Simple recurrence formulas for bipartite maps with prescribed degrees

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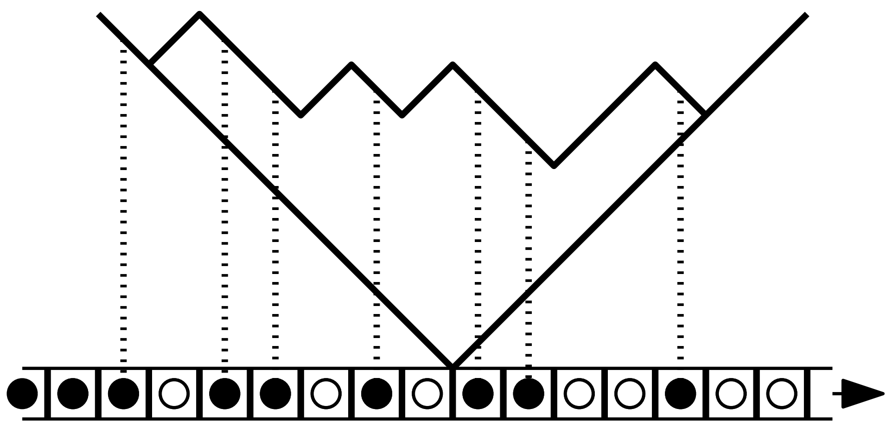

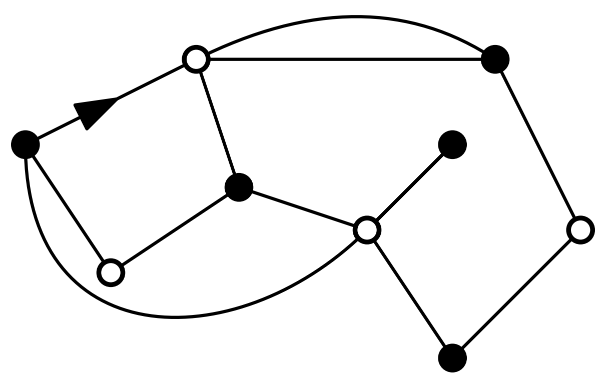

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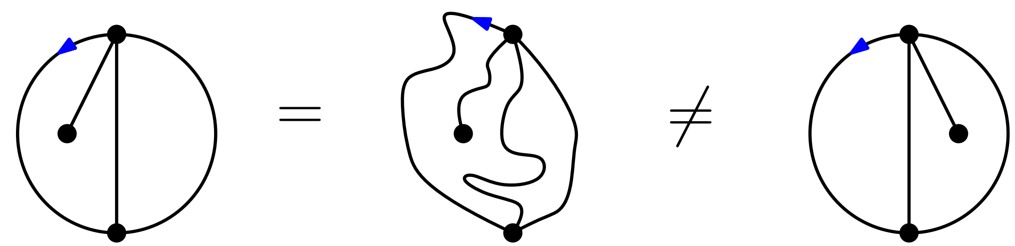
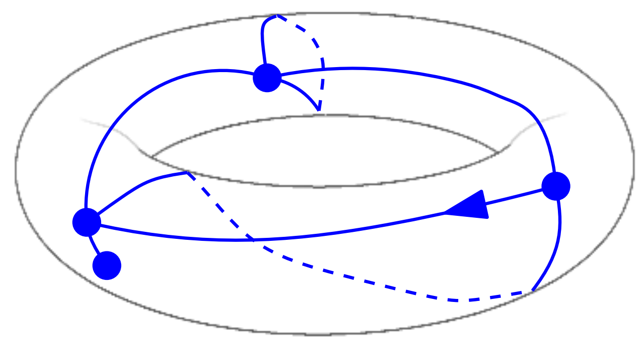
Introduction : maps

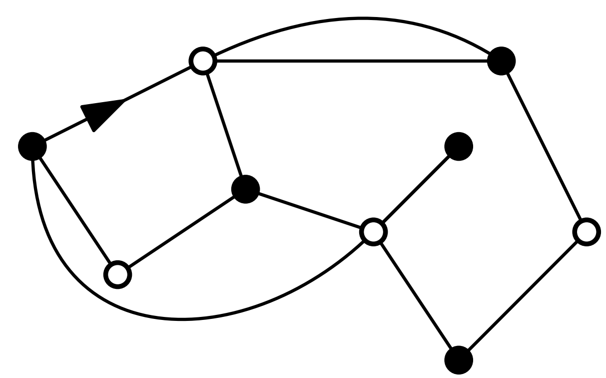
Map = embedding up to homeomorphism of a connected multigraph (loops and multiple edges allowed) in a compact connected orientable surface.
Rooted = an oriented edge is distinguished



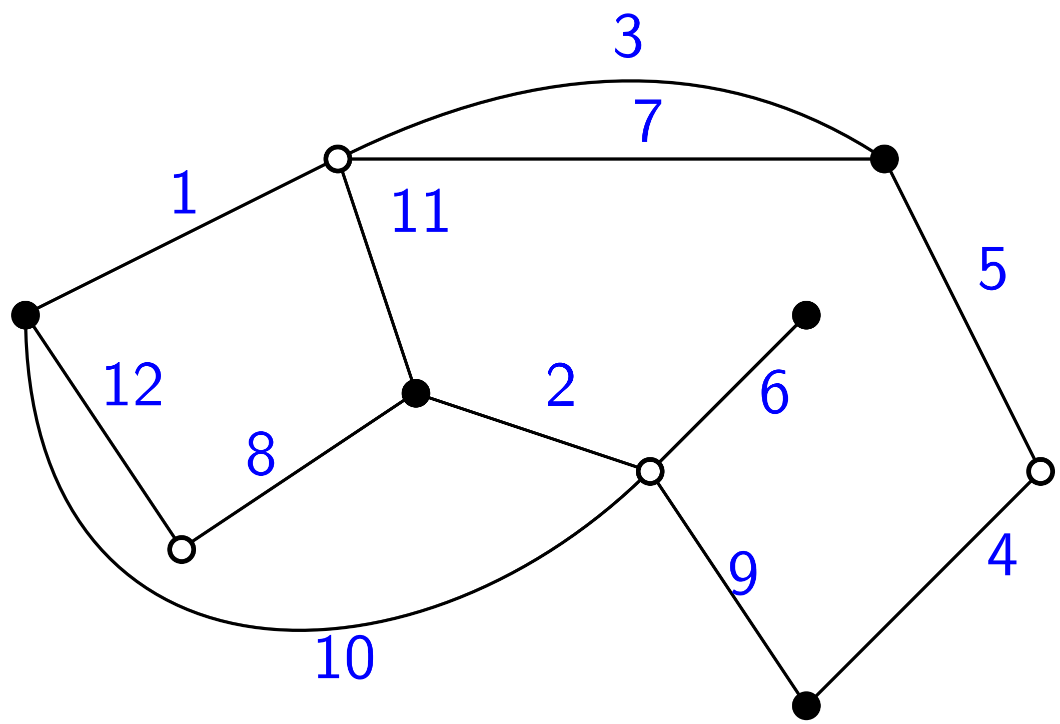
Genus g of the map = genus of the surface = # of handles



Bipartite maps (also called *hypermaps*, or *dessins d'enfants*) : vertices are either black or white, and monochromatic edges are forbidden



Bipartite maps as permutations



$\sigma_\circ = (1,3,7,11)(2,6,9,10)(4,5)(8,12)$

$\sigma_\bullet = (1,12,10)(2,8,11)(3,5,7)(4,9)(6)$

$\phi = (1,8)(2,12)(3,4,10)(5,11,6,9)(7)$

Each vertex/face is a cycle (degree=size of cycle)

$$\sigma_\circ \sigma_\bullet = \phi$$

Connectedness = transitivity of $\langle \sigma_\circ, \sigma_\bullet \rangle$

Edge labeled $\longleftrightarrow$ rooted

$(n-1)!$-to-1

KP/2-Toda hierarchies

$$\partial_x \left(\partial_t u + u\partial_x u + \epsilon^2 \partial_{xxx} u\right) + \lambda \partial_{yy} u = 0$$

"original" KP equation

KP hierarchy

Obtained from the KP equation by studying its symmetries

An infinite set of variables $(p_1, p_2, \dots)$ ... and an infinite number of equations

$$F_{3,1} = F_{2,2} + \frac{1}{2}F_{1,1}^2 + \frac{1}{12}F_{1,1,1,1}$$

$$F_{4,1} = F_{3,2} + F_{1,1}F_{2,1} + \frac{1}{6}F_{1,1,1,2}$$

$$\dots$$

2-Toda hierarchy

Extension of the KP hierarchy with two sets of infinite variables

GFs of maps are solutions [Goulden-Jackson '08], but also Hurwitz numbers, random partitions [Okounkov '0x], ...

KP/2-Toda hierarchies

Very powerful tool, gives very nice, combinatorial formulas

[Goulden–Jackson '08] $\longrightarrow$ triangulations

[Carrell–Chapuy '15] $\longrightarrow$ maps

[Kazarian–Zograf '15] $\longrightarrow$ bipartite maps

[L. '19+] $\longrightarrow$ bipartite maps with prescribed degrees (+ constellations, monotone Hurwitz numbers)

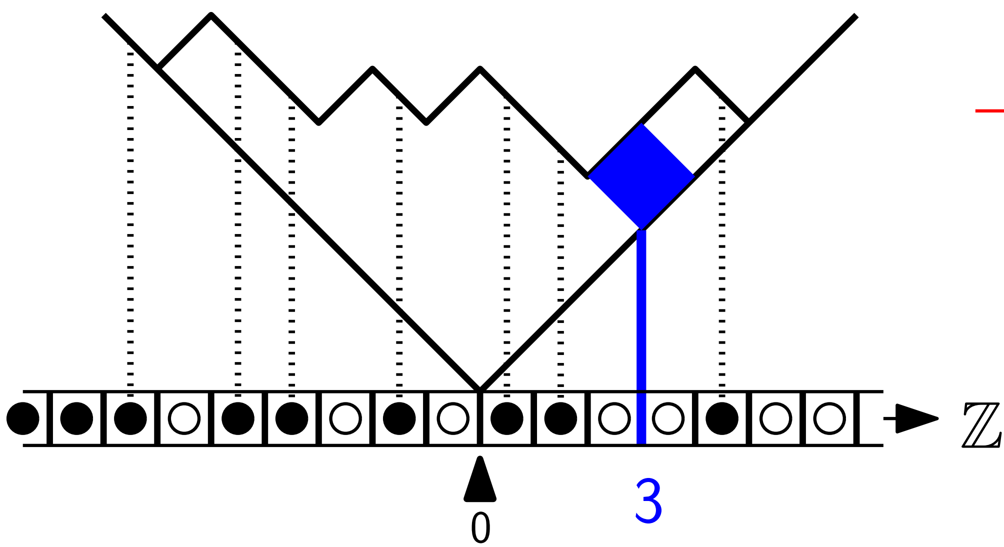
$$\boxed{\begin{aligned}\binom{n+1}{2}B_g(\mathbf{f}) = &\sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ \mathbf{s},\mathbf{t}\neq\mathbf{0}\\ g_1+g_2+g^*=g}} (1+n_1)\binom{v_2}{2g^*+2}B_{g_1}(\mathbf{s})B_{g_2}(\mathbf{t})\\ &+\sum_{g^*\geq 0}\binom{v+2g^*}{2g^*+2}B_{g-g^*}(\mathbf{f})\end{aligned}}$$

$B_g(\mathbf{f})=$ number of bipartite maps of genus g with f_i faces of degree $2i$, $\mathbf{f}=(f_1,f_2,\dots)$

What is it useful for ?

- Counting (very fast, simplest way known)
- Finding bijections that explain the structure of maps ([Chapuy–Féray–Fusy '13], [L. 18+])
- Asymptotic study of random high genus maps + convergence towards random hyperbolic maps ([Budzinski, L. 19+])

The semi-infinite wedge space $\Lambda^{\frac{\infty}{2}}$



"Balanced" diagrams are in bijection with integer partitions

(every diagram is balanced up to a shift)

Maya diagram: $\mathbb{Z} + \frac{1}{2}$ decorated with particles and antiparticles

$\Lambda^{\frac{\infty}{2}} =$ vector space whose orthonormal basis is the Maya diagrams

Operators on $\Lambda^{\frac{\infty}{2}}$

Fermions ψ_k / ψ_k^* : add/remove a particle in position k (up to a sign)

Bosons α_n / α_{-n} : add/remove a ribbon of size n to a partition

Energy H : counts the size of a partition

can be expressed in terms of fermions

The GF of bipartite maps

$W(l, \lambda, \mu)$=number of 4-uples of permutations $(\sigma_1, \sigma_2, \sigma_\lambda, \sigma_\mu)$s.t.

- σ_1, σ_2 have l cycles in total
- $(\sigma_\lambda, \sigma_\mu)$ have cycle types (λ, μ)

$$\boxed{\tau(z, \mathbf{p}, \mathbf{q}, u) = \sum_{\substack{|\lambda|=|\mu|=n>0 \\ l>0}} W(l, \lambda, \mu) \frac{z^n}{n!} u^{2n-l} p_\lambda q_\mu}$$

Remark:

Setting $q_i = \delta_{i=1}$, one recovers the exponential GF of (non necessarily connected, edge labeled) bipartite maps, counted by edges, vertices, and faces of each degree

$\log \tau$ is the GF of connected maps.

The GF of bipartite maps is a solution to the 2-Toda hierarchy

We have

$$\boxed{\tau = \langle \emptyset | \Gamma_+(\mathbf{p}) z^H \Lambda \Gamma_-(\mathbf{q}) | \emptyset \rangle} \qquad \left(\Gamma_\pm(\mathbf{p}) = \exp\left(\sum_{n=1}^\infty \frac{p_n}{n} \alpha_{\pm n}\right)\right)$$

with

$$\Lambda |\nu\rangle = \left(\prod_{\square \in \nu} (1 + uc(\square)) \right)^2 |\nu\rangle$$

(Classical form of solutions : $\langle \emptyset | \Gamma_+(\mathbf{p}) A \Gamma_-(\mathbf{q}) | \emptyset \rangle$ with $[A \bigotimes A, \Omega] = 0$ and $\Omega = \sum_k \psi_k \bigotimes \psi_k^*$)

Proof includes :

- Jucys-Murphy elements
- Representation theory of $\mathfrak{S}_n$
- Schur functions
- the Jacobi-Trudi rule

Outline of the proof of the formula

Because τ is a solution of the 2-Toda hierarchy, the following equation holds:

$$\frac{\partial^2}{\partial p_1 \partial q_1} \log \tau = \frac{\tau_1 \tau_{-1}}{\tau^2} \qquad (1)$$

where τ_1 and τ_{-1} are auxilliary functions related to τ

First, express τ_1 and τ_{-1} in terms of τ, then transform (1) in a quadratic equation in $\log \tau$ (using algebraic tricks)

Then, interpret $\frac{\partial^2}{\partial p_1 \partial q_1}$ combinatorially

Finally, extracting coefficients in the equation yields the recurrence formula

Bijections ? More formulas ?

Take home message:

- The calculus of fermions is a good algebraic framework to work on partitions
- The KP/2-Toda hierarchies are very powerful, apply to many combinatorial models and involve a lot of nice algebraic combinatorics

Thank you !