Kostka-Foulkes polynomials in type C_n

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Main topic

This talk is about the following positivity phenomenon in type C_n:

$$s_\lambda^{C_n} = \sum_\mu K_{\lambda,\mu}^{C_n}(q) P_\mu^{C_n}(x;q)$$

where $s_\lambda^{C_n}$ is the Schur function and $P_\mu^{C_n}(x;q)$ is the Hall-Littlewood function. The polynomials $K_{\lambda,\mu}^{C_n}(q)$ are known as Kostka-Foulkes polynomials.

The charge of a semistandard Young tableau

In type A_n, Lascoux-Schützenberger found a statistic

$$\mathsf{ch} : \mathsf{SSYT}_n \to \mathbb{Z}_{\geqslant 0}$$

on semistandard Young tableaux called **charge** which gives the following formula:

$$K_{\lambda,\mu}(q) = \sum_{\mathrm{T} \in \mathsf{SSYT}_n(\lambda,\mu)} q^{\mathsf{ch}(\mathrm{T})}.$$

The charge of a semistandard young tableau T was originally defined directly on word(T), its word.

- Extract standard subwords from word(T)
- Define the charge of a standard word.
- Add up the charges of the standard subwords. This is the charge of word(T).

Alternative definition

- Define a graph structure on SSYT_n by setting $\mathrm{T} \to \mathrm{T}'$ whenever there exists a word u and a letter $x \neq 1$ such that

$$\mathsf{word}(\mathrm{T}) \equiv xu \text{ and } \mathsf{word}(\mathrm{T}') \equiv ux.$$

where $\equiv$ denotes plactic equivalence on words.

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- If the shape of T' is a row, it cannot be obtained from some T in this way.
- Fix a weight μ, and let T_μ be the unique tableau with row shape and content/weight μ. Then, all paths joining a tableau T of weight μ to T_μ have the same length (and there exists at least one) n_{T}. Then

$$\mathsf{ch}(\mathrm{T}) := \sum_i (i-1)\mu_i - n_{\mathrm{T}}.$$

Type C_n

Semistandard Young tableaux are replaced by Kashiwara-Nakashima tableaux KN_n, which are semistandard Young tableaux on the ordered alphabet

$$\mathcal{C}_n = \left\{ \bar{n} < \cdots < \bar{1} < 1 < \ldots < n \right\}$$

satisfying certain conditions.

- Lecouvey has defined a cyclage algorithm and with it a directed graph structure on the set

$$\mathsf{KN} = \bigcup_{n>0} \mathsf{KN}_n$$

in such a way that all sinks are columns of weight zero, and such that, for every $\mathrm{T} \in \mathsf{KN}$, there always exists a finite path to a unique sink C_{T}, and all paths from T to C_{T} have the same length n_{T}.

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- Let C be a column of weight zero. Define

$$\mathsf{ch}_n(\mathrm{C}) := 2 \sum_{i \in \mathrm{E}_C} (n - i),$$

  where

$$\mathrm{E}_\mathrm{C} = \{i \geqslant 1 | i \in \mathrm{C}, i + 1 \notin \mathrm{C}\}.$$

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- Let $\mathrm{T} \in \mathsf{KN}_n$. Then

$$\mathrm{ch}_n(\mathrm{T}) := \mathrm{ch}_n(\mathrm{C}_\mathrm{T}) + n_\mathrm{T}.$$

Conjecture (Lecouvey, 2000)

Let $\mathsf{KN}_n(\lambda, \mu)$ denote the set of Kashiwara-Nakashima tableaux of shape λ and weight μ. The following formula holds:

$$K_{\lambda,\mu}(q) = \sum_{\mathrm{T} \in \mathsf{KN}_n(\lambda,\mu)} q^{\mathsf{ch}_n(\mathrm{T})}.$$

Theorem (Dołęga-Gerber-T, 2019+)

Lecouvey's conjecture is true for λ of row shape. For $\mathrm{T} \in \mathsf{KN}_n((2r), 0)$, given by

$$\mathrm{T} = \boxed{\bar{i}_r} \boxed{\ldots} \boxed{\bar{i}_1} \boxed{i_1} \boxed{\ldots} \boxed{i_r}$$

for positive integers $i_1 \leqslant \cdots \leqslant i_r$, we have

$$\mathrm{ch}_n(\mathrm{T}) = r + 2 \sum_{k=1}^{r} (n - i_k).$$

Thank you for your attention!