On a Fourfold Refined Enumeration of Alternating Sign Trapezoids

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3 September 2019

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and of Column Strict Shifted Plane Partitions

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Alternating Sign Trapezoids

Definition (Ayyer; Aigner; Behrend, Fischer)

An (n, l)-alternating sign trapezoid (AST) (for $l \geq 2$) is a trapezoidal array of integers with n rows, l entries in the bottom row and entries -1, 0 or $+1$ such that

- the sum of the entries in each row equals 1,
- the nonzero entries alternate in sign along each row and each column,
- the top-most nonzero entry in each column is 1,
- the entries in the central $l-2$ columns sum up to 0.

0	0	0	1	0	0	0	0	0	0	0	0
	1	0	-1	0	0	0	1	0	0	0	
		0	1	0	0	0	-1	0	1		
			0	0	0	1	0	0			
				1	0	-1	1				

$(5, 4)$-AST

Statistics on ASTs

0	0	0	1	0	0	0	0	0	0	0	0
	1	0	-1	0	0	0	1	0	0	0	
		0	1	0	0	0	-1	0	1		
			0	0	0	1	0	0			
				1	0	-1	1				

1-column vector: $\mathbf{c} = (-4, -2, -1, 1, 3)$, length n

1-columns on the left: R^3

10-columns on the left: S^1

10-columns on the right: T^0

-1s: Q^3

Statistics on ASTs

−5	−4	−3	−2	−1			1	2	3	4	5
0	0	0	1	0	0	0	0	0	0	0	0
	1	0	−1	0	0	0	1	0	0	0	
		0	1	0	0	0	−1	0	1		
			0	0	0	1	0	0			
				1	0	−1	1				

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Statistics on ASTs

0	0	0	1	0	0	0	0	0	0	0	0
	1	0	−1	0	0	0	1	0	0	0	
		0	1	0	0	0	−1	0	1		
			0	0	0	1	0	0			
				1	0	−1	1				

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Statistics on ASTs

0	0	0	1	0	0	0	0	0	0	0	0
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		0	1	0	0	0	−1	0	1		
			0	0	0	1	0	0			
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Statistics on ASTs

0	0	0	1	0	0	0	0	0	0	0	0
	1	0	−1	0	0	0	1	0	0	0	
		0	1	0	0	0	−1	0	1		
			0	0	0	1	0	0			
				1	0	−1	1				

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Statistics on ASTs

0	0	0	1	0	0	0	0	0	0	0	0
	1	0	−1	0	0	0	1	0	0	0	
		0	1	0	0	0	−1	0	1		
			0	0	0	1	0	0			
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1-column vector: $\mathbf{c} = (-4, -2, -1, 1, 3)$, length n

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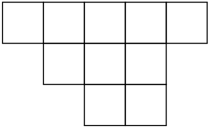
\# 10-columns on the right: T^0

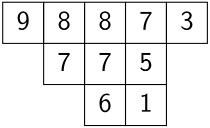
\# -1s: Q^3

Column Strict Shifted Plane Partitions

Definition (Mills, Robbins, Rumsey 1987)

A column strict shifted plane partition (CSSPP) is filling of a shifted Young diagram with positive integers such that the entries decrease along each row and strictly decrease down each column. It is of class k if the first entry of each row i is exactly k plus its row length.





CSSPP of class 4

Observation

CSSPPs of class 2 with no row length larger than n

$$\longleftrightarrow$$

descending plane partitions with no part larger than $n+1$

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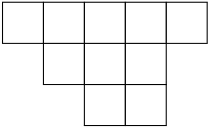

9	8	8	7	3
	7	7	5	
		6	1	

CSSPP of class 4

Observation

CSSPPs of class 2 with no row length larger than n

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Statistics on CSSPPs (of class k)

9	8	8	7	3
	7	7	5	
		6	1	

CSSPP of class $k = 4$

For a fixed parameter $d \in \{1, \ldots, k\}$:

- # rows: R^3
- # parts $p_{i,j}$ equal to $j - i + d$: S^1
- # 1s: T^1
- # parts $p_{i,j}$ less or equal to $j - i + k$ but not 1 or $j - i + d$: Q^2

Statistics on CSSPPs (of class k)

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Statistics on CSSPPs (of class k)

9	8	8	7	3
	7	7	5	
		6	1	

CSSPP of class $k = 4$

3	4	5	6	7
	3	4	5	
		3	4	

filling with $j - i + d$, $d = 3$

For a fixed parameter $d \in \{1, \ldots, k\}$:

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Statistics on CSSPPs (of class k)

9	8	8	7	3
	7	7	5	
		6	1	

CSSPP of class $k = 4$

4	5	6	7	8
	4	5	6	
		4	5	

filling with $j - i + k$, $k = 4$

For a fixed parameter $d \in \{1, \dots, k\}$:

- # rows: R^3
- # parts $p_{i,j}$ equal to $j - i + d$: S^1
- # 1s: T^1
- # parts $p_{i,j}$ less or equal to $j - i + k$ but not 1 or $j - i + d$: Q^2

(Joint) distributions

Theorem (Behrend, Di Francesco, Zinn-Justin 2012 + Ayyer, Behrend, Fischer 2016)

The distribution of the Q-statistic on CSSPPs of class 2 with no row length larger than n and on $(n, 3)$-ASTs coincide.

Theorem (Fischer 2018)

The joint distribution of the R-, S-, and T-statistics on CSSPPs of class $l-1$ with no row length larger than n and on (n, l)-ASTs coincide.

Theorem (H.)

The joint distribution of the Q-, R-, S-, and T-statistics on CSSPPs of class $l-1$ with no row length larger than n and on (n, l)-ASTs coincide.

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The proof

Spoiler Ahead:

Generating function for ASTs:

- ASTs $\longleftrightarrow$ Trees
- Operator formula for ASTs with given 1-column vector
- Constant term expression
- Summation over all possible 1-column vectors

Generating function for CSSPPs:

- CSSPPs $\longleftrightarrow$ Family of nonintersecting lattice paths
- Lindström-Gessel-Viennot lemma

The proof

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This is not a bijective proof!

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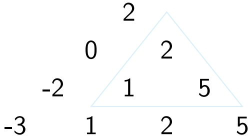
Monotone Triangles

Definition

A monotone triangle (MT) of order n is a triangular array of integers with n rows such that the entries

- strictly increase along rows,
- weakly increase along $\nearrow$-diagonals, and
- weakly increase along $\searrow$-diagonals.



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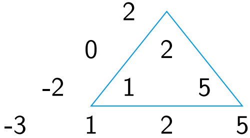
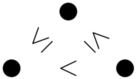
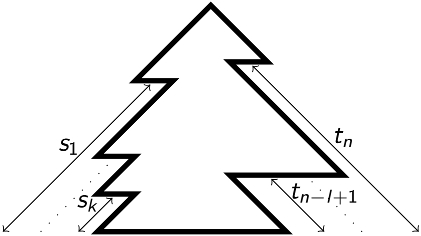


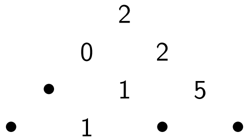


Trees

Definition

For a weakly decreasing sequence $\mathbf{s} = (s_1, \dots, s_k)$ and a weakly increasing sequence $\mathbf{t} = (t_{n-l+1}, \dots, t_n)$ of nonnegative integers $(k + l \leq n)$, a $(\mathbf{s}, \mathbf{t})$-tree is a MT of order n with truncated diagonals: the s_i bottom entries of the ith $\nearrow$-diagonal and the t_j bottom entries of the jth $\searrow$-diagonal are deleted.





$((2), (1, 1))$-tree,
bottom row $(0, 1, 1, 5)$

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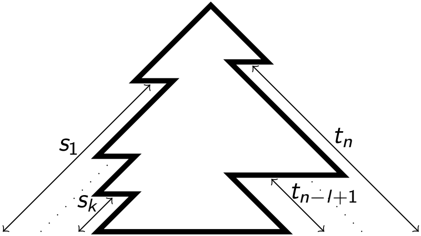

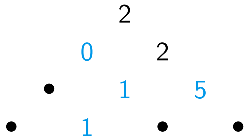

$((2), (1, 1))$-tree,
bottom row $(0, 1, 1, 5)$

ASTs and Trees

$$\begin{array}{cccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
 & & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & & \\
 & & & 0 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & & 1 & 0 & -1 & 1 & & & &
\end{array}$$

- Add 0s for a rectangular shape
- Replace entries by partial column sums
- Record posititions of 1s in shape of a MT
- Remove additional entries

ASTs and Trees

0	0	0	1	0	0	0	0	0	0	0	0
0	1	0	−1	0	0	0	1	0	0	0	0
0	0	0	1	0	0	0	−1	0	1	0	0
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	1	0	−1	1	0	0	0	0

- Add 0s for a rectangular shape
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ASTs and Trees

0	0	0	1	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	1	0	0	0	0
0	1	0	1	0	0	0	0	0	1	0	0
0	1	0	1	0	0	1	0	0	1	0	0
0	1	0	1	1	0	0	1	0	1	0	0

- Add 0s for a rectangular shape
- Replace entries by partial column sums
- Record posititions of 1s in shape of a MT
- Remove additional entries

ASTs and Trees

−5	−4	−3	−2	−1	0	1	2	3	4	5	6
0	0	0	1	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	1	0	0	0	0
0	1	0	1	0	0	0	0	0	1	0	0
0	1	0	1	0	0	1	0	0	1	0	0
0	1	0	1	1	0	0	1	0	1	0	0

$$\longleftrightarrow$$

$$\begin{array}{ccccccccc} & & & & -2 & & & & \\ & & & -4 & & 2 & & & \\ & & -4 & & -2 & & 4 & & \\ & -4 & & -2 & & 1 & & 4 & \\ -4 & & -2 & & -1 & & 2 & & 4 \end{array}$$

- Add 0s for a rectangular shape
- Replace entries by partial column sums
- Record posititions of 1s in shape of a MT
- Remove additional entries

ASTs and Trees

$$\begin{array}{cccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
 & & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & & \\
 & & & 0 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & & 1 & 0 & -1 & 1 & & & &
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{cccc}
 & -2 & & \\
-4 & & 2 & \\
 & -2 & & 4 \\
-2 & & 1 & \\
 & -1 & & 2
\end{array}$$

$(5,4)$-AST

tree with 5 rows

- R-weight
- 1-column vector $\mathbf{c}$
- Q-weight
- S-weight (T-weight similar)

ASTs and Trees

$$\begin{array}{cccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
 & & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & & \\
 & & & 0 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & & 1 & 0 & -1 & 1 & & & &
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{cccc}
 & -2 & & \\
-4 & & 2 & \\
 & -2 & & 4 \\
-2 & & 1 & \\
 & -1 & & 2
\end{array}$$

(n, l)-AST

tree with n rows

- R-weight
- 1-column vector $\mathbf{c}$
- Q-weight
- S-weight (T-weight similar)

ASTs and Trees

$$
\begin{array}{cccccccccccc}
 & -4 & & -2 & -1 & & & 1 & & 3 & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
 & & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & & \\
 & & & 0 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & & 1 & 0 & -1 & 1 & & & &
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{cccc}
 & -2 & & \\
-4 & & 2 & \\
 & -2 & & 4 \\
-2 & & 1 & \\
 & -1 & & 2
\end{array}
$$

$\mathbf{c} = (-4, -2, -1, 1, 3)$

$((3,1),(2))$-tree,
bottom row $(-4, -2, -1, 2, 4)$

- R-weight
- 1-column vector $\mathbf{c}$
- Q-weight
- S-weight (T-weight similar)

ASTs and Trees

$$
\begin{array}{cccccccccccc}
 & -4 & & -2 & -1 & & & 1 & & 3 & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
 & & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & & \\
 & & & 0 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & & 1 & 0 & -1 & 1 & & & &
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{cccc}
 & -2 & & \\
-4 & & 2 & \\
 & -2 & & 4 \\
-2 & & 1 & \\
 & -1 & & 2
\end{array}
$$

$$\mathbf{c} = (\underbrace{c_1, \ldots, c_m}_{<0}, \underbrace{c_{m+1}, \ldots, c_n}_{>0})$$

$(\mathbf{s}, \mathbf{t})$-tree, bottom row

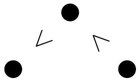
$$\tilde{\mathbf{c}} = (c_1, \ldots, c_m, c_{m+1} + l - 3, \ldots, c_n + l - 3),$$
$$\mathbf{s} = (-c_1 - 1, \ldots, -c_m - 1),$$
$$\mathbf{t} = (c_{m+1} - 1, \ldots, c_n - 1)$$

- R-weight
- 1-column vector $\mathbf{c}$
- Q-weight
- S-weight (T-weight similar)

ASTs and Trees

$$
\begin{array}{cccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
 & & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & & \\
 & & & 0 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & & 1 & 0 & -1 & 1 & & & &
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{cccc}
 & -2 & & \\
-4 & & 2 & \\
 & -2 & & 4 \\
-2 & & -1 & \\
 & -1 & & 2
\end{array}
$$

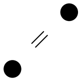
- R-weight
- 1-column vector **c**
- Q-weight
- S-weight (T-weight similar)



ASTs and Trees

$$
\begin{array}{cccccccccccc}
 & & & \color{orange}{-2} & & & & & & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\
 & & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & & \\
 & & & 0 & 0 & 0 & 1 & 0 & 0 & & & \\
 & & & & 1 & 0 & -1 & 1 & & & &
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{cccc}
 & -2 & & \\
-4 & & 2 & \\
 & \color{orange}{-2} & & 4 \\
\color{orange}{-2} & & 1 & \\
 & -1 & & 2
\end{array}
$$

- R-weight
- 1-column vector $\mathbf{c}$
- Q-weight
- S-weight (T-weight similar)



at the bottom row

Main Tool: Operator Formulae

Main tool for enumeration:

- shift operator: $\mathrm{E}_x\, f(x) := f(x+1)$
- forward difference operator: $\Delta_x\, f(x) := \mathrm{E}_x - \mathrm{id}_x$
- backward difference operator: $\delta_x\, f(x) := \mathrm{id}_x - \mathrm{E}_x^{-1}$

Abbreviation: $\mathrm{E}_k\, f(k) := \mathrm{E}_x\, f(x)\big|_{x=k}$

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- forward difference operator: $\Delta_x\, f(x) := \mathrm{E}_x - \mathrm{id}_x$
- backward difference operator: $\delta_x\, f(x) := \mathrm{id}_x - \mathrm{E}_x^{-1}$

Abbreviation: $\mathrm{E}_k\, f(k) := \mathrm{E}_x\, f(x)\big|_{x=k}$

Enumeration of MTs

Theorem (Fischer 2006)

The number $\mathbf{MT}_n(k_1, \ldots, k_n)$ of MTs of order n with bottom row $(k_1, \ldots, k_n)$ is given by

$$\prod_{1 \leq s < t \leq n} (\mathrm{E}_{k_s} + \mathrm{E}_{k_t}^{-1} - \mathrm{E}_{k_s} \mathrm{E}_{k_t}^{-1}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i}.$$

Theorem (Fischer 2010)

The Q-enumeration ${}^{Q}\mathbf{MT}_n(k_1, \ldots, k_n)$ of MTs of order n with bottom row $(k_1, \ldots, k_n)$ is given by

$$\prod_{1 \leq s < t \leq n} (\mathrm{E}_{k_s} + \mathrm{E}_{k_t}^{-1} - (2 - Q) \mathrm{E}_{k_s} \mathrm{E}_{k_t}^{-1}) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i}.$$

Enumeration of Trees

Observation
The difference operators $-\Delta_x = \mathrm{id}_x - \mathrm{E}_x$ and $\delta_x = \mathrm{id}_x - \mathrm{E}_x^{-1}$ cut off entries!

Theorem (Fischer 2011)
The number of $(\mathbf{s}, \mathbf{t})$-trees with n rows and bottom row $(k_1, \ldots, k_n)$ is given by

$$\prod_{i=1}^{k} (-\Delta_{k_i})^{s_i} \prod_{j=n-l+1}^{n} (\delta_{k_j})^{t_j} \, \mathbf{MT}_n(k_1, \ldots, k_n).$$

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Enumeration of Trees

Theorem (Fischer 2018)

The Q-enumeration of $(\mathbf{s},\mathbf{t})$-trees with n rows and bottom row $(k_1,\ldots,k_n)$ is given by

$$\prod_{i=1}^{k}\left(-{}^{\mathrm{Q}}\,\Delta_{k_i}\right)^{s_i}\prod_{j=n-l+1}^{n}\left({}^{\mathrm{Q}}\,\delta_{k_j}\right)^{t_j}\,{}^{\mathrm{Q}}\,\mathbf{MT}_n\,(k_1,\ldots,k_n)\,,$$

where

$$\begin{aligned}{}^{\mathrm{Q}}\,\Delta_x &:= (Q-(1-Q)\,\Delta_x)^{-1}\,\Delta_x,\\ {}^{\mathrm{Q}}\,\delta_x &:= (Q-(Q-1)\,\delta_x)^{-1}\,\delta_x\,.\end{aligned}$$

Constant term expression

Define the (anti)symmetriser of a formal Laurent series:

$$\mathbf{Sym}_{x_1,\ldots,x_m} f(x_1,\ldots,x_m) := \sum_{\sigma\in\mathfrak{S}_m} f(x_{\sigma(1)},\ldots,x_{\sigma(m)})$$

$$\mathbf{ASym}_{x_1,\ldots,x_m} f(x_1,\ldots,x_m) := \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}(\sigma) f(x_{\sigma(1)},\ldots,x_{\sigma(m)})$$

Theorem (Fischer, Riegler 2015)
${}^{Q}\,\mathbf{MT}_n(x_1,\ldots,x_n)$ is equivalent to

$$\mathrm{CT}_{\mathbf{Y}}\,\mathbf{Sym}_{\mathbf{Y}}\left(\prod_{i=1}^{n}(1+Y_i)^{x_i}\prod_{1\le i<j\le n}(Q-(1-Q)Y_i+Y_j+Y_iY_j)\right.$$
$$\left.\times\prod_{1\le i<j\le n}(Y_j-Y_i)^{-1}\right).$$

ASTs with given 1-column vector

Theorem (H.)

The Q-enumeration of (n, l)-ASTs with 1-columns in positions $\mathbf{c} = (c_1, \ldots, c_n)$ is given by

$$\prod_{i=1}^{m} \left(- {}^{Q}\Delta_{\tilde{c}_i}\right)^{-c_i-1} \prod_{j=m+1}^{n} {}^{Q}\delta_{\tilde{c}_j}^{c_j-1} \, {}^{Q}\mathbf{MT}_n(\tilde{c}_1, \ldots, \tilde{c}_n),$$

where $\tilde{\mathbf{c}} = (c_1, \ldots, c_m, c_{m+1} + l - 3, \ldots, c_n + l - 3)$.

To incorporate S and T, add additional difference operators:

$$\prod_{i=1}^{m} \left(\mathrm{id} - \frac{S}{Q}\delta_{\tilde{c}_i}\right)\left(\mathrm{id} + {}^{Q}\Delta_{\tilde{c}_i}\right)\left(- {}^{Q}\Delta_{\tilde{c}_i}\right)^{-c_i-1}$$

$$\times \prod_{i=m+1}^{n} \left(\mathrm{id} + \frac{T}{Q}\Delta_{\tilde{c}_i}\right)\left(\mathrm{id} - {}^{Q}\delta_{\tilde{c}_i}\right) {}^{Q}\delta_{\tilde{c}_i}^{c_i-1} M_n(\tilde{c}_1, \ldots, \tilde{c}_n).$$

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To incorporate S and T, add additional difference operators:

$$\prod_{i=1}^{m} \left(\mathrm{id} - \frac{S}{Q}\, \delta_{\tilde{c}_i} \right) \left(\mathrm{id} + {}^{\mathrm{Q}}\,\Delta_{\tilde{c}_i} \right) \left(- {}^{\mathrm{Q}}\,\Delta_{\tilde{c}_i} \right)^{-c_i - 1}$$

$$\times \prod_{i=m+1}^{n} \left(\mathrm{id} + \frac{T}{Q}\, \Delta_{\tilde{c}_i} \right) \left(\mathrm{id} - {}^{\mathrm{Q}}\,\delta_{\tilde{c}_i} \right) {}^{\mathrm{Q}}\,\delta_{\tilde{c}_i}^{c_i - 1}\, M_n \left(\tilde{c}_1, \dots, \tilde{c}_n \right).$$

Constant term expression

This can be transformed into the constant term of

$$\mathbf{Sym_Y}\left(\prod_{i=1}^{m}(-Y_i)^{-c_i-1}(1+Y_i)^{c_i}(Q-(1-Q)Y_i)^{c_i+1}\frac{Q-(S-Q)Y_i}{Q-(1-Q)Y_i}\right.$$
$$\times\prod_{i=m+1}^{n}Y_i^{c_i-1}(1+Y_i)^{c_i+l-3}(Q+Y_i)^{-c_i+1}\frac{Q+TY_i}{Q+Y_i}$$
$$\left.\times\prod_{1\le i<j\le n}(Q-(1-Q)Y_i+Y_j+Y_iY_j)\prod_{1\le i<j\le n}(Y_j-Y_i)^{-1}\right).$$

Sum over 1-columns

Remaining step: Sum over all possible 1-column vectors.

- First, sum over all $-n \le c_1 < \cdots < c_m < 0 < c_{m+1} < \cdots < c_n \le n$.
- Then, sum over all $0 \le m \le n$.

$$\mathbf{Sym_Y}\left(\sum_{c_1<\cdots<c_m<0}\left(\prod_{i=1}^{m}(-Y_i)^{-c_i-1}(1+Y_i)^{c_i}(Q-(1-Q)Y_i)^{c_i+1}\left(\frac{Q-(S-Q)Y_i}{Q-(1-Q)Y_i}\right)\right)\right.$$
$$\times\sum_{0<c_{m+1}<\cdots<c_n}\left(\prod_{i=m+1}^{n}Y_i^{c_i-1}(1+Y_i)^{c_i+l-3}(Q+Y_i)^{-c_i+1}\left(\frac{Q+TY_i}{Q+Y_i}\right)\right)$$
$$\left.\times\prod_{1\le i<j\le n}\left(Q-(1-Q)Y_i+Y_j+Y_iY_j\right)\left(Y_j-Y_i\right)^{-1}\right),$$

which is equal to...

$$\mathbf{Sym_Y}\left(\prod_{i=1}^{m}\frac{1}{1+Y_i}\left(\frac{-Y_i}{(1+Y_i)\,(Q-(1-Q)\,Y_i)}\right)^{m-i}\right.$$

$$\times\left(1-\prod_{j=1}^{i}\left(\frac{-Y_j}{(1+Y_j)\,(Q-(1-Q)\,Y_j)}\right)\right)^{-1}\frac{Q-(S-Q)\,Y_i}{Q-(1-Q)\,Y_i}$$

$$\times\prod_{i=m+1}^{n}(1+Y_i)^{l-2}\left(\frac{Y_i\,(1+Y_i)}{Q+Y_i}\right)^{i-m-1}\left(1-\prod_{j=i}^{n}\left(\frac{Y_j\,(1+Y_j)}{Q+Y_j}\right)\right)^{-1}\frac{Q+TY_i}{Q+Y_i}$$

$$\left.\times\prod_{1\leq i<j\leq n}(Q-(1-Q)\,Y_i+Y_j+Y_iY_j)\,(Y_j-Y_i)^{-1}\right),$$

by using the following geometric series evaluation:

$$\sum_{0\leq c_1<\cdots<c_m}Y_1^{c_1}\ldots Y_m^{c_m}=\prod_{i=1}^{m}Y_i^{i-1}\left(1-\prod_{j=i}^{m}Y_j\right)^{-1}.$$

The **Subsets** operator

We define the *subsets* operator as

$$\mathbf{Subsets}_{X_1,\dots,X_m}^{X_{m+1},\dots,X_n} f(X_1,\dots,X_n) := \sum_{\substack{\sigma \in \mathfrak{S}_n,\\ \sigma(1) < \cdots < \sigma(m),\\ \sigma(m+1) < \cdots < \sigma(n)}} f\left(X_{\sigma(1)},\dots,X_{\sigma(n)}\right).$$

It follows that

$$\begin{aligned}&\mathbf{Sym}_{X_1,\dots,X_n} f(X_1,\dots,X_n)\\ &\qquad = \mathbf{Subsets}_{X_1,\dots,X_m}^{X_{m+1},\dots,X_n} \mathbf{Sym}_{X_1,\dots,X_m} \mathbf{Sym}_{X_{m+1},\dots,X_n} f(X_1,\dots,X_n).\end{aligned}$$

Antisymmetriser lemma

We can compute the (anti)symmetriser!

Lemma (H.)
Let $m \geq 1$, then

$$\mathbf{ASym}_{X_1,\dots,X_m}\left(\prod_{r=1}^{m}\frac{\left(\frac{X_r(1+X_r)}{Q+X_r}\right)^{r-1}}{1-\prod_{j=r}^{m}\frac{X_j(1+X_j)}{Q+X_j}}\prod_{1\leq s<t\leq m}(Q+(Q-1)X_s+X_t+X_sX_t)\right)$$
$$=\prod_{r=1}^{m}\frac{Q+X_r}{Q-X_r^2}\prod_{1\leq s<t\leq m}\frac{(Q(1+X_s)(1+X_t)-X_sX_t)(X_t-X_s)}{Q-X_sX_t}.$$

Generating function for ASTs

After some manipulations involving the antisymmetriser lemma, the Cauchy determinant and some linearity properties, we obtain:

Theorem (H.)

The generating function of (n, l)-ASTs is given by

$$\det_{0\le i,j\le n-1}\left(R\sum_{k=0}^{i}T^{i-k}\sum_{m=0}^{j}\binom{j}{m}Q^{k-m}\left(\binom{k+l-3}{k-m}+\binom{k+l-3}{k-m-1}SQ^{-1}\right)+\delta_{i,j}\right).$$

But what about the CSSPPs?

Generating function for ASTs

After some manipulations involving the antisymmetriser lemma, the Cauchy determinant and some linearity properties, we obtain:

Theorem (H.)

The generating function of (n, l)-ASTs is given by

$$\det_{0 \le i,j \le n-1} \left(R \sum_{k=0}^{i} T^{i-k} \sum_{m=0}^{j} \binom{j}{m} Q^{k-m} \left(\binom{k+l-3}{k-m} + \binom{k+l-3}{k-m-1} S Q^{-1} \right) + \delta_{i,j} \right).$$

But what about the CSSPPs?

Andrews' determinant

If we set $Q = R = S = T = 1$,

$$\det_{0 \le i,j \le n-1} \left(R \sum_{k=0}^{i} T^{i-k} \sum_{m=0}^{j} \binom{j}{m} Q^{k-m} \left(\binom{k+l-3}{k-m} + \binom{k+l-3}{k-m-1} S Q^{-1} \right) + \delta_{i,j} \right)$$

simplifies to

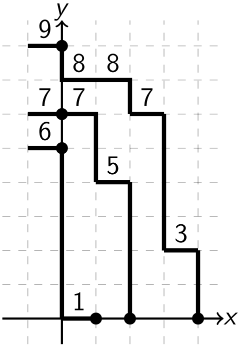
$$\det_{0 \le i,j \le n-1} \left(\binom{i+j+l-1}{i} + \delta_{i,j} \right),$$

which enumerates descending plane partitions with no part larger than $n+1$ (Andrews 1979). This can be proved by a nonintersecting lattice paths approach and the Lindström-Gessel-Viennot lemma!

CSSPPs and nonintersecting lattics paths

9	8	8	7	3
	7	7	5	
		6	1	

CSSPP of class k=4



For a fixed parameter $d \in \{1, \ldots, k\}$:

- R-weight
- S-weight: # paths that cross the line $y = x + d$ by a vertical step from the right
- T-weight: # steps at height 0
- Q-weight: # vertical steps below the line $y = x + k$ not already counted

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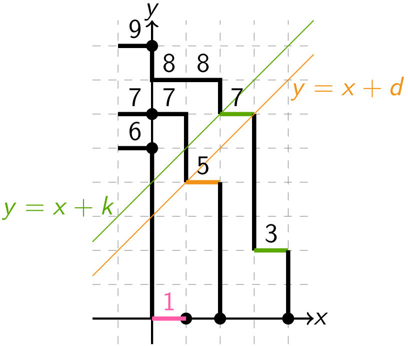

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Final result

By the Lindström-Gessel-Viennot lemma, we obtain as generating function

$$\sum_{r=0}^{n} R^r \sum_{0 \le u_1 < \cdots < u_r \le n-1} \det_{0 \le i,j \le r} \left(\sum_{k=0}^{u_i} T^{u_i - k} \sum_{m=0}^{u_j} \binom{u_j}{m} Q^{k-m} \left(\binom{k+l-3}{k-m} + \binom{k+l-3}{k-m-1} S Q^{-1} \right) \right).$$

Theorem (H.)

The generating function of (n,l)-ASTs and of CSSPPs of class $l-1$ with no row length larger than n coincide and is given by

$$\det_{0 \le i,j \le n-1} \left(R \sum_{k=0}^{i} T^{i-k} \sum_{m=0}^{j} \binom{j}{m} Q^{k-m} \left(\binom{k+l-3}{k-m} + \binom{k+l-3}{k-m-1} S Q^{-1} \right) + \delta_{i,j} \right).$$

Thank you for your attention!