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Promotion on oscillating and alternating tableaux and rotation of matchings and permutations

Stephan Pfannerer, (Martin Rubey, Bruce Westbury)

Institute of Discrete Mathematics and Geometry
Vienna University of Technology

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Alternating tableaux

Definition (Alternating tableaux)

A GL_n*-alternating tableau* $\mathcal{A}$ of length r and shape μ is a sequence of $2r+1$ weakly decreasing vectors in $\mathbb{Z}^n$

$$\mathcal{A} = (\emptyset = \mu^0, \mu^1, \dots, \mu^{2r} = \mu)$$

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$n = 4$, $r = 4$

$\mathcal{A} =$ 0000 1000 $100\bar{1}$ $200\bar{1}$ $20\bar{1}\bar{1}$ $200\bar{1}$ $20\bar{1}\bar{1}$ $200\bar{1}$ $100\bar{1}$

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$n = 4$, $r = 5$

$\mathcal{A} =$ 0000 1000 $100\bar{1}$ $200\bar{1}$ $20\bar{1}\bar{1}$ $200\bar{1}$ $20\bar{1}\bar{1}$ $200\bar{1}$ $100\bar{1}$ 1000 0000

Promotion and evacuation

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- calculated via local rules (van Leeuwen, Lenart)

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
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$$\begin{array}{llllllllllll}
\mathcal{A} = & 0000 & 1000 & 100\bar{1} & 200\bar{1} & 20\overline{11} & 200\bar{1} & 20\overline{11} & 200\bar{1} & 100\bar{1} & 1000 & 0000 \\
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First result

Theorem (P, Rubey, Westbury)

Let $n \geq r$. Then GL_n-alternating tableaux of empty shape and length r are in bijection with permutations of $\{1, \ldots, r\}$.

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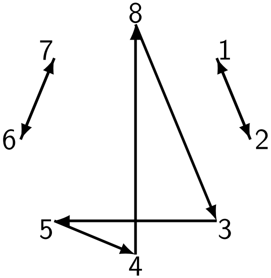
Let $n \geq r$. Then GL_n-alternating tableaux of empty shape and length r are in bijection with permutations of $\{1, \dots, r\}$.The bijection intertwines

$$\begin{aligned} \textit{promotion} &\quad \leftrightarrow \quad \textit{rotation} \\ \textit{evacuation} &\quad \leftrightarrow \quad \textit{reverse-complement} \end{aligned}$$

Rotation and reverse-complement

π

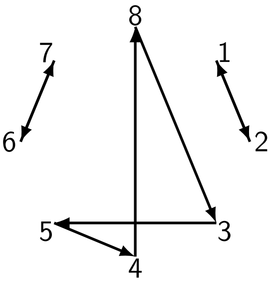


rot π

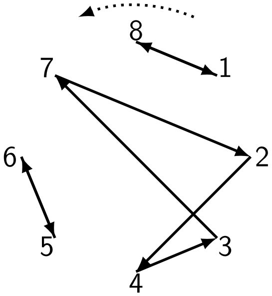
rc π

Rotation and reverse-complement

π



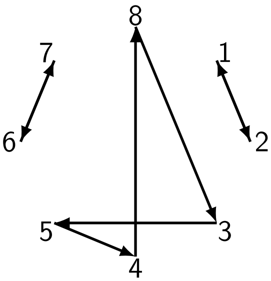
$\operatorname{rot}\pi$



$\operatorname{rc}\pi$

Rotation and reverse-complement

π



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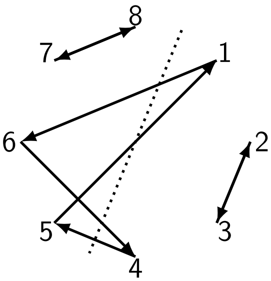
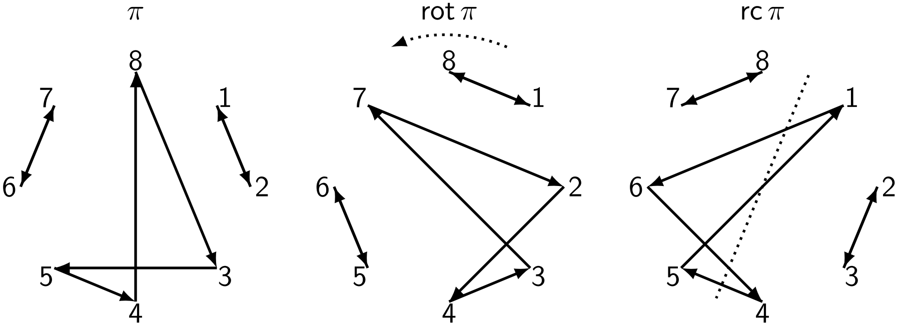


Rotation and reverse-complement



Classical growth diagrams (Fomin)

	1	2	3	4	5	6	7	8
1								
2								
3								
4								
5								
6								
7								
8								

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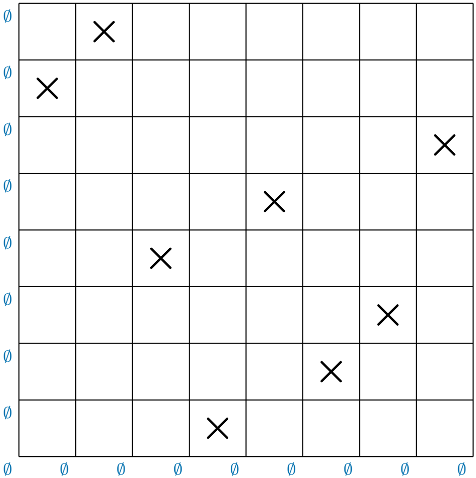
	1	2	3	4	5	6	7	8
1		×						
2	×							
3								×
4					×			
5			×					
6							×	
7						×		
8				×				

Classical growth diagrams (Fomin)



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$$\mu' = \text{sort}(\kappa' + \nu' - \lambda')$$

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$$\mu = \lambda + e_1$$

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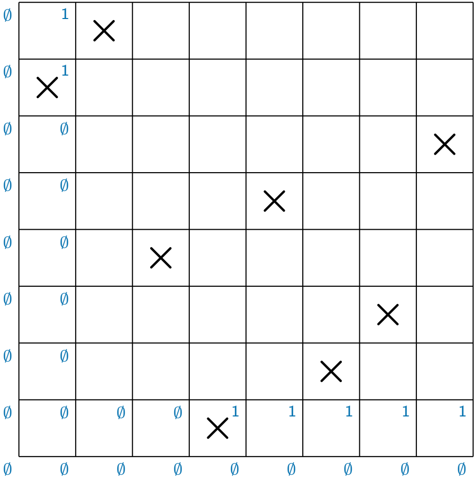

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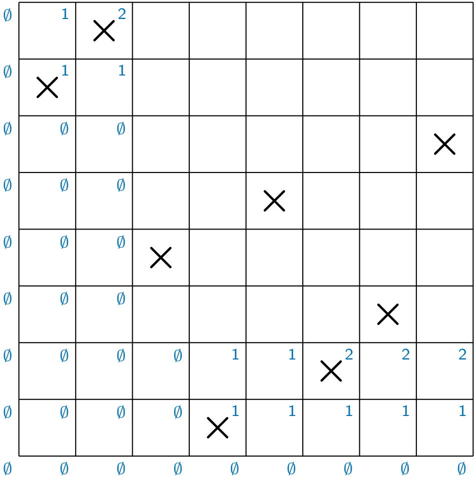

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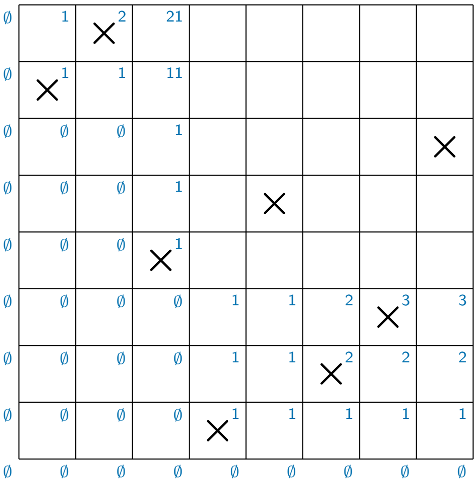

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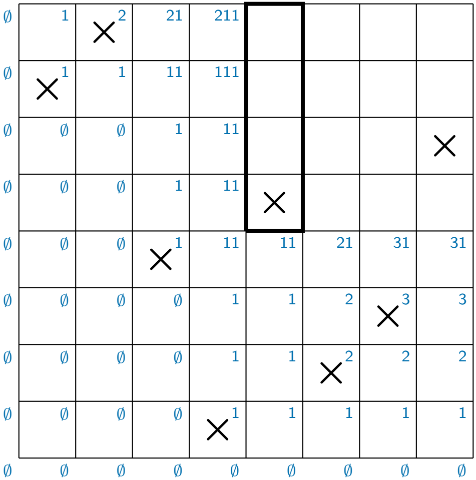

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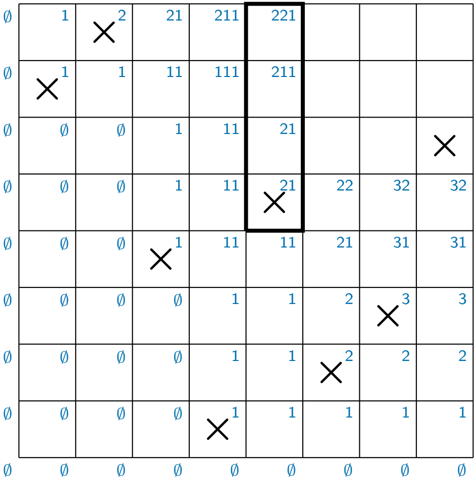
Jump to example

Classical growth diagrams (Fomin)



∅	1	2	21	211	221			
∅	1	1	11	111	211			
∅	∅	∅	1	11	21			
∅	∅	∅	1	11	21	22	32	32
∅	∅	∅	1	11	11	21	31	31
∅	∅	∅	∅	1	1	2	3	3
∅	∅	∅	∅	1	1	2	2	2
∅	∅	∅	∅	1	1	1	1	1
∅	∅	∅	∅	∅	∅	∅	∅	∅

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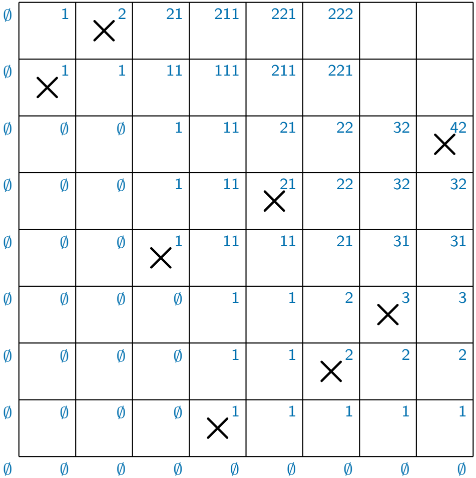

∅	1	2	21	211	221	222		
∅	1	1	11	111	211	221		
∅	∅	∅	1	11	21	22	32	42
∅	∅	∅	1	11	21	22	32	32
∅	∅	∅	1	11	11	21	31	31
∅	∅	∅	∅	1	1	2	3	3
∅	∅	∅	∅	1	1	2	2	2
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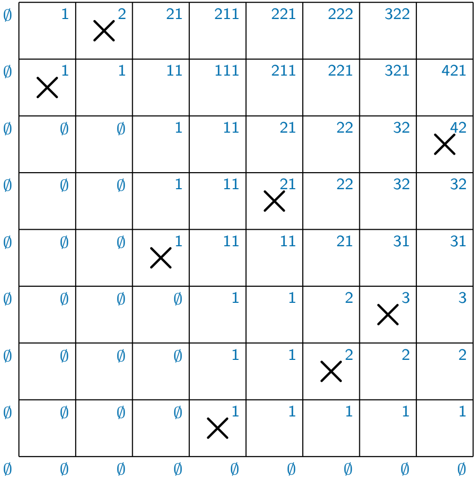

∅	1	2	21	211	221	222	322	
∅	1	1	11	111	211	221	321	421
∅	∅	∅	1	11	21	22	32	42
∅	∅	∅	1	11	21	22	32	32
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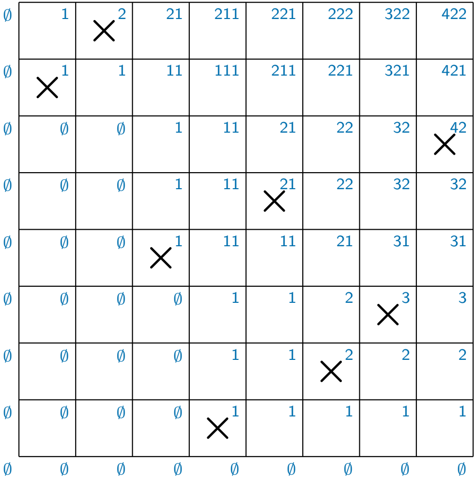

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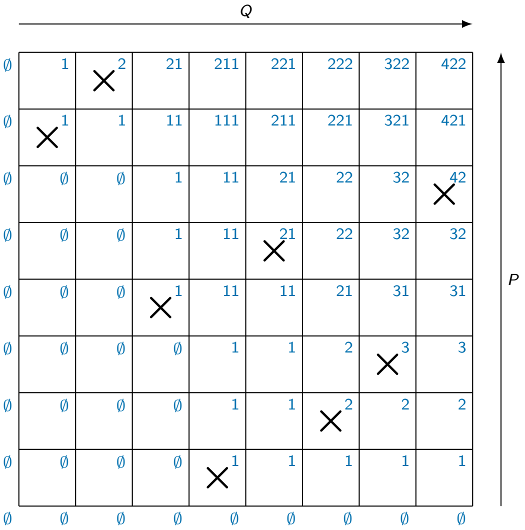
Classical growth diagrams (Fomin)



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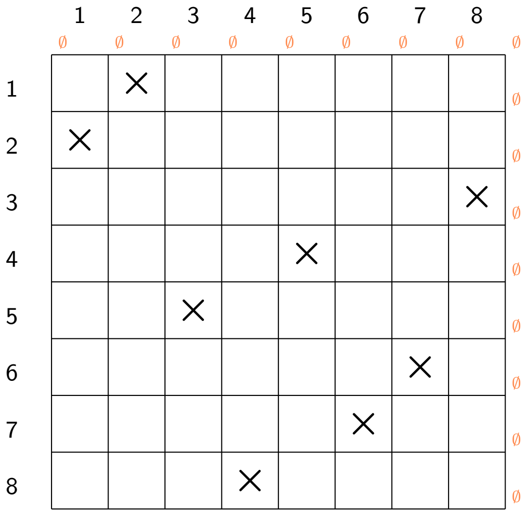
$$\lambda = \mu - e_1$$

Growing again



$$c = \begin{tikzcd}\kappa & \lambda \arrow[l] \arrow[d] \\ \mu \arrow[from=u] \arrow[ur, phantom, "\square"] & \nu \arrow[l]\end{tikzcd}$$

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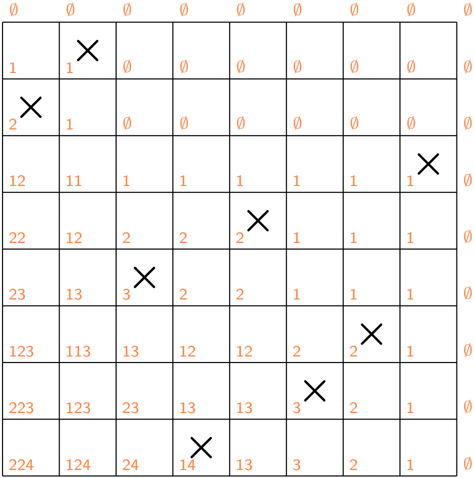
$$\mu = \lambda + e_1$$

Growing again



∅	∅	∅	∅	∅	∅	∅	∅	∅
1	1 ×	∅	∅	∅	∅	∅	∅	∅
2 ×	1	∅	∅	∅	∅	∅	∅	∅
12	11	1	1	1	1	1	1 ×	∅
22	12	2	2	2 ×	1	1	1	∅
23	13	3 ×	2	2	1	1	1	∅
123	113	13	12	12	2	2 ×	1	∅
223	123	23	13	13	3 ×	2	1	∅
224	124	24	14 ×	13	3	2	1	∅

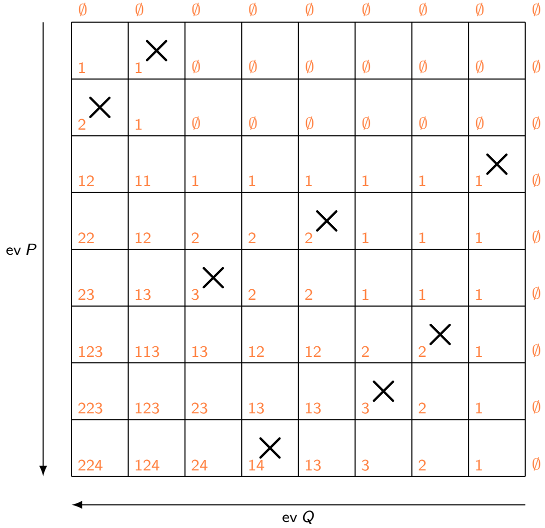
$$c = \begin{tikzcd} \kappa \arrow[d] \arrow[dr, phantom, "\square"] & \lambda \arrow[l] \arrow[d] \\ \mu & \nu \arrow[l] \end{tikzcd}$$

$$\mu' = \mathrm{sort}(\kappa' + \nu' - \lambda')$$

$$c = \begin{tikzcd} \lambda \arrow[d] \arrow[dr, phantom, "\boxed{\times}"] & \lambda \arrow[l] \arrow[d] \\ \mu & \lambda \arrow[l] \end{tikzcd}$$

$$\mu = \lambda + e_1$$

Growing again



∅	∅	∅	∅	∅	∅	∅	∅	∅
1	1 ✕	∅	∅	∅	∅	∅	∅	∅
2 ✕	1	∅	∅	∅	∅	∅	∅	∅
12	11	1	1	1	1	1	1 ✕	∅
22	12	2	2	2 ✕	1	1	1	∅
23	13	3 ✕	2	2	1	1	1	∅
123	113	13	12	12	2	2 ✕	1	∅
223	123	23	13	13	3 ✕	2	1	∅
224	124	24	14 ✕	13	3	2	1	∅

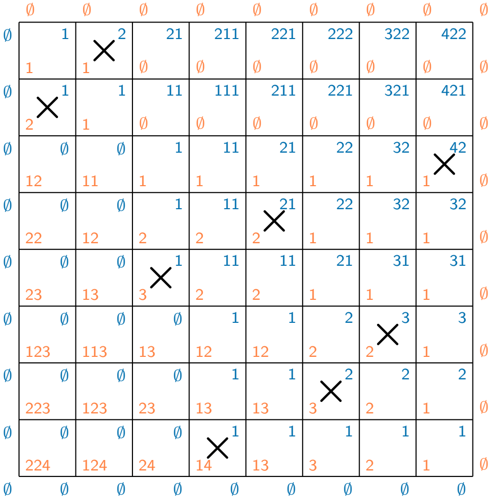
$$c = \begin{tikzcd} \kappa \arrow[d] \arrow[dr, phantom, "\square"] & \lambda \arrow[l] \arrow[d] \\ \mu & \nu \arrow[l] \end{tikzcd}$$

$$\mu' = \text{sort}(\kappa' + \nu' - \lambda')$$

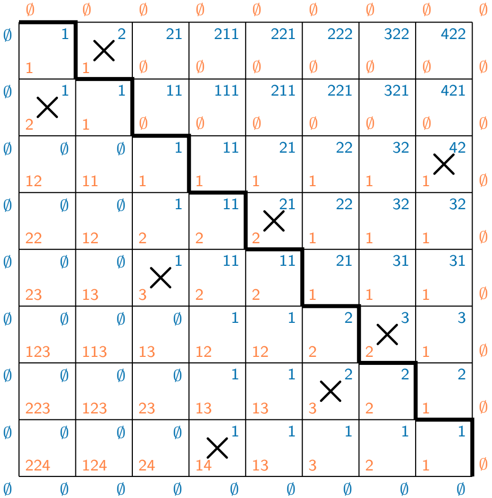
$$c = \begin{tikzcd} \lambda \arrow[d] \arrow[dr, phantom, "\boxed{\times}"] & \lambda \arrow[l] \arrow[d] \\ \mu & \lambda \arrow[l] \end{tikzcd}$$

$$\mu = \lambda + e_1$$

Growing again

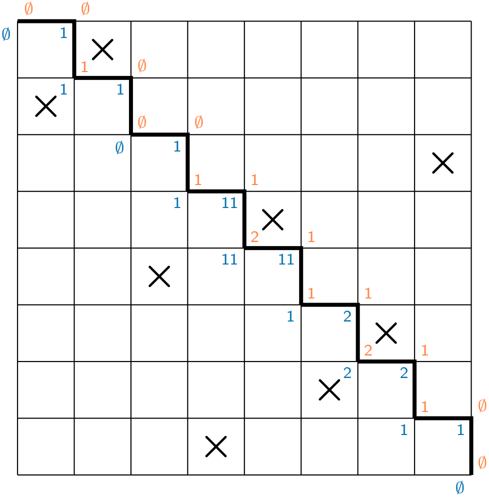


Growing again

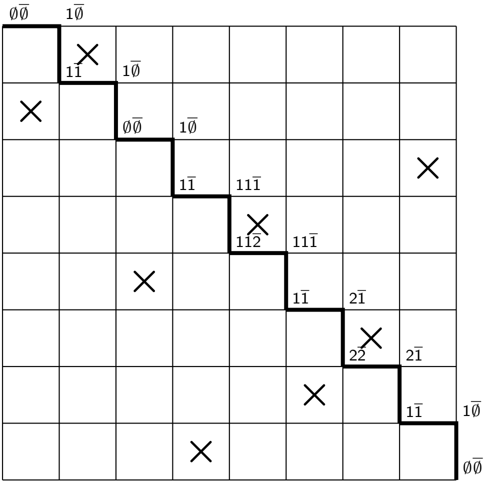


	∅	∅	∅	∅	∅	∅	∅	∅	∅
∅	1 1	2 × 1	21 ∅	211 ∅	221 ∅	222 ∅	322 ∅	422 ∅	∅
∅	1 × 2	1 1	11 ∅	111 ∅	211 ∅	221 ∅	321 ∅	421 ∅	∅
∅	∅ 12	∅ 11	1 1	11 1	21 1	22 1	32 1	42 × 1	∅
∅	∅ 22	∅ 12	1 2	11 2	21 × 2	22 1	32 1	32 1	∅
∅	∅ 23	∅ 13	1 × 3	11 2	11 2	21 1	31 1	31 1	∅
∅	∅ 123	∅ 113	∅ 13	1 12	1 12	2 2	3 × 2	3 1	∅
∅	∅ 223	∅ 123	∅ 23	1 13	1 13	2 × 3	2 2	2 1	∅
∅	∅ 224	∅ 124	∅ 24	1 × 14	1 13	1 3	1 2	1 1	∅
∅	∅	∅	∅	∅	∅	∅	∅	∅	

Growing again



Growing again



Alternating Tableaux, Promotion, Evacuation

$\mathcal{A} = 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

Alternating Tableaux, Promotion, Evacuation

$$\begin{array}{lllllllllllllllll} \mathcal{A} = 000 & 100 & 10\overline{1} & 100 & 000 & 100 & 10\overline{1} & 11\overline{1} & 11\overline{2} & 11\overline{1} & 10\overline{1} & 20\overline{1} & 20\overline{2} & 20\overline{1} & 10\overline{1} & 100 & 000 \\ & & 100 & & & & & & & & & & & & & & \\ & & 000 & & & & & & & & & & & & & & \end{array}$$

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
\kappa \arrow[r] & \mu
\end{tikzcd}$$

$$\mu = \mathsf{dom}_{\mathfrak{S}_n}(\kappa + \nu - \lambda)$$

Alternating Tableaux, Promotion, Evacuation

$$\mathcal{A} = 000\ 100\ \begin{array}[t]{c}10\bar{1}\\ 100\\ 000\end{array}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$$

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
\kappa \arrow[r] & \mu
\end{tikzcd}$$

$$\mu = \mathsf{sort}(\kappa + \nu - \lambda)$$

Alternating Tableaux, Promotion, Evacuation

$$\begin{aligned}\mathcal{A} = 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000 \\ 100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100 \\ 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\end{aligned}$$

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
\kappa \arrow[r] & \mu
\end{tikzcd}$$

$$\mu = \mathsf{sort}(\kappa + \nu - \lambda)$$

Alternating Tableaux, Promotion, Evacuation

$$\begin{aligned}
\mathcal{A} &= 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000\\
&100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100\\
\operatorname{pr}\mathcal{A} &= 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000
\end{aligned}$$

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
\kappa \arrow[r] & \mu
\end{tikzcd}$$

$$\mu = \operatorname{sort}(\kappa + \nu - \lambda)$$

Alternating Tableaux, Promotion, Evacuation

$$\mathcal{A} = 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$$

$$100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$$

$$\operatorname{pr}\mathcal{A} = 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$$

$$100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$$

$$000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$$

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
\kappa \arrow[r] & \mu
\end{tikzcd}$$

$$\mu = \operatorname{sort}(\kappa + \nu - \lambda)$$

Alternating Tableaux, Promotion, Evacuation

$\mathcal{A} = 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$\operatorname{pr}\mathcal{A} = 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 110\ 11\bar{1}\ 110\ 100\ 200\ 20\bar{1}\ 200\ 100\ 110\ 100$

$000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}$

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
\kappa \arrow[r] & \mu
\end{tikzcd}$$

$$\mu = \operatorname{sort}(\kappa + \nu - \lambda)$$

Alternating Tableaux, Promotion, Evacuation

$$\mathcal{A} = 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$$

$$100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$$

$$\operatorname{pr}\mathcal{A} = 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$$

$$100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$$

$$000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$$

$$100\ 110\ 11\bar{1}\ 110\ 100\ 200\ 20\bar{1}\ 200\ 100\ 110\ 100$$

$$000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}$$

$$100\ 110\ 11\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 11\bar{1}\ 21\bar{1}\ 20\bar{1}$$

$$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 20\bar{1}\ 2\overline{11}$$

$$100\ 200\ 20\bar{1}\ 200\ 100\ 200\ 20\bar{1}$$

$$000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}$$

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
\kappa \arrow[r] & \mu
\end{tikzcd}$$

$$\mu = \operatorname{sort}(\kappa + \nu - \lambda)$$

Alternating Tableaux, Promotion, Evacuation

$$\mathcal{A} = 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$$

$$100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$$

$$\mathrm{pr}\,\mathcal{A} = 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$$

$$100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$$

$$000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$$

$$100\ 110\ 11\bar{1}\ 110\ 100\ 200\ 20\bar{1}\ 200\ 100\ 110\ 100$$

$$000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}$$

$$100\ 110\ 11\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 11\bar{1}\ 21\bar{1}\ 20\bar{1}$$

$$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 20\bar{1}\ 2\overline{11}$$

$$100\ 200\ 20\bar{1}\ 200\ 100\ 200\ 20\bar{1}$$

$$000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}$$

$$100\ 110\ 100\ 200\ 20\bar{1}$$

$$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}$$

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
\kappa \arrow[r] & \mu
\end{tikzcd}$$

$$\mu = \mathrm{sort}(\kappa + \nu - \lambda)$$

Alternating Tableaux, Promotion, Evacuation

$$\begin{array}{rr}
\mathcal{A} = & 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000 \\
 & 100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100 \\
\mathrm{pr}\,\mathcal{A} = & 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000 \\
 & 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100 \\
 & 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000 \\
 & 100\ 110\ 11\bar{1}\ 110\ 100\ 200\ 20\bar{1}\ 200\ 100\ 110\ 100 \\
 & 000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1} \\
 & 100\ 110\ 11\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 11\bar{1}\ 21\bar{1}\ 20\bar{1} \\
 & 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 20\bar{1}\ 2\bar{1}\bar{1} \\
 & 100\ 200\ 20\bar{1}\ 200\ 100\ 200\ 20\bar{1} \\
 & 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1} \\
 & 100\ 110\ 100\ 200\ 20\bar{1} \\
 & 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2} \\
 & 100\ 200\ 20\bar{1} \\
 & 000\ 100\ 10\bar{1}
\end{array}$$

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
\kappa \arrow[r] & \mu
\end{tikzcd}$$

$$\mu = \mathrm{sort}(\kappa + \nu - \lambda)$$

Alternating Tableaux, Promotion, Evacuation

$$\begin{array}{rccccccccccccccccccc}
\mathcal{A} = & 000 & 100 & 10\bar{1} & 100 & 000 & 100 & 10\bar{1} & 11\bar{1} & 11\bar{2} & 11\bar{1} & 10\bar{1} & 20\bar{1} & 20\bar{2} & 20\bar{1} & 10\bar{1} & 100 & 000 & & \\
 & & & 100 & 110 & 100 & 200 & 20\bar{1} & 21\bar{1} & 21\bar{2} & 21\bar{1} & 20\bar{1} & 30\bar{1} & 30\bar{2} & 30\bar{1} & 20\bar{1} & 200 & 100 & & \\
\mathrm{pr}\,\mathcal{A} = & & & 000 & 100 & 10\bar{1} & 20\bar{1} & 20\bar{2} & 21\bar{2} & 21\bar{3} & 21\bar{2} & 20\bar{2} & 30\bar{2} & 30\bar{3} & 30\bar{2} & 20\bar{2} & 20\bar{1} & 10\bar{1} & 100 & 000 \\
 & & & & & 100 & 200 & 20\bar{1} & 21\bar{1} & 21\bar{2} & 21\bar{1} & 20\bar{1} & 30\bar{1} & 30\bar{2} & 30\bar{1} & 20\bar{1} & 200 & 100 & & \\
 & & & & & 000 & 100 & 10\bar{1} & 11\bar{1} & 11\bar{2} & 11\bar{1} & 10\bar{1} & 20\bar{1} & 20\bar{2} & 20\bar{1} & 10\bar{1} & 100 & 000 & & \\
 & & & & & & & 100 & 110 & 11\bar{1} & 110 & 100 & 200 & 20\bar{1} & 200 & 100 & 110 & 100 & & \\
 & & & & & & & 000 & 100 & 10\bar{1} & 100 & 10\bar{1} & 20\bar{1} & 20\bar{2} & 20\bar{1} & 10\bar{1} & 11\bar{1} & 10\bar{1} & & \\
 & & & & & & & & & 100 & 110 & 11\bar{1} & 21\bar{1} & 21\bar{2} & 21\bar{1} & 11\bar{1} & 21\bar{1} & 20\bar{1} & & \\
 & & & & & & & & & 000 & 100 & 10\bar{1} & 20\bar{1} & 20\bar{2} & 20\bar{1} & 10\bar{1} & 20\bar{1} & 2\bar{1}\bar{1} & & \\
 & & & & & & & & & & & 100 & 200 & 20\bar{1} & 200 & 100 & 200 & 20\bar{1} & & \\
 & & & & & & & & & & & 000 & 100 & 10\bar{1} & 100 & 000 & 100 & 10\bar{1} & & \\
 & & & & & & & & & & & & & 100 & 110 & 100 & 200 & 20\bar{1} & & \\
 & & & & & & & & & & & & & 000 & 100 & 10\bar{1} & 20\bar{1} & 20\bar{2} & & \\
 & & & & & & & & & & & & & & & 100 & 200 & 20\bar{1} & & \\
 & & & & & & & & & & & & & & & 000 & 100 & 10\bar{1} & & \\
 & & & & & & & & & & & & & & & & & 100 & & \\
 & & & & & & & & & & & & & & & & & 000 & &
\end{array}$$

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
\kappa \arrow[r] & \mu
\end{tikzcd}$$

$$\mu = \mathrm{sort}(\kappa + \nu - \lambda)$$

Alternating Tableaux, Promotion, Evacuation

$\mathcal{A} = 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$\operatorname{pr}\mathcal{A} = 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 110\ 11\bar{1}\ 110\ 100\ 200\ 20\bar{1}\ 200\ 100\ 110\ 100$

$000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}$

$100\ 110\ 11\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 11\bar{1}\ 21\bar{1}\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 20\bar{1}\ 2\overline{11}$

$100\ 200\ 20\bar{1}\ 200\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}$

$100\ 110\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}$

$100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}$

100

000

$\operatorname{ev}\mathcal{A} =$

Alternating Tableaux, Promotion, Evacuation

$\mathcal{A} =$ $000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

pr $\mathcal{A} =$ $000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 110\ 11\bar{1}\ 110\ 100\ 200\ 20\bar{1}\ 200\ 100\ 110\ 100$

$000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}$

$100\ 110\ 11\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 11\bar{1}\ 21\bar{1}\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 20\bar{1}\ 2\overline{11}$

$100\ 200\ 20\bar{1}\ 200\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}$

$100\ 110\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}$

$100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}$

100

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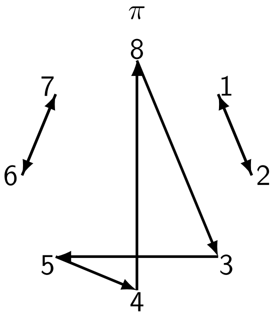
ev $\mathcal{A} =$



Alternating Tableaux, Promotion, Evacuation

$\mathcal{A} = 000\ 100\ 10\bar1\ 100\ 000\ 100\ 10\bar1\ 11\bar1\ 11\bar2\ 11\bar1\ 10\bar1\ 20\bar1\ 20\bar2\ 20\bar1\ 10\bar1\ 100\ 000$

$100\ 110\ 100\ 200\ 20\bar1\ 21\bar1\ 21\bar2\ 21\bar1\ 20\bar1\ 30\bar1\ 30\bar2\ 30\bar1\ 20\bar1\ 200\ 100$

$\mathrm{pr}\,\mathcal{A} = 000\ 100\ 10\bar1\ 20\bar1\ 20\bar2\ 21\bar2\ 21\bar3\ 21\bar2\ 20\bar2\ 30\bar2\ 30\bar3\ 30\bar2\ 20\bar2\ 20\bar1\ 10\bar1\ 100\ 000$

$100\ 200\ 20\bar1\ 21\bar1\ 21\bar2\ 21\bar1\ 20\bar1\ 30\bar1\ 30\bar2\ 30\bar1\ 20\bar1\ 200\ 100$

$000\ 100\ 10\bar1\ 11\bar1\ 11\bar2\ 11\bar1\ 10\bar1\ 20\bar1\ 20\bar2\ 20\bar1\ 10\bar1\ 100\ 000$

$100\ 110\ 11\bar1\ 110\ 100\ 200\ 20\bar1\ 200\ 100\ 110\ 100$

$000\ 100\ 10\bar1\ 100\ 10\bar1\ 20\bar1\ 20\bar2\ 20\bar1\ 10\bar1\ 11\bar1\ 10\bar1$

$100\ 110\ 11\bar1\ 21\bar1\ 21\bar2\ 21\bar1\ 11\bar1\ 21\bar1\ 20\bar1$

$000\ 100\ 10\bar1\ 20\bar1\ 20\bar2\ 20\bar1\ 10\bar1\ 20\bar1\ 2\bar1\bar1$

$100\ 200\ 20\bar1\ 200\ 100\ 200\ 20\bar1$

$000\ 100\ 10\bar1\ 100\ 000\ 100\ 10\bar1$

$100\ 110\ 100\ 200\ 20\bar1$

$000\ 100\ 10\bar1\ 20\bar1\ 20\bar2$

$100\ 200\ 20\bar1$

$000\ 100\ 10\bar1$

100

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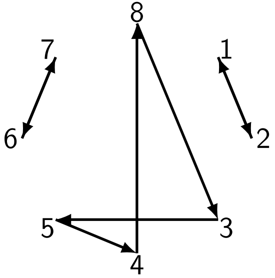
$\mathrm{ev}\,\mathcal{A} =$

π



Alternating Tableaux, Promotion, Evacuation

$\mathcal{A} =$ $000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

1 $100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$\mathrm{pr}\,\mathcal{A} =$ $000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

2 $100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

3 $100\ 110\ 11\bar{1}\ 110\ 100\ 200\ 20\bar{1}\ 200\ 100\ 110\ 100$

$000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}$

4 $100\ 110\ 11\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 11\bar{1}\ 21\bar{1}\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 20\bar{1}\ 2\bar{1}\bar{1}$

5 $100\ 200\ 20\bar{1}\ 200\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}$

6 $100\ 110\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}$

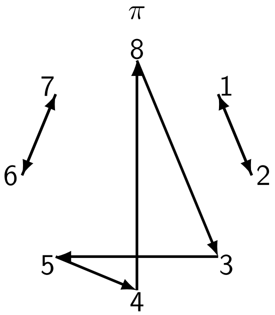
7 $100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}$

8 100

000

$\mathrm{ev}\,\mathcal{A} =$



Alternating Tableaux, Promotion, Evacuation

$\mathcal{A} =$ $000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

1 $100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

pr $\mathcal{A} =$ $000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

2 $100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

3 $100\ 110\ 11\bar{1}\ 110\ 100\ 200\ 20\bar{1}\ 200\ 100\ 110\ 100$

$000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}$

4 $100\ 110\ 11\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 11\bar{1}\ 21\bar{1}\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 20\bar{1}\ 2\bar{1}\bar{1}$

5 $100\ 200\ 20\bar{1}\ 200\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}$

6 $100\ 110\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}$

7 $100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}$

8 100

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ev $\mathcal{A} =$

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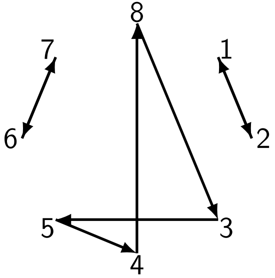
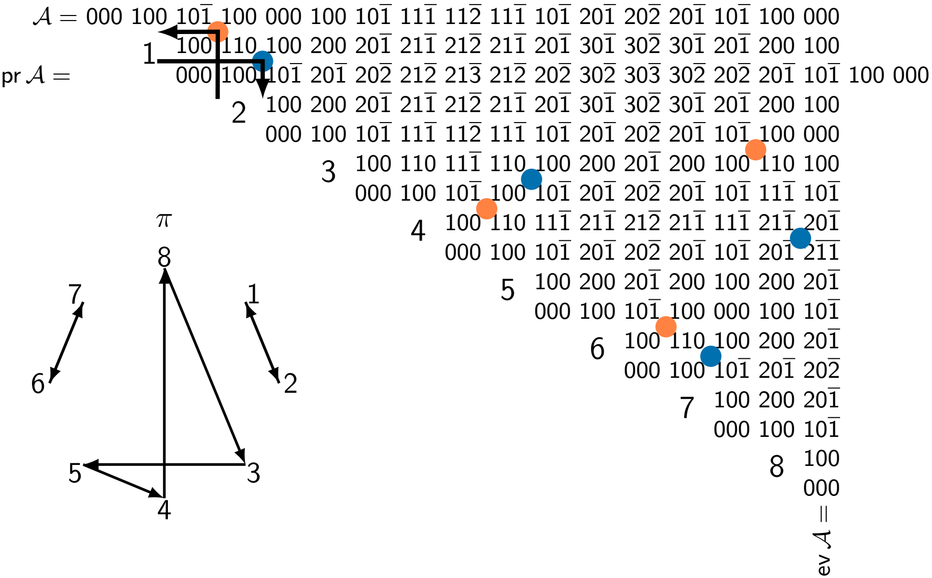


Alternating Tableaux, Promotion, Evacuation



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$100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

1

$\operatorname{pr}\mathcal{A} = 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

2 $100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

3 $100\ 110\ 11\bar{1}\ 110\ 100\ 200\ 20\bar{1}\ 200\ 100\ 110\ 100$

$000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}$

4 $100\ 110\ 11\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 11\bar{1}\ 21\bar{1}\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 20\bar{1}\ 2\bar{1}\bar{1}$

5 $100\ 200\ 20\bar{1}\ 200\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}$

6 $100\ 110\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}$

7 $100\ 200\ 20\bar{1}$

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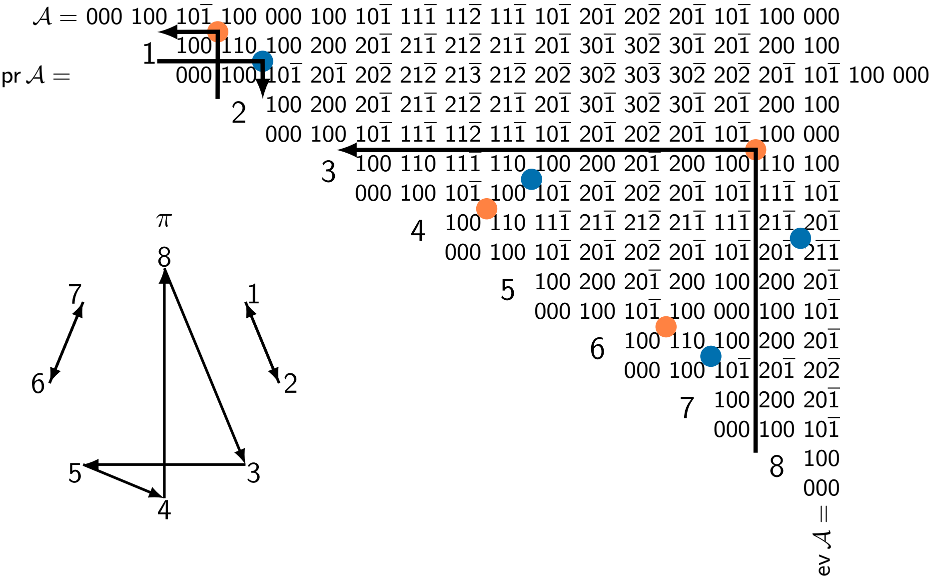
Alternating Tableaux, Promotion, Evacuation



$\mathcal{A} =$ 000 100 $10\bar{1}$ 100 000 100 $10\bar{1}$ $11\bar{1}$ $11\bar{2}$ $11\bar{1}$ $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ 100 000

1 100 110 100 200 $20\bar{1}$ $21\bar{1}$ $21\bar{2}$ $21\bar{1}$ $20\bar{1}$ $30\bar{1}$ $30\bar{2}$ $30\bar{1}$ $20\bar{1}$ 200 100

$\mathrm{pr}\,\mathcal{A} =$ 000 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $21\bar{2}$ $21\bar{3}$ $21\bar{2}$ $20\bar{2}$ $30\bar{2}$ $30\bar{3}$ $30\bar{2}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ 100 000

2 100 200 $20\bar{1}$ $21\bar{1}$ $21\bar{2}$ $21\bar{1}$ $20\bar{1}$ $30\bar{1}$ $30\bar{2}$ $30\bar{1}$ $20\bar{1}$ 200 100

000 100 $10\bar{1}$ $11\bar{1}$ $11\bar{2}$ $11\bar{1}$ $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ 100 000

3 100 110 $11\bar{1}$ 110 100 200 $20\bar{1}$ 200 100 110 100

000 100 $10\bar{1}$ 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ $11\bar{1}$ $10\bar{1}$

4 100 110 $11\bar{1}$ $21\bar{1}$ $21\bar{2}$ $21\bar{1}$ $11\bar{1}$ $21\bar{1}$ $20\bar{1}$

000 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ $20\bar{1}$ $2\overline{11}$

5 100 200 $20\bar{1}$ 200 100 200 $20\bar{1}$

000 100 $10\bar{1}$ 100 000 100 $10\bar{1}$

6 100 110 100 200 $20\bar{1}$

000 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$

7 100 200 $20\bar{1}$

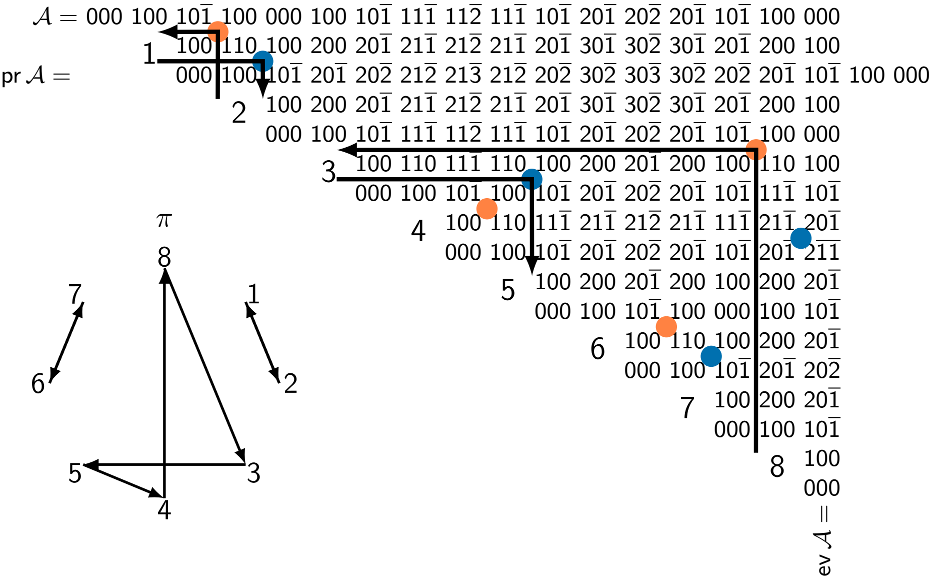
000 100 $10\bar{1}$

8 100

000

$\mathrm{ev}\,\mathcal{A} =$

Alternating Tableaux, Promotion, Evacuation



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$100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$\mathrm{pr}\,\mathcal{A} = 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 110\ 11\bar{1}\ 110\ 100\ 200\ 20\bar{1}\ 200\ 100\ 110\ 100$

$000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}$

$100\ 110\ 11\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 11\bar{1}\ 21\bar{1}\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 20\bar{1}\ 21\bar{1}$

$100\ 200\ 20\bar{1}\ 200\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}$

$100\ 110\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}$

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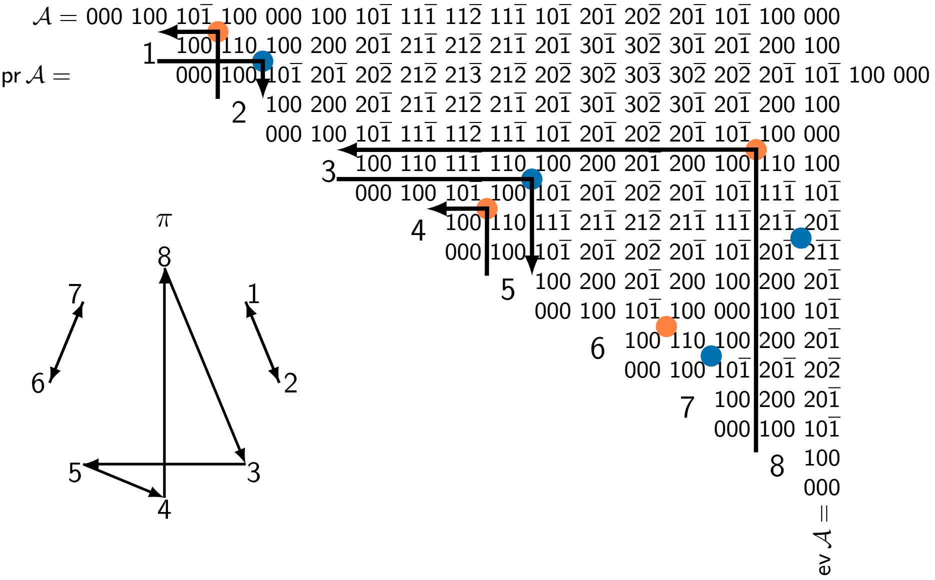
Alternating Tableaux, Promotion, Evacuation



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100 110 100 200 $20\bar{1}$ $21\bar{1}$ $21\bar{2}$ $21\bar{1}$ $20\bar{1}$ $30\bar{1}$ $30\bar{2}$ $30\bar{1}$ $20\bar{1}$ 200 100

1

$\mathrm{pr}\,\mathcal{A} =$ 000 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $21\bar{2}$ $21\bar{3}$ $21\bar{2}$ $20\bar{2}$ $30\bar{2}$ $30\bar{3}$ $30\bar{2}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ 100 000

2 100 200 $20\bar{1}$ $21\bar{1}$ $21\bar{2}$ $21\bar{1}$ $20\bar{1}$ $30\bar{1}$ $30\bar{2}$ $30\bar{1}$ $20\bar{1}$ 200 100

000 100 $10\bar{1}$ $11\bar{1}$ $11\bar{2}$ $11\bar{1}$ $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ 100 000

100 110 $11\bar{1}$ 110 100 200 $20\bar{1}$ 200 100 110 100

3 000 100 $10\bar{1}$ 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ $11\bar{1}$ $10\bar{1}$

4 100 110 $11\bar{1}$ $21\bar{1}$ $21\bar{2}$ $21\bar{1}$ $11\bar{1}$ $21\bar{1}$ $20\bar{1}$

000 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ $20\bar{1}$ $2\bar{1}\bar{1}$

5 100 200 $20\bar{1}$ 200 100 200 $20\bar{1}$

000 100 $10\bar{1}$ 100 000 100 $10\bar{1}$

6 100 110 100 200 $20\bar{1}$

000 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$

7 100 200 $20\bar{1}$

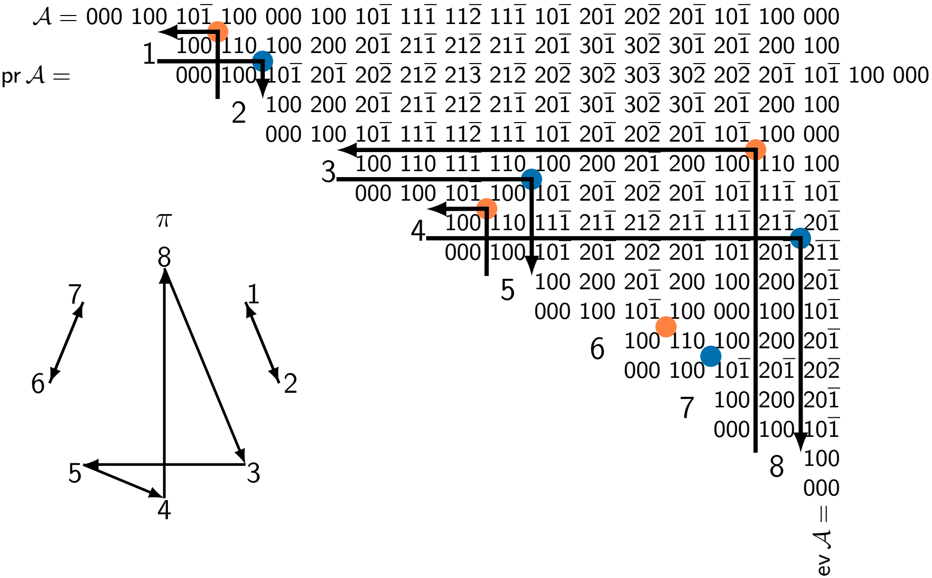
000 100 $10\bar{1}$

8 100

000

$\mathrm{ev}\,\mathcal{A} =$

Alternating Tableaux, Promotion, Evacuation



$\mathcal{A} = 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$\operatorname{pr}\mathcal{A} = 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 110\ 11\bar{1}\ 110\ 100\ 200\ 20\bar{1}\ 200\ 100\ 110\ 100$

$000\ 100\ 10\bar{1}\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 11\bar{1}\ 10\bar{1}$

$100\ 110\ 11\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 11\bar{1}\ 21\bar{1}\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 20\bar{1}\ 21\bar{1}$

$100\ 200\ 20\bar{1}\ 200\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}$

$100\ 110\ 100\ 200\ 20\bar{1}$

$000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}$

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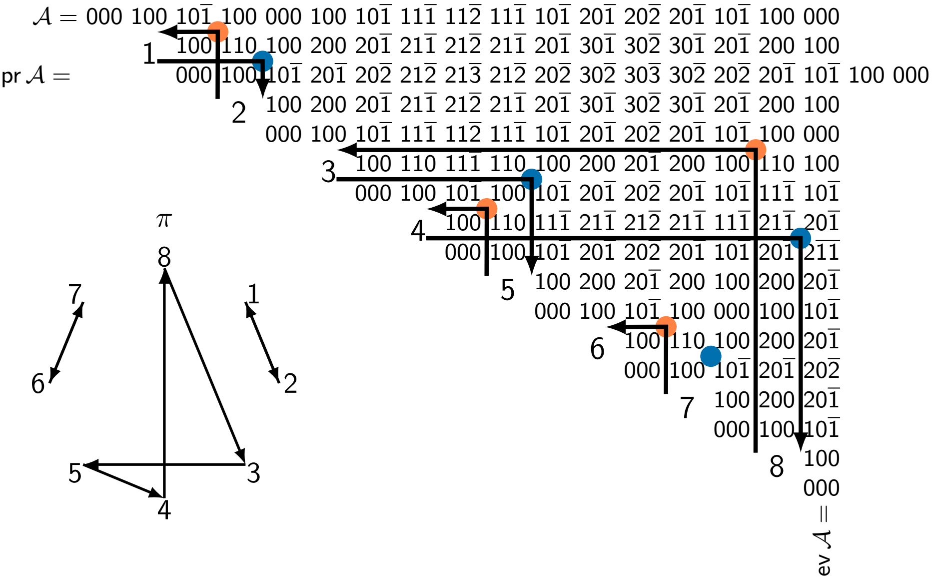
$\operatorname{ev}\mathcal{A} =$

Alternating Tableaux, Promotion, Evacuation



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100 110 100 200 $20\bar{1}$ $21\bar{1}$ $21\bar{2}$ $21\bar{1}$ $20\bar{1}$ $30\bar{1}$ $30\bar{2}$ $30\bar{1}$ $20\bar{1}$ 200 100

pr $\mathcal{A} =$ 000 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $21\bar{2}$ $21\bar{3}$ $21\bar{2}$ $20\bar{2}$ $30\bar{2}$ $30\bar{3}$ $30\bar{2}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ 100 000

100 200 $20\bar{1}$ $21\bar{1}$ $21\bar{2}$ $21\bar{1}$ $20\bar{1}$ $30\bar{1}$ $30\bar{2}$ $30\bar{1}$ $20\bar{1}$ 200 100

000 100 $10\bar{1}$ $11\bar{1}$ $11\bar{2}$ $11\bar{1}$ $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ 100 000

100 110 $11\bar{1}$ 110 100 200 $20\bar{1}$ 200 100 110 100

000 100 $10\bar{1}$ 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ $11\bar{1}$ $10\bar{1}$

100 110 $11\bar{1}$ $21\bar{1}$ $21\bar{2}$ $21\bar{1}$ $11\bar{1}$ $21\bar{1}$ $20\bar{1}$

000 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$ $20\bar{1}$ $10\bar{1}$ $20\bar{1}$ $2\bar{1}\bar{1}$

100 200 $20\bar{1}$ 200 100 200 $20\bar{1}$

000 100 $10\bar{1}$ 100 000 100 $10\bar{1}$

100 110 100 200 $20\bar{1}$

000 100 $10\bar{1}$ $20\bar{1}$ $20\bar{2}$

100 200 $20\bar{1}$

000 100 $10\bar{1}$

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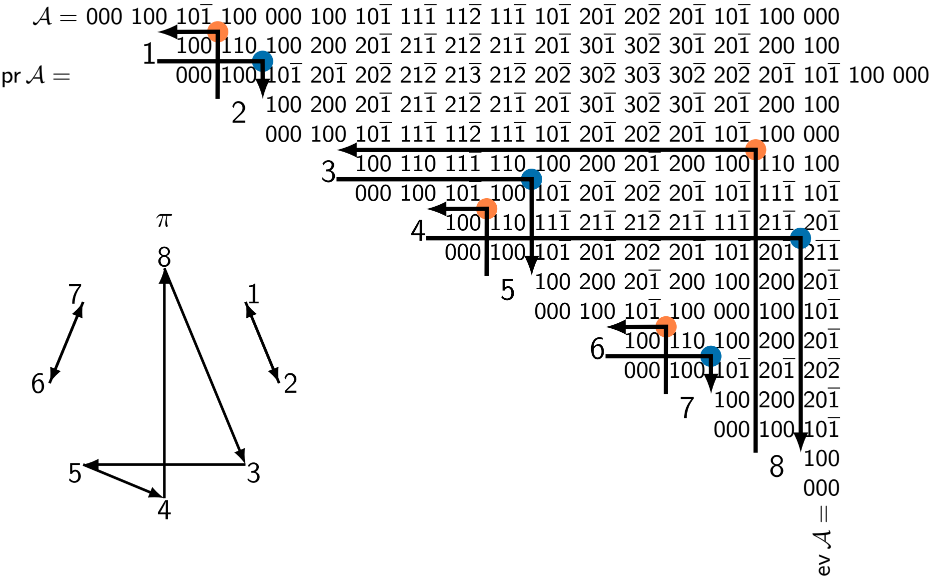
ev $\mathcal{A} =$

Alternating Tableaux, Promotion, Evacuation



$\mathcal{A} = 000\ 100\ 10\bar{1}\ 100\ 000\ 100\ 10\bar{1}\ 11\bar{1}\ 11\bar{2}\ 11\bar{1}\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

$100\ 110\ 100\ 200\ 20\bar{1}\ 21\bar{1}\ 21\bar{2}\ 21\bar{1}\ 20\bar{1}\ 30\bar{1}\ 30\bar{2}\ 30\bar{1}\ 20\bar{1}\ 200\ 100$

$\operatorname{pr}\mathcal{A} = 000\ 100\ 10\bar{1}\ 20\bar{1}\ 20\bar{2}\ 21\bar{2}\ 21\bar{3}\ 21\bar{2}\ 20\bar{2}\ 30\bar{2}\ 30\bar{3}\ 30\bar{2}\ 20\bar{2}\ 20\bar{1}\ 10\bar{1}\ 100\ 000$

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100

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Motivation from classical invariant theory

- G a group of matrices - GL_n, SL_n, SO_n, Spin_n, Sp_{2n}, $\mathfrak{S}_n$, ...
- V a $\mathbb{C}G$-module - vector, adjoint representation, ...

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The symmetric group $\mathfrak{S}_r$ acts on $V^{\otimes r}$ via

$$\sigma \cdot \vec{w}_1 \otimes \cdots \otimes \vec{w}_r = \vec{w}_{\sigma^{-1}1} \otimes \cdots \otimes \vec{w}_{\sigma^{-1}r},$$

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and this commutes with the diagonal action of G.

It would be nice to …

Find a (diagrammatic) basis of the space of invariant tensors $(V^{\otimes r})^G$, that is preserved by the action of the long cycle.

Background from representation theory

Consider the decomposition

$$V^{\otimes r} \cong \bigoplus_{\mu \in \Lambda^+} V(\mu)^{\oplus n_\mu}, \text{ where}$$

- Λ^+ are the dominant weights of G,
- $V(\mu)$ is the irreducible rep. of G with dominant weight μ,
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The space of invariant tensors $(V^{\otimes r})^G$ is isomorphic to the direct sum of the *trivial (one dimensional) representations* appearing in the decomposition.

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  - vertices labelled with words $c_1 \otimes \cdots \otimes c_r$
  - $\mathrm{wt}(c_1 \otimes \cdots \otimes c_r) = \mathrm{wt}(c_1) + \cdots + \mathrm{wt}(c_r)$
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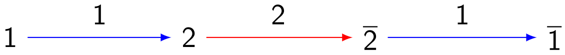
- hww of weight 0 $\leftrightarrow$ isolated vertices $\leftrightarrow$ invariant tensors

Crystal graphs - Sp_{2n}



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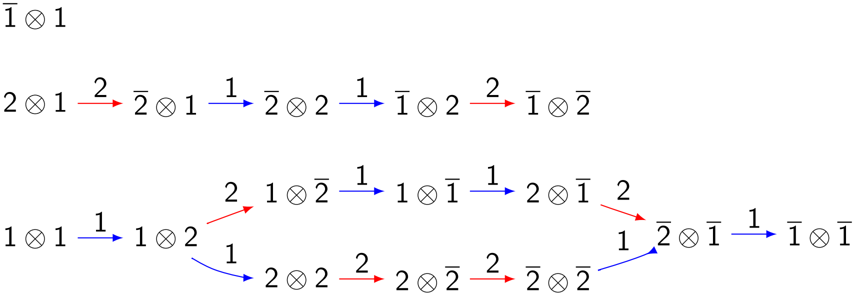

General goal

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Describe the action of evacuation.

Thank you!

Appendix

Our goal revisited

The general goal

Fix a representation. Find a diagrammatic basis of the space of invariant tensors. Give an explicit bijection between hww of weight 0 and the basis, that intertwines

$$\text{promotion} \quad \leftrightarrow \quad \text{rotation.}$$

1. invariant tensors – symmetric group action
2. diagram categories – rotation
3. crystal graphs – promotion

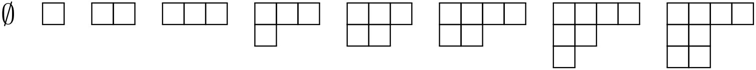
Standard Tableaux and RSK

$$P = \quad \emptyset \quad 1 \quad 2 \quad 3 \quad 31 \quad 32 \quad 42 \quad 421 \quad 422$$

A sequence of partitions

Standard Tableaux and RSK

$P = \quad \emptyset \quad 1 \quad 2 \quad 3 \quad 31 \quad 32 \quad 42 \quad 421 \quad 422$



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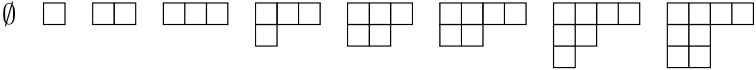
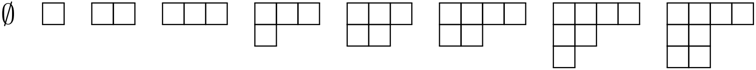


A sequence of partitions as a *Standard Young Tableaux.*

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2} 7 & 8 \\ \cline{1-2}\end{array}$$

Standard Tableaux and RSK

$P =$	$\emptyset$	1	2	3	31	32	42	421	422
$Q =$	$\emptyset$	1	2	21	211	221	222	322	422



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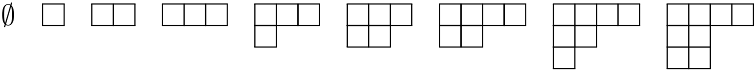
$P =$

1	2	3	6
4	5		
7	8		

Standard Tableaux and RSK

$P =$ ∅ 1 2 3 31 32 42 421 422



$Q =$ ∅ 1 2 21 211 221 222 322 422

A sequence of partitions as a *Standard Young Tableaux.*

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2} 7 & 8 \\ \cline{1-2} \end{array} \qquad Q = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 7 & 8 \\ \hline 3 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2} \end{array}$$

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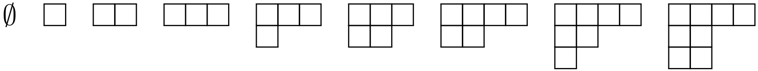

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RSK Correspondence (Robinson, Schensted, Knuth)

Bijection between permutations and pairs of SYT of the same shape.

Schützenberger Promotion and Evacuation

$P = 000\ 100\ 200\ 300\ 310\ 320\ 420\ 421\ 422$

Schützenberger Promotion and Evacuation

$P =$ 000 100 200 300 310 320 420 421 422

pr $P =$ 000

$$\begin{tikzcd}
\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \nu \arrow[d] \\
\kappa \arrow[r] & \mu
\end{tikzcd}$$

$$\mu = \mathrm{dom}_{\mathfrak{S}_n}(\kappa + \nu - \lambda)$$

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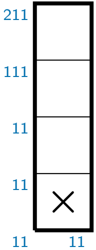
No whiteboard? No problem!



$$c = \begin{tikzcd} \kappa \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \nu \arrow[u] \end{tikzcd}$$

$$\mu' = \mathsf{sort}(\kappa' + \nu' - \lambda')$$

$$c = \begin{tikzcd} \lambda \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\boxtimes"] & \lambda \arrow[u] \end{tikzcd}$$

$$\mu = \lambda + e_1$$

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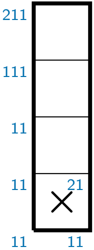

$$11 + 1 = 21$$

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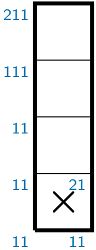
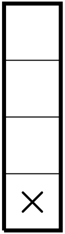


$\overset{\prime}{\leftrightarrow}$



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$c =$ the square above

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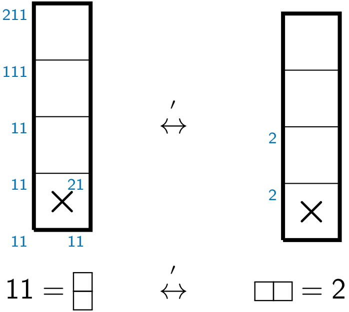

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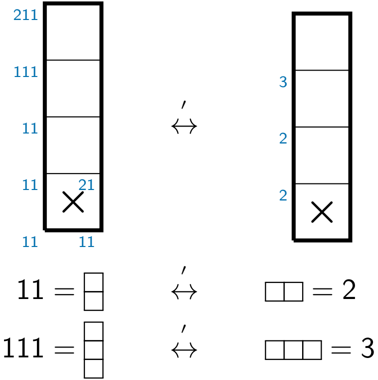

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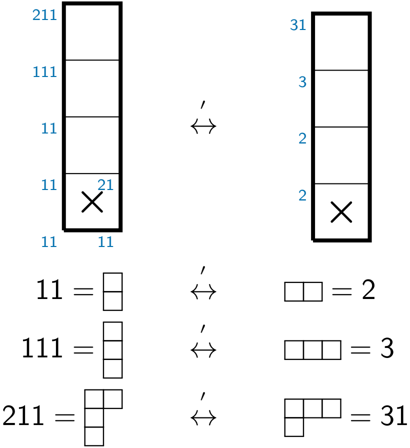

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$$\mu = \lambda + e_1$$

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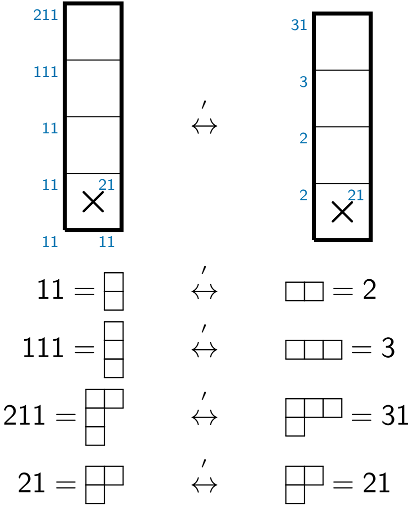

$$c = \begin{tikzcd} \kappa \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \nu \arrow[u] \end{tikzcd}$$

$$\mu' = \mathrm{sort}(\kappa' + \nu' - \lambda')$$

$$c = \begin{tikzcd} \lambda \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\boxtimes"] & \lambda \arrow[u] \end{tikzcd}$$

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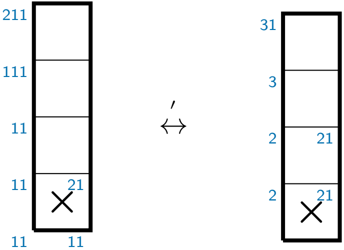
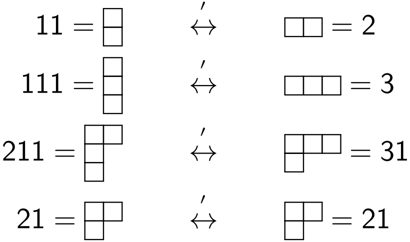


2 + 21 − 2 = 21

$$c = \begin{tikzcd} \kappa \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \nu \arrow[u] \end{tikzcd}$$

$$\mu' = \mathrm{sort}(\kappa' + \nu' - \lambda')$$



$11 =$ $\overset{\prime}{\leftrightarrow}$ $= 2$

$111 =$ $\overset{\prime}{\leftrightarrow}$ $= 3$

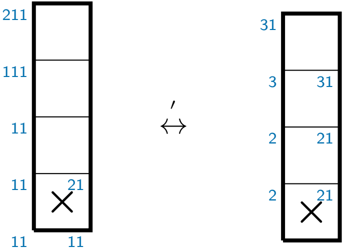
$211 =$ $\overset{\prime}{\leftrightarrow}$ $= 31$

$21 =$ $\overset{\prime}{\leftrightarrow}$ $= 21$

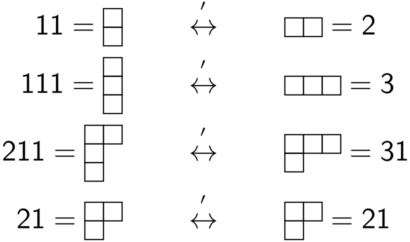
$$c = \begin{tikzcd} \lambda \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\boxed{\times}"] & \lambda \arrow[u] \end{tikzcd}$$

$$\mu = \lambda + e_1$$

No whiteboard? No problem!



$3 + 21 - 2 = 31$



$$c = \begin{tikzcd} \kappa \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \nu \arrow[u] \end{tikzcd}$$

$$\mu' = \mathrm{sort}(\kappa' + \nu' - \lambda')$$

$$c = \begin{tikzcd} \lambda \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\boxtimes"] & \lambda \arrow[u] \end{tikzcd}$$

$$\mu = \lambda + e_1$$

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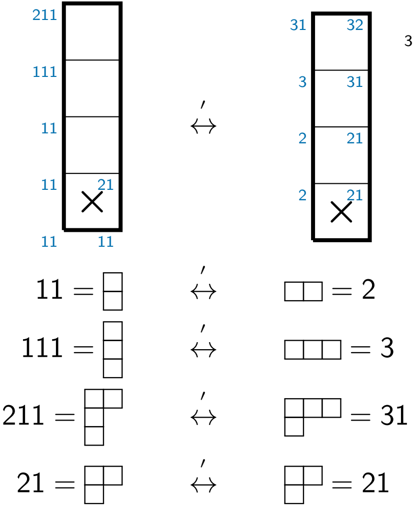


31 + 31 − 3 = 32

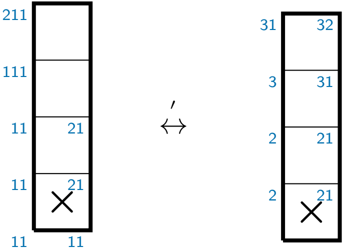
$$c = \begin{tikzcd} \kappa \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \nu \arrow[u] \end{tikzcd}$$

$$\mu' = \mathrm{sort}(\kappa' + \nu' - \lambda')$$

$$c = \begin{tikzcd} \lambda \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\boxtimes"] & \lambda \arrow[u] \end{tikzcd}$$

$$\mu = \lambda + e_1$$

No whiteboard? No problem!



$$11 = \ydiagram{1,1} \overset{\prime}{\leftrightarrow} \ydiagram{2} = 2$$

$$111 = \ydiagram{1,1,1} \overset{\prime}{\leftrightarrow} \ydiagram{3} = 3$$

$$211 = \ydiagram{2,1,1} \overset{\prime}{\leftrightarrow} \ydiagram{3,1} = 31$$

$$21 = \ydiagram{2,1} \overset{\prime}{\leftrightarrow} \ydiagram{2,1} = 21$$

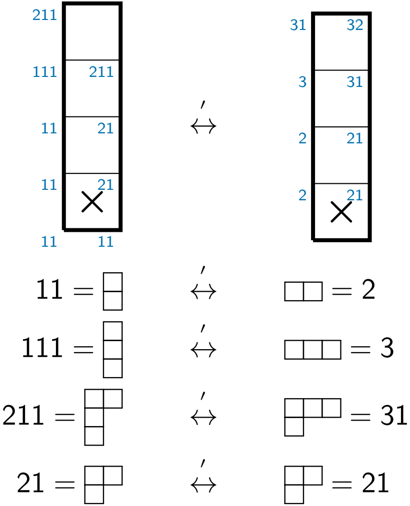
$$c = \begin{tikzcd}\kappa \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \nu \arrow[u]\end{tikzcd}$$

$$\mu' = \mathrm{sort}(\kappa' + \nu' - \lambda')$$

$$c = \begin{tikzcd}\lambda \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\boxtimes"] & \lambda \arrow[u]\end{tikzcd}$$

$$\mu = \lambda + e_1$$

No whiteboard? No problem!



$$c = \begin{tikzcd} \kappa \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \nu \arrow[u] \end{tikzcd}$$

$$\mu' = \mathrm{sort}(\kappa' + \nu' - \lambda')$$

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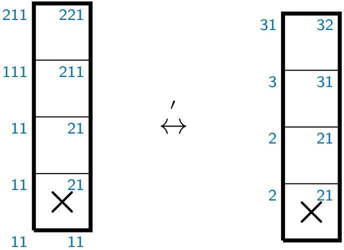
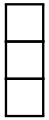
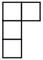
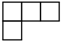
$$\mu = \lambda + e_1$$

No whiteboard? No problem!



$11 =$  $\overset{\prime}{\leftrightarrow}$  $= 2$

$111 =$  $\overset{\prime}{\leftrightarrow}$  $= 3$

$211 =$  $\overset{\prime}{\leftrightarrow}$  $= 31$

$21 =$  $\overset{\prime}{\leftrightarrow}$  $= 21$

$$c = \begin{tikzcd} \kappa \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \nu \arrow[u] \end{tikzcd}$$

$$\mu' = \mathrm{sort}(\kappa' + \nu' - \lambda')$$

$$c = \begin{tikzcd} \lambda \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\boxtimes"] & \lambda \arrow[u] \end{tikzcd}$$

$$\mu = \lambda + e_1$$

Go back

Thank you!