SYMMETRIC FUNCTIONS AND TOEPLITZ+HANKEL MATRICES

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Joint work with Miguel Tierz (arXiv:1706.02574 and 1901.08922)

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We say that a matrix is Toeplitz or Hankel if it is constant along its diagonals or antidiagonals, respectively

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We will denote these matrices by

$$T_N(f) = (f_{j-k})_{j,k=1}^{N}, \qquad H_N(f) = (f_{j+k-1})_{j,k=1}^{N},$$

where $f(z) = \sum_{k\in\mathbb{Z}} f_k z^k$.

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where $f(z) = \sum_{k\in\mathbb{Z}} f_k z^k$.

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T+H MATRICES IN COMBINATORICS AND REPRESENTATION THEORY

Toeplitz and Hankel matrices appear in many different contexts, such as random matrices, orthogonal polynomials, statistical mechanics...

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- Jacobi-Trudi identities:

$$s_\mu(x) = \det\left(h_{j-k+\mu_k}(x)\right)_{j,k=1}^{l(\mu)} = \det\left(e_{j-k+\mu'_k}(x)\right)_{j,k=1}^{\mu'_1}.$$

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FACTORIZATIONS OF CHARACTERS INDEXED BY RECTANGULAR SHAPES

Choose the following function f in the matrices above

$$f(z) = \prod_{j=1}^{K}(1 + x_j z)(1 + x_j z^{-1}) = \left(\prod_{j=1}^{K} x_j\right) \sum_{j=-K}^{K} e_{K+j}(x_1, \ldots, x_K, x_1^{-1}, \ldots, x_K^{-1}) z^j.$$

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Theorem (Ciucu-Krattenthaler'09, GG-Tierz'19)

Consider a finite set of variables $x = (x_1, x_2, \ldots, x_K)$. The following relations hold between the symmetric functions associated to the characters of the groups $G(N)$:

$$\begin{aligned} s_{(2N-1)^K}(x,x^{-1}) &= sp_{(N-1)^K}(x)o^{even}_{(N^K)}(x) \\ &= \frac{(-1)^{NK}}{2}\left[o^{odd}_{(N-1)^K}(x)o^{odd}_{(N^K)}(-x) + o^{odd}_{(N^K)}(x)o^{odd}_{(N-1)^K}(-x)\right], \\ s_{(2N)^K}(x,x^{-1}) &= (-1)^{NK}o^{odd}_{(N^K)}(x)o^{odd}_{(N^K)}(-x) \\ &= \frac{1}{2}\left[sp_{(N^K)}(x)o^{even}_{(N^K)}(x) + sp_{(N-1)^K}(x)o^{even}_{(N+1)^K}(x)\right]. \end{aligned}$$

Theorem (GG-Tierz'17)

Any minor of a semi-banded Toeplitz matrix can be expressed as the specialization of a single skew Schur polynomial

$$\det T_N^{\lambda,\mu}\left(\prod_{k=1}^{d}(1+y_k z^{-1})\prod_{j=1}^{\infty}(1+x_j z)\right)=\left(\prod_{k=1}^{d}y_k^N\right)s_{((d^N)+\mu/\lambda)'}(y_1^{-1},\dots,y_d^{-1},x).$$

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We combine this with the usual formula for the inverse of a matrix in terms of its cofactors. That is, in terms of minors of the original matrix where only one row and column have been removed. For the case of Toeplitz matrices, this reads

$$\left[T_N^{-1}(f)\right]_{j,k}=(-1)^{j+k}\det T_{N-1}^{(1^{k-1}),(1^{j-1})}(f)\,/\,\det T_N(f).$$

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Therefore, computing the inverse and determinant of a Toeplitz matrix amounts to computing a particular specialization of the above skew Schur polynomials. For instance,

$$\begin{aligned}s_{(N,j)/(k)}(x,y^{-1})&=(xy^{-1})^{(N+j-k)/2}U_{\min(j,k)}(c)U_{N-\max(j,k)}(c)\\&=\frac{1}{x^k y^{N+j-k}}\sum_{r=0}^{\min(j,k)}(xy)^r\sum_{r=\max(j,k)}^{N}(xy)^r,\end{aligned}$$

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$$s_{(\underbrace{N,\dots,N}_{d},j)/(k)}(1^M)= G(N+2)\frac{G(M+N+2)}{G(M+1)}\frac{G(M-d+1)}{G(M-d+N+2)}\frac{G(d+1)}{G(d+N+2)}\times$$

$$\frac{\Gamma(M-d+j+1)}{\Gamma(j+1)}\frac{\Gamma(d+k+1)}{\Gamma(k+1)}\sum_{r=\max(j,k)}^{N}\frac{\Gamma(r+1)}{\Gamma(M+r+1)}\binom{M-d+r-k-1}{r-k}\binom{d+r-j-1}{r-j},$$

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$$s_{(\underbrace{N,\dots,N}_{d},j)/(k)}(1,q,q^2,\dots)=$$

$$\frac{q^{(d-1)j-dk+d(d-1)N/2}}{(1-q)^{d(N+1)}}\frac{G_q(N+2)G_q(d+1)}{G_q(d+N+2)}\frac{(q;q)_{d+k}}{(q;q)_j}\sum_{r=\max(j,k)}^{N}q^r\begin{bmatrix}r\\r-k\end{bmatrix}_q\begin{bmatrix}d+r-j-1\\r-j\end{bmatrix}_q.$$

TOEPLITZ AND HANKEL MATRICES

Many more results follow from this approach:

- Explicit solutions of random matrix models.
- Generalizations of Gessel's identity to minors of Toeplitz$\pm$Hankel matrices.
- Expansions of determinants of Toeplitz$\pm$Hankel matrices as sums of minors of Toeplitz matrices. Equivalently: expansions of characters indexed by rectangular shapes as sums of skew Schur polynomials.
- Asymptotics of minors of Toeplitz$\pm$Hankel matrices. Equivalently: study of the large-N regime of gauge theories with symmetries other than unitary.

Thank you!