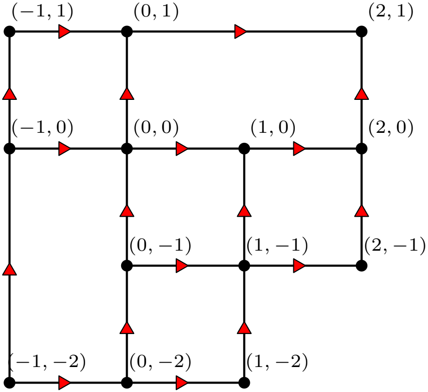
$$\begin{array}{ccc} \mathcal{TID}_n & \xrightarrow{\chi} & \mathcal{IP}_n \\ \uparrow \phi & & \downarrow \rho \\ \mathcal{CC}_n & \xleftarrow{\psi} & \mathcal{TI}_n \end{array}$$

Cubic realization of $\mathcal{CC}_3$



The elements of $\mathcal{CC}_3$ are vertices and the cover relations are arrows orientated to the covering cubic coordinates.

Cells

Let $c \in \mathcal{CC}_n$. Suppose that there is $c' \in \mathcal{CC}_n$ such that $c'_i > c_i$ and $c'_j = c_j$ for all $j \neq i$, with $i, j \in [n-1]$. We define then the map of minimal increase $\uparrow_i$ as follows

$$\uparrow_i (c) = (c_1, \ldots, c_{i-1}, \widehat{c}_i, c_{i+1}, \ldots, c_{n-1}),$$

such that $c \lessdot \uparrow_i (c)$ and $c_i < \widehat{c}_i \leqslant c'_i$.

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Example

$c^m = (0, -1, 1, -1, -5, 0, 1, -1, -3)$ is minimal-cellular.

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Lemma

Let c^m be a minimal-cellular cubic coordinate of size n and $i \in [n-1]$. If

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Let $c^M \in \mathcal{CC}_n$, then c^M is the maximal-cellular correspondent of c^m if

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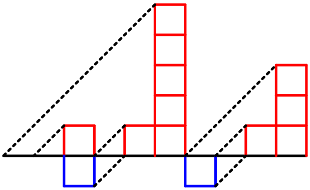
We denote by $\langle c^m, c^M \rangle$ the corresponding cell.

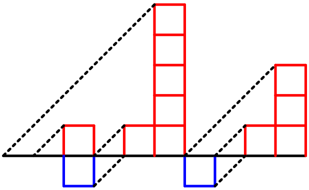
Cells

Example

$c^m = (0, -1, 1, -1, -5, 0, 1, -1, -3)$ is minimal-cellular, and its maximal-cellular correspondent is $c^M = (1, 0, 2, 0, -4, 3, 2, 0, -2)$.



c^m



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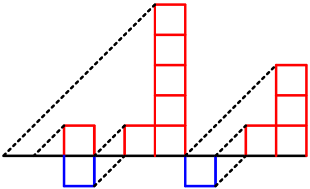

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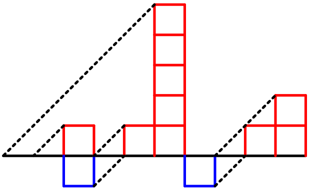

$\uparrow_9 (c^m)$

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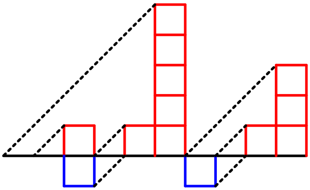

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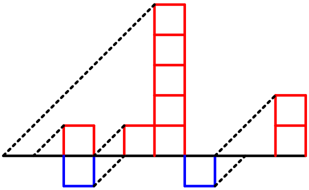

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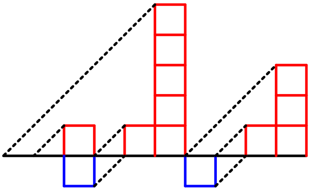
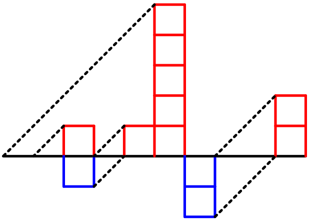


c^m



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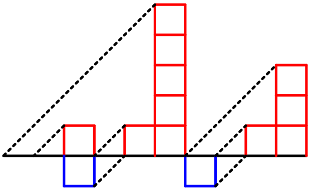

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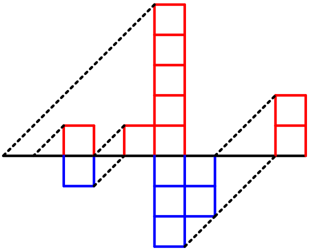

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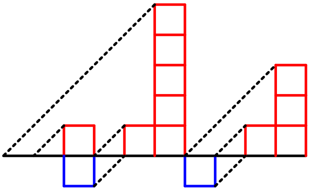
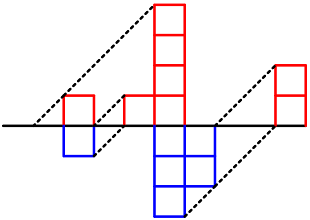


c^m

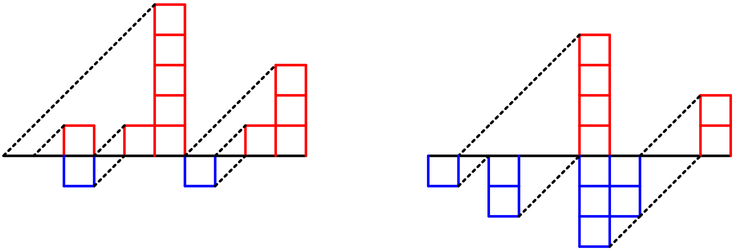


$\uparrow_5 (\uparrow_6 (\uparrow_7 (\uparrow_8 (\uparrow_9 (c^m)))))$

Cells

Example

$c^m = (0, -1, 1, -1, -5, 0, 1, -1, -3)$ is minimal-cellular, and its maximal-cellular correspondent is $c^M = (1, 0, 2, 0, -4, 3, 2, 0, -2)$.



c^m c^M

Bijection

Let γ be the map defined for all $i \in [n-1]$ by

$$\gamma(c_i^m, c_i^M) = \begin{cases} c_i^m & \text{if } c_i^m < 0, \\ c_i^M & \text{if } c_i^m \geqslant 0, \end{cases}$$

and Γ be the map from the set of cells of size n to the set of $(n-1)$-tuples defined by

$$\Gamma(\langle c^m, c^M \rangle) = (\gamma(c_1^m, c_1^M), \gamma(c_2^m, c_2^M), \dots, \gamma(c_{n-1}^m, c_{n-1}^M)).$$

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Example

The cell $\langle (0, -1, 1, -1, -5, 0, 1, -1, -3), (1, 0, 2, 0, -4, 3, 2, 0, -2) \rangle$ is sent to $(1, -1, 2, -1, -5, 3, 2, -1, -3)$.

Bijection

Let γ be the map defined for all $i \in [n-1]$ by

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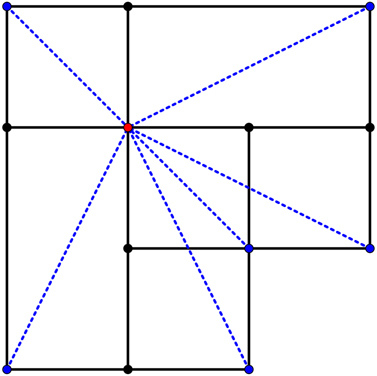
Example

The cell $\langle (0,-1,1,-1,-5,0,1,-1,-3), (1,0,2,0,-4,3,2,0,-2) \rangle$ is sent to $(1,-1,2,-1,-5,3,2,-1,-3)$.

Theorem [C., 2019]

The map Γ is a bijection from the set of cells of size n to $\mathcal{CC}_n^{sync}$.

Cells and synchronized



- ⋆ Blue dots: synchronized cubic coordinates.
- ⋆ Red dot: cubic coordinate $(0, 0)$.

EL-Shellability

Generalization of Björner and Wachs results on Tamari:

Let $c, c' \in \mathcal{CC}_n$ such that $c \lessdot c'$ with $c_i < c'_i$ for $i \in [n-1]$. Let $\lambda : \mathcal{E}(\mathcal{CC}_n) \to \mathbb{Z}^3$ the edge-labeling:

$$\lambda(c, c') = (\varepsilon, i, c_i),$$

where $\varepsilon = \begin{cases} -1 & \text{if } c_i < 0, \\ 1 & \text{otherwise.} \end{cases}$

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Theorem [C., 2019]

The map λ gives an EL-labeling of $\mathcal{CC}_n$. Moreover, there is at most one falling chain in each interval of $\mathcal{CC}_n$.

Cubic realization of $\mathcal{CC}_4$