Polyhedral and Algebraic Approaches to the k-dimensional Multiplication Table Problem

·	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

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SLC82

Lehrstuhl II für Mathematik

RWTH AACHEN UNIVERSITY

Overview

Question

How many different entries does a multiplication table have?

·	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

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Answer for the 10×10*-table: 42*

Definition

For $k, n \in \mathbb{N}$

$$P(k,n) := \left\{ \prod_{i=1}^{k} m_i \,\middle|\, m_i \in \{1,\ldots,n\} \forall i \in \{1,\ldots,k\} \right\}$$

$$p(k,n) := |P(k,n)|$$

Example ($k = 2, n = 4$)

$\cdot$	*1*	*2*	*3*	*4*
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4	*4*	*8*	*12*	*16*

$P(2,4) = \{1, 2, 3, 4, 6, 8, 9, 12, 16\}$,
$p(2,4) = 9$

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$$6 = 2^1 3^1 5^0 7^0 \rightsquigarrow (1, 1, 0, 0) \in \mathbb{N}_0^4$$

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For every full dimensional lattice polytope Φ and every $t \in \mathbb{N}$ we have

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Definition

If $t \star \Phi = int(t\Phi)$ holds for every $t \in \mathbb{N}$, the polytope Φ is called **integrally closed**.

Application to our Problem

Definition

Let Γ_n *denote the polytope* $conv(M(1,n)) \subseteq \mathbb{R}^d$ *with* $d = \pi(n)$.

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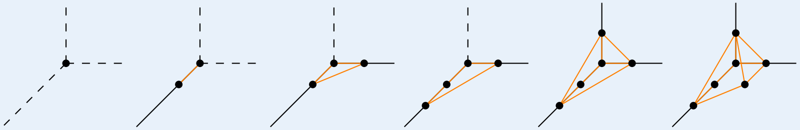
Example (Γ_n)



$n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$ $n = 6$

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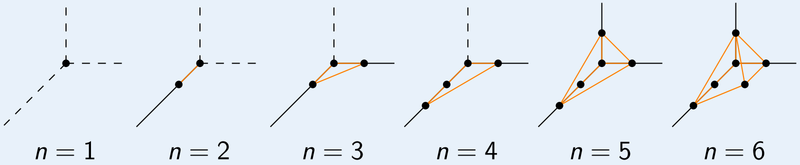

Remark

- *In general: Γ_n is not integrally closed.*
- $k \star \Gamma_n \subseteq \mathrm{int}(k\Gamma_n)$ *and* $|k \star \Gamma_n| \leq L_{\Gamma_n}(k)$.
- $\mathrm{int}(\Gamma_n) = M(1,n)$, *therefore* $k \star \Gamma_n = M(k,n)$.
- *In conclusion:* $p(k,n) = |M(k,n)| = |k \star \Gamma_n| \leq L_{\Gamma_n}(k)$.
- *If Γ_n is integrally closed (e.g. for $1 \leq n \leq 27$), we have* $p(k,n) = L_{\Gamma_n}(k)$.

Theorem (Ehrhart)

For every integral polytope Φ of dimension d, $L_\Phi(t)$ is a polynomial, which has degree d, and the leading coefficient is the d-dimensional volume of Φ.

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By calculations with polytopes we get the following theorem:

Theorem (Scheidweiler & Triesch)

For all $n, k \in \mathbb{N}$ and for $d = \pi(n)$ the inequalities

$$p(k, n) \leq L_{\Gamma_n}(k) \leq p(k + d, n)$$

hold.

Definition

$$X_n := \left\{ t \cdot \prod_{j=1}^{\pi(n)} x_j^{\alpha_j} \;\middle|\; (\alpha_1, \ldots, \alpha_{\pi(n)}) \in M(1,n) \right\} \subseteq K[x_1, \ldots, x_{\pi(n)}, t]$$

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$A_n := \langle X_n \rangle_K$ the K-algebra generated by X_n.

Proposition

A_n is a graded K-algebra, which means $A_n = \oplus_{k=0}^{\infty} A_{n,k}$, with

$$A_{n,k} = A_n \cap t^k \cdot K[x_1, \ldots, x_{\pi(n)}].$$

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Lemma

$$p(k,n) = \dim_K(A_{n,k})$$

Definition

The **(projective) Hilbert function** *of a graded K-algebra $B = \oplus_{k=0}^{\infty} B_k$ is defined as $HF_B(k) = \dim_K(B_k)$. For short: $HF_n(k) := HF_{A_n}(k) = p(k, n)$.*

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Theorem (Hilbert)

For a graded K*-algebra* $B = \oplus_{k=0}^{\infty} B_k$ *there exists a polynomial* q_B *(***Hilbert polynomial***), and a number* $k_B \in \mathbb{N}_0$ *(***regularity index***), such that the equation* $HF_B(k) = q_B(k)$ *is fulfilled for every* $k \geq k_B$*.*

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Corollary

For every $n \in \mathbb{N}$ *there is a number* $k_n \in \mathbb{N}_0$ *and a polynomial* q_n *such that* $p(k, n) = q_n(k)$ *is true for every* $k \geq k_n$.

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For every $n \in \mathbb{N}$ there is a number $k_n \in \mathbb{N}_0$ and a polynomial q_n such that $p(k,n) = q_n(k)$ is true for every $k \geq k_n$.

Conjecture (L., Scheidweiler, Triesch)

For every $n \in \mathbb{N}$, the regularity index k_n equals 0 and, therefore, $p(k,n)$ is a polynomial in k.

Fusion of the Theories

Theorem (Ehrhart, Repetition)

For every integral polytope Φ of dimension d, $L_\Phi(t)$ is a polynomial in t which has degree d and the leading coefficient is the d-dimensional volume of Φ.

Theorem (Repetition)

For all $n, k \in \mathbb{N}$ and for $d = \pi(n)$ the inequalities $p(k, n) \leq L_{\Gamma_n}(k) \leq p(k + d, n)$ hold.

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Theorem (Scheidweiler, Triesch)

a) *The Hilbert polynomial q_n has degree $d = \pi(n)$ and the leading coefficient is equal to the d-dimensional volume of the polytope Γ_n.*

b) *If Γ_n is integrally closed, $q_n(k) = p(k,n) = L_{\Gamma_n}(k)$.*

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b) *If Γ_n is integrally closed, $q_n(k) = p(k,n) = L_{\Gamma_n}(k)$.*

Corollary

For fixed $n \in \mathbb{N}$: $p(k,n) = \Theta(k^{\pi(n)})$.

Definition

The **(projective) Hilbert series** *of a graded K-algebra $B = \oplus_{k=0}^{\infty} B_k$ is defined as the formal power series $HS_B := \sum_{k=0}^{\infty} \dim_K(B_k) z^k$.*
For short: $HS_n := HS_{A_n}$

Analogously, we define the Ehrhart series.

Definition

For a full dimensional lattice polytope $\Phi \subset \mathbb{R}^d$, the **Ehrhart series** *is defined as the formal power series $ES_\Phi := \sum_{k=0}^{\infty} L_\Phi(k) z^k$.*
For short: $ES_n := ES_{\Gamma_n}$.

Hilbert Series as Fraction

Theorem

For every graded K-algebra B which is generated by elements of degree 1, there is a polynomial r_B, which such that $HS_B(z) = \frac{r_B(z)}{(1-z)^\delta}$ and δ is the Krull-dimension of B. Furthermore $HF_B = q_B$ if and only if $\deg(r_B) \leq \delta - 1$.

Corollary

For every $n \in \mathbb{N}$ there is a polynomial r_n such that $HS_n(z) = \frac{r_n(z)}{(1-z)^{\pi(n)+1}}$.

Ehrhart Series as Fraction

Lemma

If q is a polynomial of degree d, there is a polynomial r of degree $e \leq d$ such that

$$\sum_{k=0}^{\infty} q(k) z^k = \frac{r(z)}{(1-z)^{d+1}}.$$

Corollary

For every $n \in \mathbb{N}$, there is a polynomial $\hat{r}_n$ of degree $\hat{e}_n \leq \pi(n)$ such that

$$\sum_{k=0}^{\infty} L_{\Gamma_n}(k) z^k = \frac{\hat{r}_n(z)}{(1-z)^{\pi(n)+1}}.$$

Thank you for your attention.