Joins, ears and Castelnuovo–Mumford regularity

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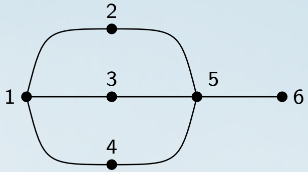
- $\eta \colon K[E_G] \to K[V_G]$ defined by $t_{ij} \mapsto x_i x_j$.
- Defn. Let $I(X_G) \subset K[E_G]$ be the ideal given by:

$$I(X_G) = \eta^{-1}(x_i^2 - x_j^2 \mid_{i,j \in V_G}).$$

$I(X_G)$ has the following minimal generating set:



- $t_{13}^2 - t_{12}^2, \ldots, t_{56}^2 - t_{12}^2,$

  $\eta(t_{ij}^2 - t_{k\ell}^2) = x_i^2 x_j^2 - x_k^2 x_\ell^2 \in (x_i^2 - x_j^2 \mid_{i,j \in V_G})$

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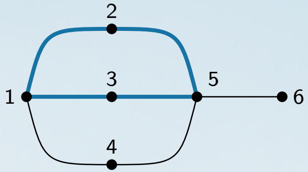

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$\eta(t_{13}t_{25} - t_{12}t_{35}) = x_1x_3x_2x_5 - x_1x_2x_3x_5 = 0 \in (x_i^2 - x_j^2 \mid_{i,j \in V_G})$

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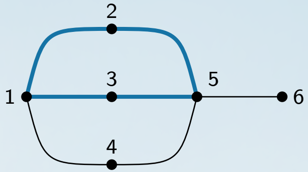

- $t_{13}^2 - t_{12}^2, \ldots, t_{56}^2 - t_{12}^2,$
- $t_{13}t_{25} - t_{12}t_{35}, \quad t_{12}t_{25} - t_{13}t_{35},$

  $\eta(t_{12}t_{25} - t_{13}t_{35}) = x_1x_2^2x_5 - x_1x_3^2x_5 \in (x_i^2 - x_j^2 \mid_{i,j \in V_G})$

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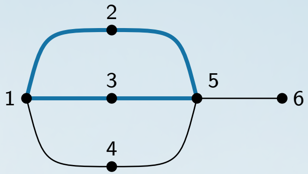

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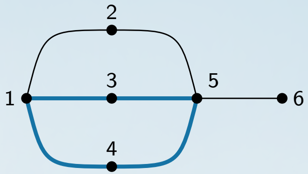

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Example [Using Macaulay2]

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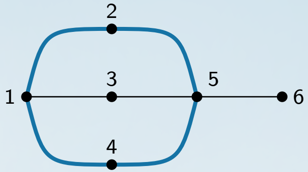

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- $I(X_G)$ is a binomial, homogeneous ideal of $K[E_G]$.

Basic Properties

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- Theorem [N., Vaz Pinto, Villarreal]

$$|X_G| = \begin{cases} 2^{n-b_0} \text{ (bipartite)} & \text{or} \\ 2^{n-b_0-1} \text{ (non-bipartite)}. \end{cases}$$

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Castelnuovo–Mumford regularity

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- Defn. $\operatorname{reg}(G) := r$, Castelnuovo–Mumford regularity.
- **Aim**: relate $\operatorname{reg}(G)$ with an invariant of G.

- $\operatorname{reg}(\mathcal{K}_{a,b}) = \max\{a, b\} - 1$;
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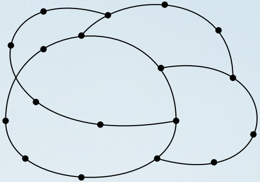
- $\operatorname{reg}(\mathcal{C}_{2k}) = k - 1$.
  [N., Vaz Pinto, Villarreal, 2015]

- G is 2-connected iff it is endowed with an ear decomposition starting from any cycle.

  [Whitney, 1932]



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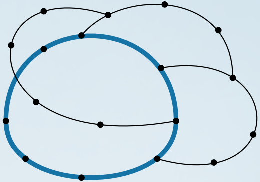

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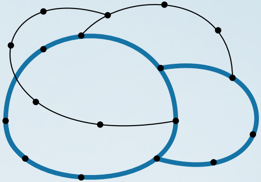

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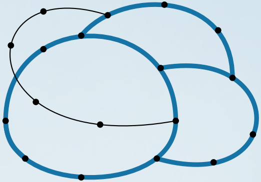

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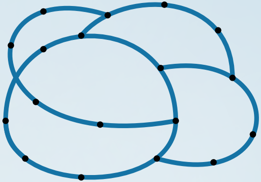

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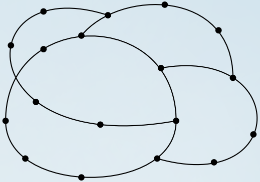
  [Whitney, 1932]
- Defn. $\varphi(G)$ is the minimum number of even length ears in an ear decomposition of G.
  [Frank, 1993]



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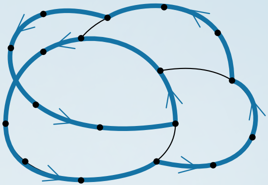

$\varphi(G) = 1$

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  [*Maximum vertex join number*, Solé and Zaslavsky, 1993]
- Theorem [Frank, 1993] If G is 2-connected, then

$$\mu(G) = \frac{n+\varphi(G)-1}{2}.$$

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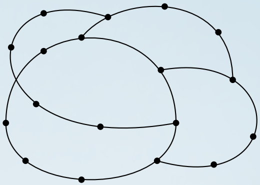
- Nested ear decompositions

  [Eppstein, 1992]

  (*i*) Ears must have both endpoints in the same previous ear.



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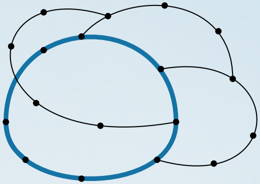

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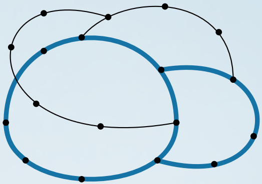

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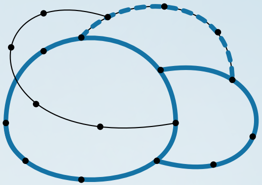

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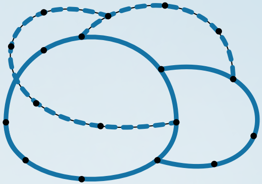

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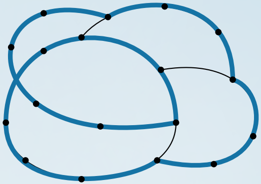
  (*ii*) Ears determine nested intervals in the ears they are attached to.



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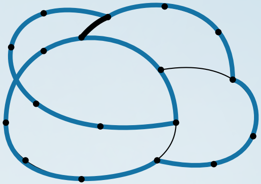

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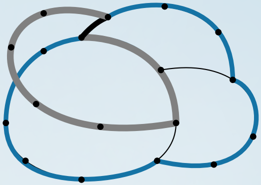

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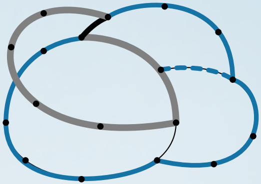

- Theorem [N.]

If G is bipartite and is endowed with a nested ear decomposition with ϵ even length ears then,

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- Corollary

In a nested ear decomposition of a bipartite graph the number of even length ears does not change.

- Theorem [N., Vaz Pinto, Villarreal]

  $\mathrm{reg}(G) \geq \mu(G) - 1$, with equality if G is bipartite.

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  $\operatorname{reg}(G) \geq \mu(G) - 1$, with equality if G is bipartite.

- Corollary

  If G is bipartite and is endowed with a nested ear decomposition then $\varphi(G)$ is attained for *any* nested ear decomposition.