Lecture 2: Total positivity and statistical mechanics

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Thanks to Pasha Galashin for some slides!

Introduction

In the last lecture, I recalled the fundamental connection

$$\left\{\begin{matrix}\text{planar networks with} \\ n \text{ sources and } n \text{ sinks}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{totally nonnegative} \\ n \times n \text{ matrices}\end{matrix}\right\}$$

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arising from counting *non-intersecting path families*.
The aim of this lecture is to do an analogous construction for $\mathrm{Gr}_{\geq 0}(k, n)$.

Dimer model	$\mathrm{Gr}_{\geq 0}(k, n)$
Electrical networks	$\mathrm{LG}_{\geq 0}(n+1, 2n)$
Ising model	$\mathrm{OG}_{\geq 0}(n, 2n)$

In terms of classical enumerative combinatorics, the first two cases are related to enumerating *perfect matchings* and *trees* in graphs. Both are known to be related to determinants: for example, recall the matrix-tree theorem.

Totally nonnegative Grassmannian

Definition (Postnikov (2006))

A point $V \in \mathrm{Gr}(k, n)$ lies in the *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(k, n)$ if $\Delta_I(V) \geq 0$ for all I.

The *totally positive Grassmannian* $\mathrm{Gr}_{>0}(k, n)$ is the locus where $\Delta_I(V) > 0$. Example:

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 3 & 1 & 1 \end{bmatrix} \qquad \begin{matrix} \Delta_{12} = 3 & \Delta_{13} = 1 & \Delta_{14} = 1 \\ \Delta_{23} = 2 & \Delta_{24} = 11 & \Delta_{34} = 3 \end{matrix}$$

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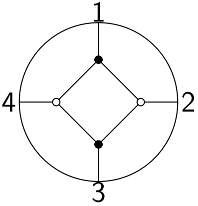
We have the Plücker relation: for $\{i_1, \ldots, i_{k-1}\}$ and $\{j_1, j_2, \ldots, j_{k+1}\}$

$$\Delta_{i_1 \cdots i_{k-1} j_1} \Delta_{j_2 \cdots j_{k+1}} - \Delta_{i_1 \ldots i_{k-1} j_2} \Delta_{j_1 j_3 \cdots j_{k+1}} + \cdots + (-1)^k \Delta_{i_1 \ldots i_{k-1} j_{k+1}} \Delta_{j_1 \cdots j_k} = 0$$

e.g. $i = 1$, $\{j_1, j_2, j_3\} = \{2, 3, 4\}$
$\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} = 3 \cdot 3 - 1 \cdot 11 + 2 \cdot 1 = 0.$

Dimer model

$N =$ 

- Bipartite graph embedded in a disk, with n boundary vertices.
- Boundary vertices are assumed to have degree one, and by convention we do not draw their colors.

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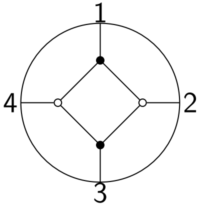
$N =$ 

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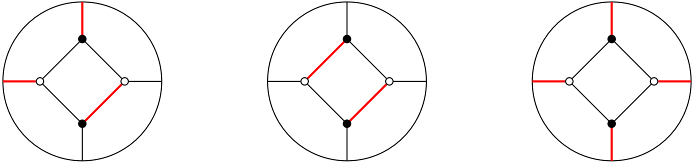
An *almost perfect matching* Π is a collection of edges that uses every interior vertex once, and may use any subset of the boundary vertices.



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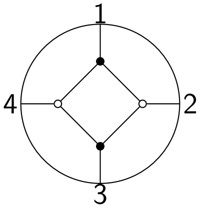
$N =$ 

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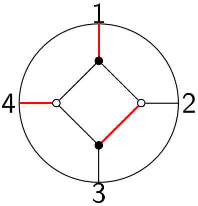

The *boundary set* $\partial(\Pi)$ is the set of black boundary vertices used union the set of white boundary vertices not used. ($\partial(\Pi) = \{3, 4\}$ in example)

We informally call these *dimers*, that is, polymers consisting of two atoms.

- Dimer model: what does a *random dimer* look like? (Kasteleyn 1967) (Fisher and Temperley 1961)

Boundary measurements

We now assume that N has positive edge weights w_e. The *weight* of an almost perfect matching Π is $\mathrm{wt}(\Pi) = \prod_{e \in \Pi} w_e$.

Definition (Dimer generating function)

For a subset $I \subset [n]$, define the *boundary measurement*

$$\Delta_I(N) := \sum_{\Pi : \partial(\Pi) = I} \mathrm{wt}(\Pi)$$

If almost perfect matchings exist, it is easy to see that there is a unique value of k such that $\Delta_I(N) \neq 0$ only if $|I| = k$.

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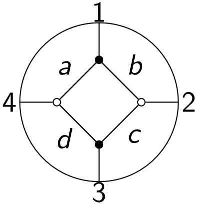

$$\Delta_{12}(N) = a \qquad \Delta_{23}(N) = d$$
$$\Delta_{13}(N) = ac + bd \qquad \Delta_{24}(N) = 1$$
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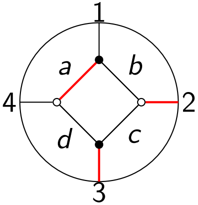

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Boundary measurement map

Theorem (Postnikov, Talaska, Postnikov-Speyer-Williams, Kuo, L.)

The map $N \to (\Delta_I(N))_{I \in \binom{[n]}{k}}$ defines a point $M(N) \in \mathrm{Gr}(k,n)$.

To prove the theorem, it suffices to check that $\Delta_I(N)$ satisfies the Plücker relations.

Theorem (Postnikov)

1. *The map $N \to M(N)$ surjects onto $\mathrm{Gr}_{\geq 0}(k,n)$.*
2. *If $M(N) = M(N')$, then N and N' are related by local moves.*
3. *For each positroid cell $\Pi_{f,>0}$, there exists a network $N(t_1, t_2, \ldots, t_d)$ with edge weights given by the parameters $t_1, \ldots, t_d$ such that the map $(t_1, t_2, \ldots, t_d) \in \mathbb{R}^d_{>0} \to M(N(t_1, t_2, \ldots, t_d))$ is a homeomorphism $\mathbb{R}^d_{>0} \cong \Pi_{f,>0}$.*

Double dimers

- Since $\Delta_I(N)\Delta_J(N)$ counts *double dimers*, the Plücker relation is equivalent to a statement about boundary connections of double dimers.

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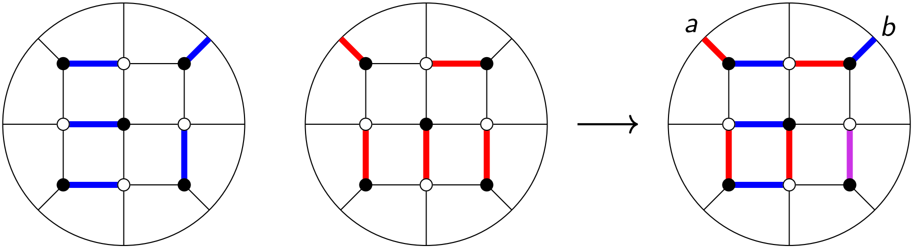

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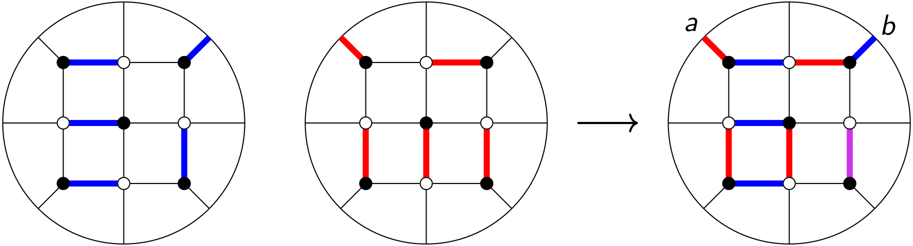

Temperley-Lieb immanant (L., cf. Rhoades–Skandera):
For a (k, n)-*partial noncrossing matching* τ,

$$F_\tau(N) = \sum \mathrm{wt}(\Sigma)$$

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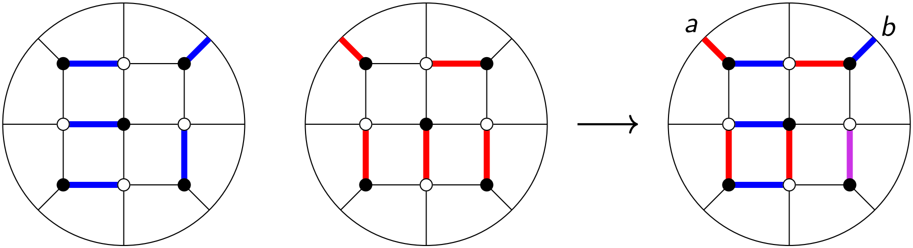

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summed over double dimers Σ with connectivity τ.
We have an identity $\Delta_I\Delta_J = \sum_{F_\tau}$, summed over τ compatible with (I, J).

Dimers and positroids

Let G be a planar bipartite graph. Then for *any* positive edge weights w_e, we have

$$M((G, w_e)) \in \Pi_{f,>0}$$

where $f = f_G$ only depends on the underlying graph G.

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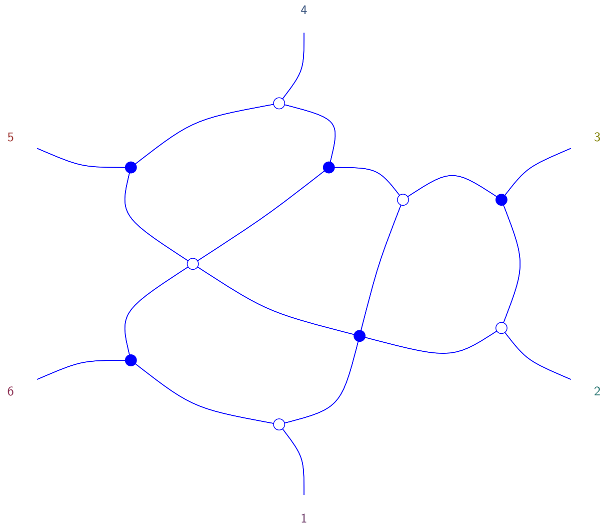
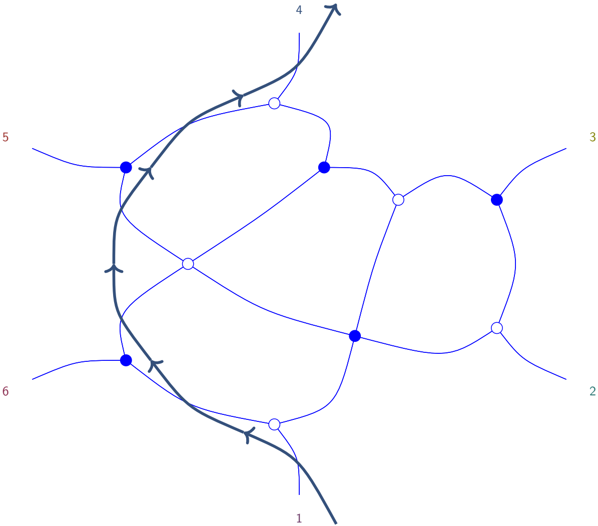
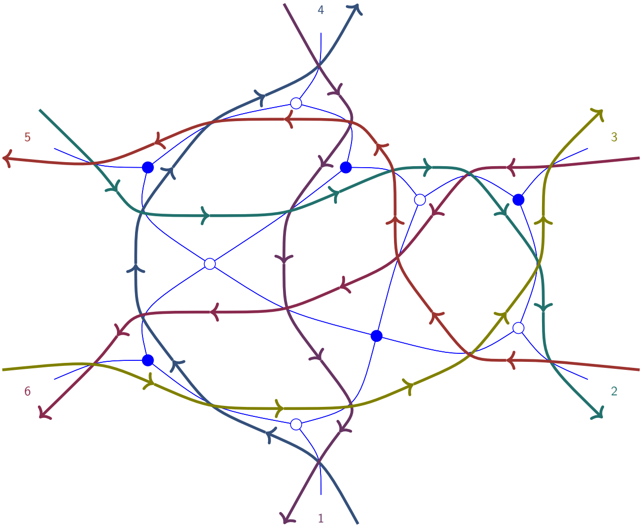
- We have $\dim(\Pi_{f,>0}) \leq \#\mathrm{Faces}(G) - 1$, and when equality holds we call G reduced.
- In the reduced case, we can read f off of G by the *rules of the road*.


4
5
3
6
2
1



$f(1) = 4$



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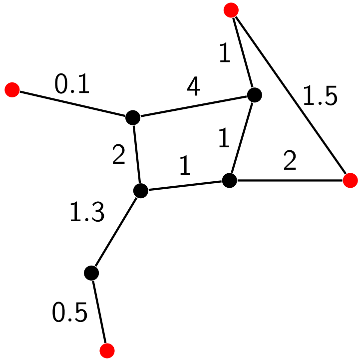
Electrical networks

An *electrical resistor network* is an undirected weighted graph Γ.



Edge weight = *conductance* = $1/\text{resistance}$
Some vertices are designated as boundary vertices. The rest are interior vertices.

Response matrix

The electrical properties are described by the *response matrix*

$$\Lambda(\Gamma) : \mathbb{R}^{\#\text{boundary vertices}} \longrightarrow \mathbb{R}^{\#\text{boundary vertices}}$$
$$\text{voltage vector} \longmapsto \text{current vector}$$

which gives the current that flows through the boundary vertices when specified voltages are applied.

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$$\Lambda_{ij} = \text{current flowing through vertex } j \text{ when}$$
$$\text{the voltage is set to 1 at vertex } i \text{ and 0 at all other vertices.}$$

Possibly surprisingly, $\Lambda(\Gamma)$ is a symmetric matrix.
If all vertices are considered boundary vertices, then $\Lambda(\Gamma)$ is simply the *Laplacian matrix* of Γ.

Axioms of electricity

The matrix $\Lambda(\Gamma)$ can be computed using only two axioms.

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Ohm's Law (1827)

For each resistor we have

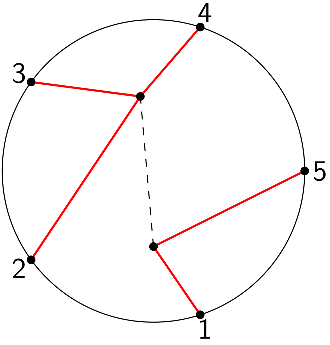
$$(V_1 - V_2) = I \times R$$

where
I = current flowing throught the resistor
V_1, V_2 = voltages at two ends of resistor
R = resistance of the resistor

To compute $\Lambda(\Gamma)$, we give variables to each edge (current through that edge) and each vertex (voltage at that vertex). Then solve a large system of linear equations.

Groves

We now assume that Γ is embedded into a disk. A *grove* F in Γ is a subforest such that every interior vertex is connected to some boundary vertex.



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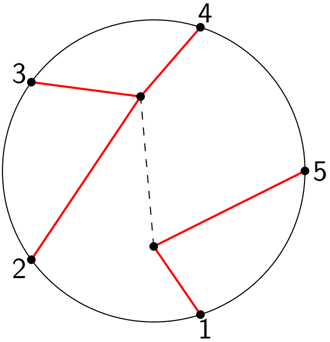

The *boundary partition* $\sigma(F)$ of a grove F is the noncrossing partition whose parts are boundary vertices belonging to the same component of F.

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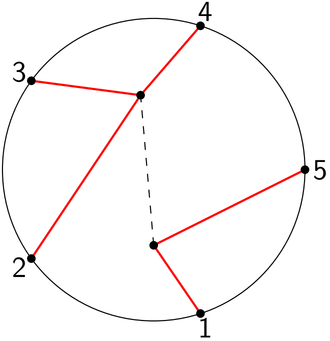

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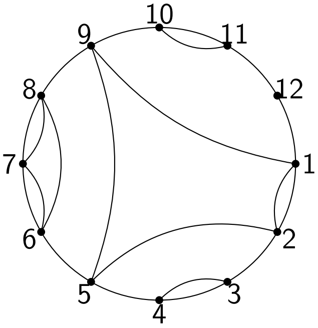
$$\sigma(F) = \{2, 3, 4|1, 5\}$$

Planarity $\Longrightarrow$ noncrossing.

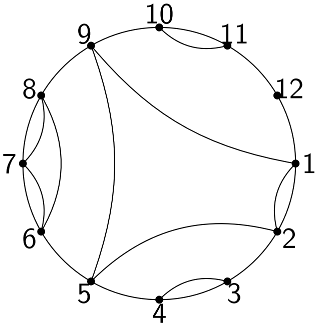
Groves were studied by Carroll–Speyer, Kenyon–Wilson, ...

Noncrossing partitions



The noncrossing partition $\sigma = \{1, 2, 5, 9|3, 4|6, 7, 8|10, 11|12\}$.

Noncrossing partitions



The noncrossing partition $\sigma = \{1, 2, 5, 9|3, 4|6, 7, 8|10, 11|12\}$.
Let $\mathcal{NC}_n$ denote the set of noncrossing partitions on $\{1, \ldots, n\}$. Then $|\mathcal{NC}_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$. For $n = 3$, we have 5 noncrossing partitions.

$$(123), (1|23), (12|3), (13|2), (1|2|3).$$

Grove measurements

Definition (Grove generating function)

For $\sigma \in \mathcal{NC}_n$, and an electrical network Γ, define

$$L_\sigma(\Gamma) = \sum_{\sigma(F)=\sigma} \mathrm{wt}(F)$$

where the weight of a grove F is the product of the weights of the edges belonging to F.

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We collect all the L_σ's together to obtain a map $\Gamma \longmapsto \mathcal{L}(\Gamma) = (L_\sigma(\Gamma))_{\sigma \in \mathcal{NC}_n} \in \mathbb{P}^{\mathcal{NC}_n}$.

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$\Lambda(\Gamma) = \Lambda(\Gamma')$ *if and only if* $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$

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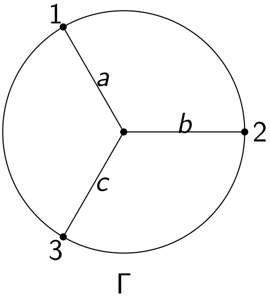
Define the *compactified space of circular planar electrical networks*:

$$\mathcal{E}_n := \overline{\{\mathcal{L}(\Gamma) \mid \Gamma \text{ planar electrical network } \}} \subset \mathbb{P}^{\mathcal{NC}_n}$$

Example: the grove embedding



$$L_{1|2|3} = a + b + c,$$
$$L_{123} = abc$$
$$L_{12|3} = ab,$$
$$L_{1|23} = bc,$$
$$L_{13|2} = ac,$$

$$\mathcal{L}(\Gamma) = (a + b + c : ab : bc : ac : abc) \in \mathbb{P}^4$$

The totally nonnegative Lagrangian Grassmannian

Consider the (degenerate) skew-symmetric bilinear form on $\mathbb{R}^{2n}$

$$\langle x, y \rangle = \sum_{k=1}^{2n} (-1)^k (x_k y_{k+1} - x_{k+1} y_k)$$

where $x_{2n+1} = (-1)^n x_1$. A subspace $U \subset \mathbb{R}^{2n}$ is *isotropic* if $\langle \cdot, \cdot \rangle$ restricts to 0 on U. We set

$$\mathrm{LG}(n+1, 2n) := \{U \subset \mathbb{R}^{2n} \mid U \text{ is maximal isotropic } \} \subset \mathrm{Gr}(n+1, 2n).$$

We have $\dim(\mathrm{Gr}(n+1, 2n)) = n^2 - 1$ but $\dim \mathrm{LG}(n+1, 2n) = n(n-1)/2$.

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Definition

The *totally nonnegative Lagrangian Grassmannian*:

$$\mathrm{LG}_{\geq 0}(n+1, 2n) := \mathrm{LG}(n+1, 2n) \cap \mathrm{Gr}_{\geq 0}(n+1, 2n).$$

Our notion differs from that of Lusztig and Karpman. (Thanks to David Speyer for a helpful discussion!)

Embedding electrical networks into the Grassmannian

Theorem (L.)

There is a homeomorphism

$$\iota : \mathcal{E}_n \longrightarrow \mathrm{LG}_{\geq 0}(n+1, 2n)$$

given by the formula

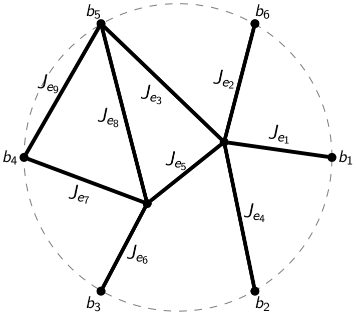
$$\Delta_I(\iota(\Gamma)) = \sum_{\sigma \in \mathcal{NC}_n} a_{I\sigma} L_\sigma(\Gamma)$$

where $a_{I\sigma}$ is a 0-1 matrix, with the 1-s given by concordant pairs (I, σ).

Earlier work: Curtis–Ingerman–Morrow (1998) and de Verdière–Gitler–Vertigan (1996).

Ising model

G = planar network in a disk (boundary vertices may have deg > 1)
J_e = weight of edge e



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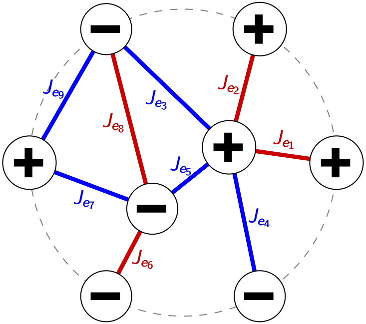

Spin configuration: a map $\sigma : V \to \{\pm 1\}$

$$\mathrm{wt}(\sigma) := \prod_{\{u,v\} \in E} \exp\left(J_{\{u,v\}} \sigma_u \sigma_v\right)$$

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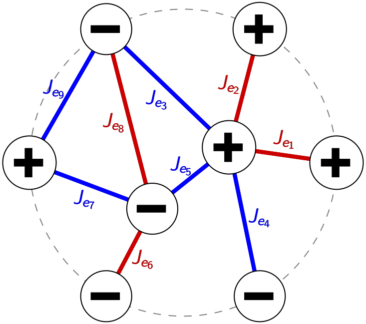

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$$\mathrm{wt}(\sigma) = \frac{\exp\left(J_{e_1} + J_{e_2} + J_{e_6} + J_{e_8}\right)}{\exp\left(J_{e_3} + J_{e_4} + J_{e_5} + J_{e_7} + J_{e_9}\right)} \qquad \mathrm{Prob}(\sigma) := \frac{\mathrm{wt}(\sigma)}{Z}$$

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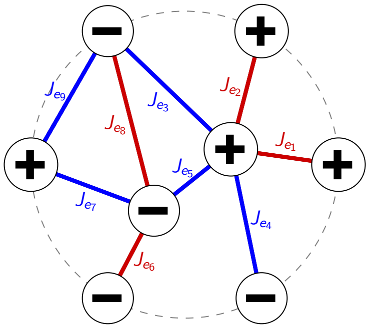
G = planar network in a disk (boundary vertices may have deg > 1)
J_e = weight of edge e



Spin configuration: a map $\sigma : V \to \{\pm 1\}$

$$\mathrm{wt}(\sigma) := \prod_{\{u,v\}\in E} \exp\left(J_{\{u,v\}}\sigma_u\sigma_v\right)$$

$$\mathrm{wt}(\sigma) = \frac{\exp\left(J_{e_1} + J_{e_2} + J_{e_6} + J_{e_8}\right)}{\exp\left(J_{e_3} + J_{e_4} + J_{e_5} + J_{e_7} + J_{e_9}\right)} \qquad \mathrm{Prob}(\sigma) := \frac{\mathrm{wt}(\sigma)}{Z}$$

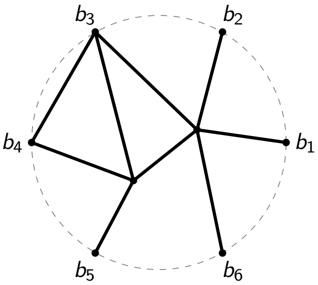
The Ising model is a model for ferromagnetism. (Lenz 1920, Ising 1925)

Boundary correlations I

Correlation: $\langle \sigma_u \sigma_v \rangle := \text{Prob}(\sigma_u = \sigma_v) - \text{Prob}(\sigma_u \neq \sigma_v)$.

Definition

Boundary correlation matrix: $M(G, J) = (m_{ij})_{i,j=1}^n$, where $m_{ij} := \langle \sigma_{b_i} \sigma_{b_j} \rangle$.



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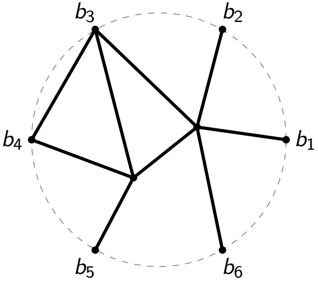

$M(G, J)$ is a symmetric matrix with 1's on the diagonal and nonnegative entries

Lives inside $\mathbb{R}^{\binom{n}{2}}$

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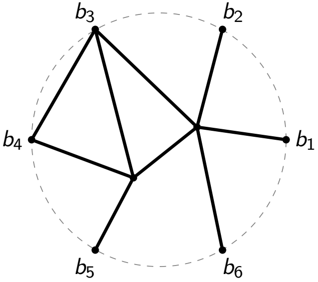

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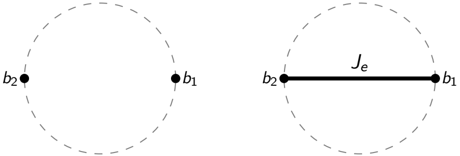

$$M(G) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}, \qquad m_{12} = \langle \sigma_1 \sigma_2 \rangle = \frac{2\exp(J_e) - 2\exp(-J_e)}{2\exp(J_e) + 2\exp(-J_e)}$$

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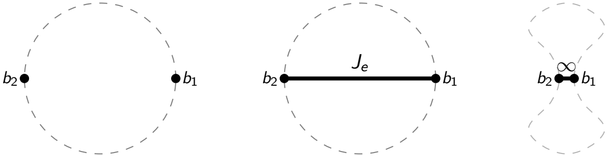

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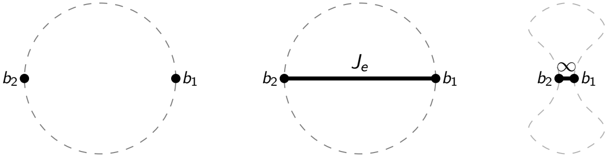

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- $\overline{\mathcal{X}}_n$ is obtained from $\mathcal{X}_n$ by allowing $J_e = \infty$ (i.e., contracting edges).

The totally nonnegative orthogonal Grassmannian

Consider the symmetric nondegenerate bilinear form on $\mathbb{R}^{2n}$ given by

$$(x, y) = \sum_{i=1}^{2n} (-1)^i x_i y_i.$$

A subspace $W \subset \mathbb{R}^{2n}$ is *isotropic* if the restriction of $(\cdot, \cdot)$ to W is identically 0. The *orthogonal Grassmannian* is given by

$$\mathrm{OG}(n, 2n) := \{W \in \mathrm{Gr}(n, 2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \text{ for all } I\}$$

and consists of a component of the isotropic subspaces of $\mathrm{Gr}(n, 2n)$. We have $\dim(\mathrm{Gr}(n, 2n)) = n^2$ but $\dim(\mathrm{OG}(n, 2n)) = \binom{n}{2} = \frac{n(n-1)}{2}$.

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Definition (Huang–Wen)

The *totally nonnegative orthogonal Grassmannian*:
$\mathrm{OG}_{\geq 0}(n,2n) := \mathrm{OG}(n,2n) \cap \mathrm{Gr}_{\geq 0}(n,2n)$, i.e.,
$\mathrm{OG}_{\geq 0}(n,2n) := \{W \in \mathrm{Gr}(n,2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \geq 0 \text{ for all } I\}$.

This notion differs from a general one of Lusztig.

Boundary correlation map

$\mathcal{X}_n := \{M(G) \mid G$ is a planar Ising network with n boundary vertices$\}$
$\overline{\mathcal{X}}_n :=$ closure of $\mathcal{X}_n$ inside the space of $n \times n$ matrices.
We have $\mathcal{X}_n, \overline{\mathcal{X}}_n \subset \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) := \left\{ \begin{array}{c} \text{symmetric } n \times n \text{ matrices} \\ \text{with 1's on the diagonal} \end{array} \right\}$.

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Theorem (Galashin–Pylyavskyy (2018))

The map ϕ restricts to a homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$.

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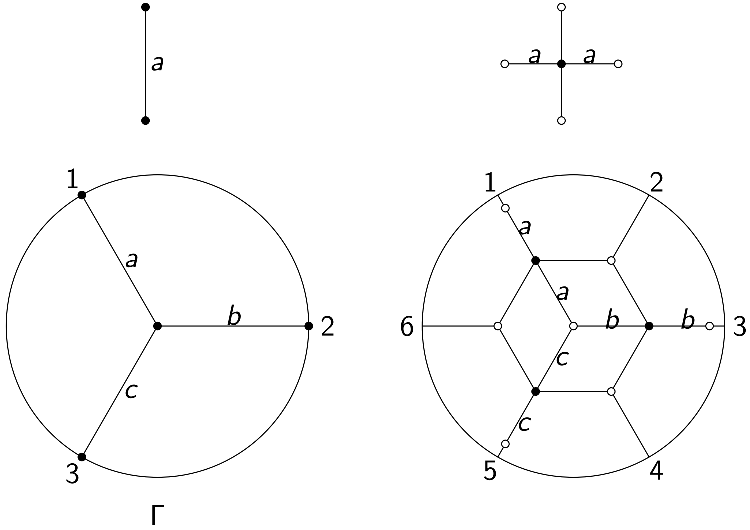
Lis (2016): boundary correlations related to $\mathrm{Gr}_{\geq 0}(n, 2n)$.

Comparison

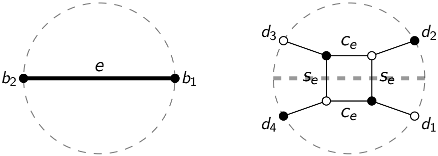
	Dimer	Electrical	Ising
vertices	bipartite	one part	one part
space	$\mathrm{Gr}_{\geq 0}(k, n)$	$\mathrm{LG}_{\geq 0}(n+1, 2n)$	$\mathrm{OG}_{\geq 0}(n, 2n)$
dimension	$k(n-k)$	$n(n-1)/2$	$n(n-1)/2$
enumeration	dimer configurations	groves	spinned flows
moves	square	$Y - \Delta$	$Y - \Delta$
strata	permutations	matchings	matchings
poset	Bruhat order	uncrossing	uncrossing

The electrical network model and Ising model have the same indexing set for strata, same closure relations, and same local moves (on the level of unweighted graphs).

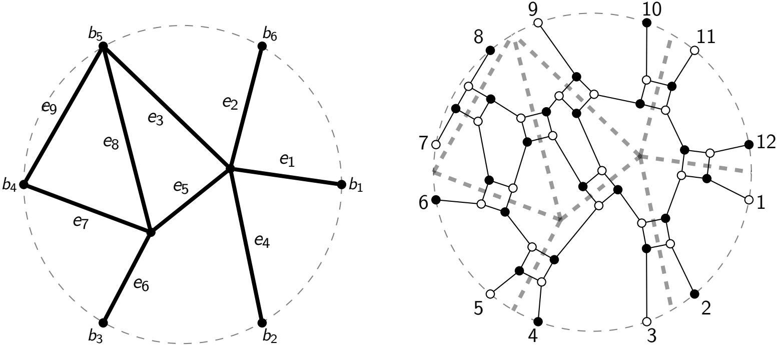
Electrical network →planar bipartite graph



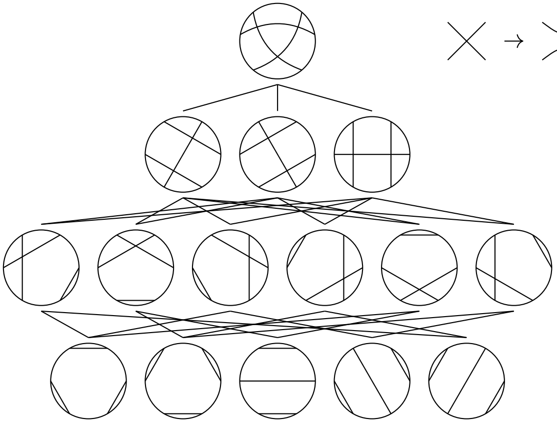
Ising network → planar bipartite graph

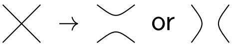


Here $s_e := \mathrm{sech}(2J_e)$, $c_e := \tanh(2J_e)$ so that $s_e^2 + c_e^2 = 1$.



Uncrossing partial order P_n





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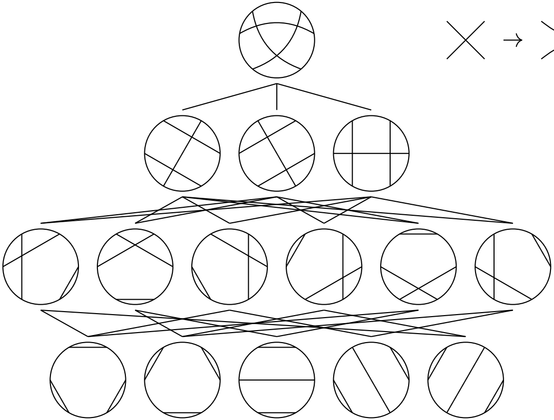

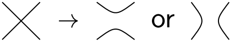

Let $\hat{P}_n$ be P_n with a minimum $\hat{0}$ added.

- $\hat{P}_n$ is Eulerian (L.)
- $\hat{P}_n$ is shellable (Kenyon–Hersh)

Further directions

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Some references:

- A. Postnikov, Total positivity, Grassmannians, and networks, `arXiv:math/0609764`.
- T. Lam, Totally nonnegative Grassmannian and Grassmann polytopes, CDM lectures 2014.
- T. Lam, Electroid varieties and a compactification of the space of planar electrical networks, *Adv. in Math.* 2018.
- P. Galashin and P. Pylyavskyy, Ising model and the positive orthogonal Grassmannian `arXiv:1807.03282`.