Polynomial invariant and reciprocity theorem on the Hopf monoid of hypergraphs

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Species

(Joyal) A species P is given by the data of:

- for each finite set I, a vector space $P[I]$,
- for each bijection $\sigma : I \to J$, a linear map

$$P[\sigma] : P[I] \to P[J],$$

such that

$$P[\tau \circ \sigma] = P[\tau] \circ P[\sigma] \text{ and } P[\mathrm{id}] = \mathrm{id}.$$

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Ex: $G[I] = \mathrm{Vect}(\{\text{graphs over } I\})$, $G[\sigma]$ relabeling

Hopf monoid

(Aguiar-Mahajan) A Hopf monoid is a species M with, for each $I = S \sqcup T$,

- a product $\mu_{S,T} : M[S] \otimes M[T] \to M[I]$,
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Co-associativity:

$$\begin{tikzcd}
 & P[R \sqcup S \sqcup T] \arrow[dl, "\Delta_{R \sqcup S, T}"'] \arrow[dr, "\Delta_{R, S \sqcup T}"] & \\
P[R \sqcup S] \otimes P[T] \arrow[dr, "\Delta_{R,S} \otimes \mathrm{id}"'] & & P[R] \otimes P[S \sqcup T] \arrow[dl, "\mathrm{id} \otimes \Delta_{S,T}"] \\
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Hopf monoid of graphs

$$\begin{aligned} \mu_{S,T} : G[S] \otimes G[T] &\to G[I] \\ g_1 \otimes g_2 &\mapsto g_1 \sqcup g_2 \end{aligned} \qquad \begin{aligned} \Delta_{S,T} : G[I] &\to G[S] \otimes G[T] \\ g &\mapsto g_{|S} \otimes g_{/S}, \end{aligned}$$

where $g_{|S}$ is the sub-graph of g induced by S and $g_{/S} = g_{|T}$.

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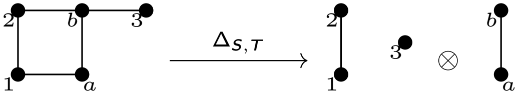
Ex: For $S = \{1, 2, 3\}$ and $T = \{a, b\}$:



Character

A Hopf monoid character $\zeta : M \to \mathbb{k}$ is a collection of linear forms

$$\zeta_I : M[I] \to \mathbb{k}$$

such that for every $I = S \sqcup T$:

$$\begin{tikzcd} M[S] \otimes M[T] \arrow[r, "\mu_{S,T}"] \arrow[d, "\zeta_S \otimes \zeta_T"] & M[I] \arrow[d, "\zeta_I"] \\ \mathbb{k} \otimes \mathbb{k} \arrow[r, "\cong"] & \mathbb{k} \end{tikzcd}$$

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Ex: For $g \in G[I]$

$$\zeta_I(g) = \begin{cases} 1 & \text{if } g \text{ is discrete (i.e has no edges)} \\ 0 & \text{if not} \end{cases}$$

Polynomial invariant

A decomposition of I, $(S_1, \ldots, S_n) \vdash I$ is a sequence of disjoint sets of I such that $\bigsqcup S_i = I$. We note $\ell(S)$ the number of elements of S.

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Theorem (Aguiar and Ardila)

Let M be a Hopf monoid, ζ a character, n an integer and $x \in M[I]$. Then

$$\chi_I(x)(n) = \sum_{S \vdash I, \ell(S)=n} \zeta_{S_1} \otimes \cdots \otimes \zeta_{S_n} \circ \Delta_{S_1,\ldots,S_n}(x)$$

is a polynomial in n such that $\chi(xy) = \chi(x)\chi(y)$ and $\chi_I(x)(1) = \zeta_I(x)$.

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Ex: With the preceding character, for g a graph, $\chi_I(g)$ is the chromatic polynomial of g.

Polynomial invariant

Ex: $I = [3]$, $G = {}_1\bullet\!\!-\!\!\underset{2}{\bullet}\!\!-\!\!\underset{3}{\bullet}$ and $n = 2$:

- $\Delta_{\{123\},\emptyset}(G) = G \otimes \emptyset \xrightarrow{\zeta_{\{123\}} \otimes \zeta_{\emptyset}} 0 \otimes 1 = 0$

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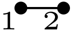
- $\Delta_{\{12\},\{3\}}(G) =$  $\otimes$  $\xrightarrow{\zeta_{\{12\}} \otimes \zeta_{\{3\}}} 0 \otimes 1 = 0$

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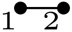
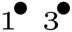
- $\Delta_{\{12\},\{3\}}(G) =$  $\otimes$  $\xrightarrow{\zeta_{\{12\}} \otimes \zeta_{\{3\}}} 0 \otimes 1 = 0$
- $\Delta_{\{13\},\{2\}}(G) =$  $\otimes$  $\xrightarrow{\zeta_{\{13\}} \otimes \zeta_{\{2\}}} 1 \otimes 1 = 1$

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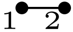
- $\Delta_{\{12\},\{3\}}(G) =$ 

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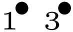
- $\Delta_{\{13\},\{2\}}(G) =$ 

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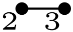
- $\Delta_{\{23\},\{1\}}(G) =$ 

$\otimes$ 

$\xrightarrow{\zeta_{\{23\}} \otimes \zeta_{\{1\}}} 1 \otimes 0 = 0$
- $\cdots + 1$

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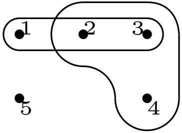
Hypergraphs

Hypergraph over I: collection of sub-sets of I called edges.

$HG[I]$=Vect({hypergraphs over I})

Ex:

$$\{\{1,2,3\},\{2,3,4\}\} \in HG[[5]]$$



Hypergraphs

Hopf monoid structure:

$$\begin{aligned} \mu_{S,T} : HG[S] \otimes HG[T] &\to HG[I] \\ H_1 \otimes H_2 &\mapsto H_1 \sqcup H_2 \end{aligned} \qquad \begin{aligned} \Delta_{S,T} : HG[I] &\to HG[S] \otimes HG[T] \\ H &\mapsto H_{|S} \otimes H_{/S} \end{aligned}$$

- $H_{|S} = \{e \in H \mid e \subseteq S\}$ restriction of H to S
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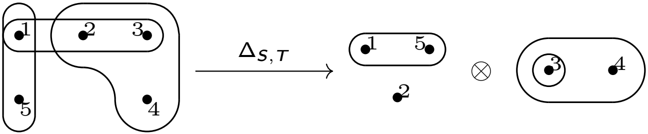
Ex: For $S = \{1, 2, 5\}$ $T = \{3, 4\}$,



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$H \in HG[I]$

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Colorings

Definition (Coloring)

A *coloring* of H with $[n]$ is a function

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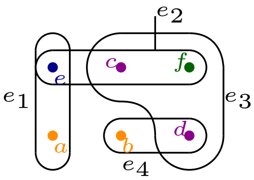
Ex: Coloring with $\{1, 2, 3, 4\}$.



a maximal in e_1, c in e_2, c and d in e_3.

Polynomial invariant

Theorem

Let I be a set and $H \in HG[I]$ a hypergraph over I.
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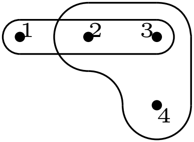

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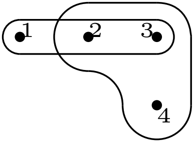

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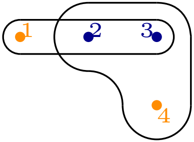

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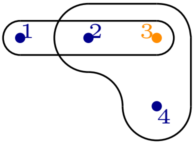

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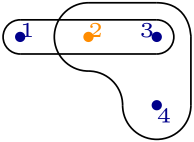

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Orientations

Definition (Orientation)

An *orientation* of H is a function $f : H \to I$ such that $f(e) \in e$ for all $e \in H$.
A *cycle* in f is a sequence $e_1, \dots, e_k$ of edges such that

$$f(e_1) \in e_2 \setminus f(e_2), \dots, f(e_k) \in e_1 \setminus f(e_1).$$

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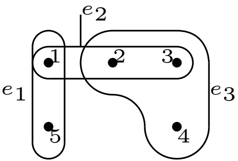

The orientation $f(e_1) = 5$, $f(e_2) = 2$, $f(e_3) = 3$ is cyclic.
The orientation $f(e_1) = 1$, $f(e_2) = 1$, $f(e_3) = 3$ is acyclic.

Reciprocity theorem

Theorem

Let I be a set and $H \in HG[I]$ be a hypergraph over I.
Then $(-1)^{|I|}\chi_I(H)(-1) = |\mathcal{A}_H|$ is the number of acyclic orientations of H.

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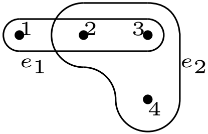

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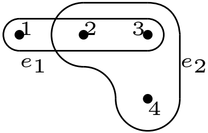

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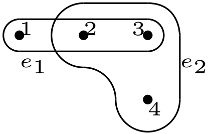

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Remark: There is a combinatorial interpretation of $(-1)^{|I|}\chi_I(H)(-n)$.

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Graphs

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proof:

Only one maximal vertex $\iff$ neighbour vertex of different colors.

Simplicial complexes (Benedetti, Hallam, Machacek)

A simplicial complex over I is a set of parts S of I such that $K \subset J \in S \Rightarrow K \in S$. We note SC the species of simplicial complexes.

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Theorem

SC is a Hopf sub-monoid of HG. Let be $C \in SC[I]$ and g be its 1-skeleton. Then $\chi_I^{SC}(C) = \chi_I^{G}(g)$.

Set of paths (Aguiar and Ardila)

A path over I is a word over I quotiented by the relation $w_1 \dots w_{|I|} \sim w_{|I|} \dots w_1$. A set of paths $s_1 | \dots | s_\ell$ over I is a partition of I in paths.

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The species F of sets of paths is a Hopf monoid:

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$$s_1 | \dots | s_\ell \otimes t_1 | \dots | t_{\ell'} \mapsto s_1 | \dots | s_\ell | t_1 | \dots | t_{\ell'}$$

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$$s_1 | \dots | s_\ell \mapsto s_1 \cap S | \dots | s_\ell \cap S \otimes {s_1}_{S\leftarrow|} | \dots | {s_\ell}_{S\leftarrow|}$$

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$$\Delta_{S,T} : F[I] \to F[S] \otimes F[T]$$
$$s_1|\dots|s_\ell \mapsto s_1 \cap S|\dots|s_\ell \cap S \otimes s_{1\,S\leftarrow|}|\dots|s_{\ell\,S\leftarrow|}$$

Ex: For $I = \{a, b, c, d, e, f, g\}$ and $S = \{b, c, e\}$ and $T = \{a, d, f, g\}$ we have :

$$\Delta_{S,T}(bfc|g|aed) = bc|e \otimes f|g|a|d$$

Set of paths (Aguiar and Ardila)

Theorem

Let α be a path over I. $\chi_I^F(\alpha)(n)$ is the number of rooted binary trees with $|I|$ vertices and colored with $[n]$ such that the color of a vertex is strictly greater than the color of its children.
Furthermore $(-1)^{|I|}\chi_I^F(\alpha)(-1) = C_{|I|}$ with $(C_k)_{k\in\mathbb{N}}$ the Catalan number sequence.

Perspectives

Generalisation to all characters on HG.

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Antipode $S : M \to M$ such that:

$$\chi_I(x)(-n) = \chi_I(S(x))(n)$$

Open question: find a (nice) proof of reciprocity theorems using the antipode.

Thank you for your attention.

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