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SMOOTHNESS OF SCHUBERT VARIETIES INDEXED BY INVOLUTIONS IN FINITE SIMPLY LACED TYPES

AXEL HULTMAN AND VINCENT UMUTABAZI

Abstract. We prove that in finite, simply laced types, every Schubert variety indexed by an involution which is not the longest element of some standard parabolic subgroup is singular.

1. Introduction

Let w be an involution in the symmetric group S_n. In [12] Hohlweg proved that the Schubert variety X_w is smooth if and only if w is the longest element of some standard parabolic subgroup of S_n. He arrived at this result by exploiting Lakshmibai and Sandhya's [15] classical pattern avoidance criterion for smoothness of type A Schubert varieties.

It is natural to wonder to what extent this surprisingly simple characterisation generalises. The main result of this paper, Theorem 3.1 below, extends Hohlweg's result to arbitrary finite, simply laced types. Namely, if W is a simply laced Weyl group, and $w \in W$ is an involution, X_w is smooth if and only if w is the longest element of a standard parabolic subgroup of W. Our proof relies solely on Carrell–Peterson type criteria for smoothness of Schubert varieties; in particular it does not depend on the classification of finite root systems.

As was anticipated in an earlier version of this manuscript [14], it is indeed also possible to arrive at Theorem 3.1 using the root system pattern avoidance criteria for smoothness pioneered by Billey and Postnikov [2]. Namely, an anonymous referee has provided a very neat argument in terms of Richmond and Slofstra's [18] so-called staircase diagrams. Let us sketch the referee's argument here. Using the fact that flipping a staircase diagram corresponds to inverting a group element [18, Theorem 3.8] and that smooth Schubert varieties in simply laced types correspond to maximally labelled staircase diagrams [18, Corollary 6.4], one deduces that Schubert varieties indexed by involutions are smooth precisely when they correspond to maximally labelled diagrams that are fixed under flipping. It is not hard to see that those are exactly the diagrams that correspond to longest elements of standard parabolic subgroups, which yields the desired conclusion. In this argument, it is in the crucial correspondence given by [18, Corollary 6.4] that pattern avoidance enters the picture. The key result which this corollary rests on is [17, Theorem 5.1] which requires a careful type by type analysis and extensive computer calculations; see the discussion in [17] for a detailed account.

In Section 2 below, we recall properties of Coxeter systems and Bruhat graphs which can be used to study smoothness of Schubert varieties in a combinatorial way. In Section 3, we prove Theorem 3.1.

2. Preliminaries

In this section some properties of Bruhat graphs of Coxeter groups are recalled. For more on these concepts, see e.g. [3] or [9].

A *Coxeter group* is a group W generated by a set S of *simple reflections* s under relations of the form $s^2 = e$ and $(ss')^{m(s,s')} = e$ for all $s, s' \in S$ where e is the identity element and $m(s', s) = m(s, s') \geq 2$ is the order of ss' for $s \neq s'$. The pair (W, S) is called a *Coxeter system*, and each element $w \in W$ is a product of generators $s_i \in S$, i.e., $w = s_1 s_2 \cdots s_j$. If j is minimal among all such expressions for w, then j is called the *length* of w, denoted $\ell(w) = j$. The Coxeter system (W, S) is *simply laced* if $m(s, s') \leq 3$ for all $s, s' \in S$; otherwise it is *multiply laced*.

If W is finite, there exists a *longest element* $w_0 \in W$. It is an involution and satisfies $\ell(v) < \ell(w_0)$ for all other elements $v \in W$. In fact w_0 is the unique element in W such that $\ell(sw_0) < \ell(w_0)$ for all $s \in S$.

From now on let us fix a Coxeter system (W, S). Let $T = \{wsw^{-1} : w \in W, s \in S\}$ be the set of *reflections* in W. For $v, w \in W$ define:

(i) $v \to w$ if $w = vt$ for some $t \in T$ with $\ell(v) < \ell(w)$.
(ii) $v \leq w$ if $v = v_0 \to v_1 \to \cdots \to v_m = w$ for some $v_i \in W$.

The *Bruhat graph* $\mathrm{Bg}_S(W)$ of (W, S) is the directed graph whose vertex set is W and whose edge set is $E_{\mathrm{Bg}_S(W)} = \{(u, w) : u \to w\}$. The *Bruhat order* is the partial order relation on W given by (ii).

Example 2.1. *Denote by $W(A_2)$ the Coxeter group of type A_2 with set of simple reflections $S(A_2) = \{s_1, s_2\}$ satisfying $m(s_1, s_2) = 3$. Then $\mathrm{Bg}_{S(A_2)}(W(A_2))$ is as shown in Figure 1.*

The map $v \mapsto v^{-1}$ is an automorphism of the Bruhat order:

Lemma 2.2. *For all $v, w \in W$, $v < w$ if and only if $v^{-1} < w^{-1}$.*

Define the *left descent set* of w as $D_L(w) = \{s \in S : \ell(sw) < \ell(w)\}$. The following fundamental result about the Bruhat order is sometimes called the *lifting property*.

Lemma 2.3 (Verma [19])**.** *Suppose $v < w$ and $s \in D_L(w) \setminus D_L(v)$. Then, $v \leq sw$ and $sv \leq w$.*

2.1. Reflection subgroups.

Maintain the Coxeter system (W, S) and its set of reflections T as defined above. Then W' is a *reflection subgroup* of W if $W' = \langle W' \cap T \rangle$. A reflection subgroup W' is called *dihedral* if $W' = \langle t, t' \rangle$ for some $t, t' \in T$, with $t \neq t'$.

Lemma 2.4 (Dyer [9])**.** *Suppose $t_1, t_2, t_3, t_4 \in T$ and $t_1 t_2 = t_3 t_4 \neq e$. Then $W' = \langle t_1, t_2, t_3, t_4 \rangle$ is a dihedral reflection subgroup of W.*

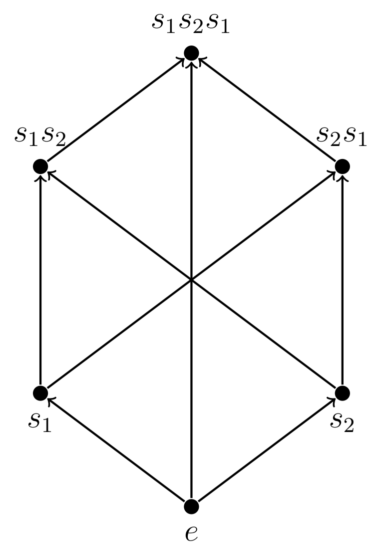

FIGURE 1. The Bruhat graph of $(W(A_2), S(A_2))$.

It turns out that reflection subgroups of W are themselves Coxeter groups. For $w \in W$, define $N(w) := \{t \in T : \ell(tw) < \ell(w)\}$. This is the set of *inversions* of w.

Theorem 2.5 (DEODHAR [7], DYER [8]). *Let W' be a reflection subgroup of W and define $X = \{t \in T : N(t) \cap W' = \{t\}\}$. Then we have:*

(1) $W' \cap T = \{ut'u^{-1} : u \in W', t' \in X\}$.
(2) *(W', X) is a Coxeter system.*

Coxeter described all types of affine groups generated by reflections and their reflection subgroups [6]. The following lemma is a very special case. It can be seen directly, e.g. by considering root lengths.

Lemma 2.6. *Every reflection subgroup of a finite simply laced group is itself simply laced.*

For any subset $Y \subseteq W$ define the Bruhat graph of Y, denoted $\mathrm{Bg}_S(Y)$, as the directed subgraph of $\mathrm{Bg}_S(W)$ induced by Y.

Theorem 2.7 (DYER [9]). *Let W' be a reflection subgroup of W and let X be as in Theorem 2.5. Then $\mathrm{Bg}_S(W') = \mathrm{Bg}_X(W')$.*

2.2. **Schubert varieties.** Let G be an algebraic group over $\mathbb{C}$ and B a Borel subgroup containing a maximal torus T. Then G/B is called the *flag variety* and it is the disjoint union of *Schubert cells* BwB/B where $w \in W$ and $W = N(T)/T$ is the *Weyl group* (which is a finite Coxeter group). The closure $X_w := \overline{BwB/B}$ is called a *Schubert variety.* Note that $G/B = X_{w_0}$ for w_0 the longest element of W. More on Schubert varieties can for example be found in [1].

Next, we review ways to detect singularities of Schubert varieties by inspecting Bruhat graphs. For a lower interval $[e, w] = \{z \in W : e \leq z \leq w\}$ write $\mathrm{Bg}_S(w)$ for the Bruhat graph of $[e, w]$. Let z be a vertex in $\mathrm{Bg}_S(w)$. The *degree* of z, denoted $\deg_w(z)$, is the number of edges incident to z in $\mathrm{Bg}_S(w)$ (where directions of edges are ignored).

The following result holds in any Coxeter group. In that generality it is due to Dyer [10]. In our context, where W is a (finite) Weyl group, other proofs given by Carrell and Peterson [4] and Polo [16] also apply.

Theorem 2.8. *Let $w \in W$. Then the degree of any vertex in $\mathrm{Bg}_S(w)$ is at least $\ell(w)$.*

In any Bruhat graph $\mathrm{Bg}_S(w)$, it is known that $\ell(w) = |N(w)| = \deg_w(w)$. In particular, if $\mathrm{Bg}_S(w)$ is regular (i.e., every vertex of $\mathrm{Bg}_S(w)$ has the same number of edges), then $\deg_w(e) = \ell(w)$.

Theorem 2.9 (Carrell–Peterson [4])**.** *The Schubert variety X_w is rationally smooth if and only if $\mathrm{Bg}_S(w)$ is regular.*

The next theorem says that smoothness and rational smoothness are equivalent for simply laced Weyl groups.

Theorem 2.10 (Carrell–Kuttler [5])**.** *Suppose W is simply laced. Then, for any $w \in W$, X_w is smooth if and only if it is rationally smooth.*

Corollary 2.11. *If W is simply laced then X_w is smooth if and only if $\mathrm{Bg}_S(w)$ is regular.*

In general smoothness is stronger than rational smoothness when W is not simply laced. For example $X_{s_1s_2s_1}$ is rationally smooth but not smooth if W is of type C_2 generated by the simple reflections s_1 and s_2 with s_1 corresponding to the short root.

The next definition provides another characterisation which can be used to prove that a given Schubert variety is not rationally smooth (see Theorem 2.14 below).

Definition 2.12 ([13])**.** *Let $x, u, v \leq w$. The Bruhat interval $[e, w]$ contains the* broken rhombus (x, u, v) *if the conditions below are satisfied:*

(1) $x \leftarrow u \rightarrow v$;
(2) *There is some $y \in W$ with $x \rightarrow y \leftarrow v$;*
(3) *If $x \rightarrow y \leftarrow v$, then $y \nleq w$.*

Example 2.13. Consider the group $W(D_4)$ of type D_4 with set of simple reflections $S(D_4) = \{s_1, s_2, s_3, s_4\}$ where $m(s_i, s_2) = 3$ for $i = 1, 3, 4$ and $m(s_i, s_j) = 2$ for $i, j \neq 2$. Figure 2 shows $\mathrm{Bg}_{S(D_4)}(w)$ for $w = s_2s_1s_3s_4s_2$. We use $1, 2, 3, 4$ for s_1, s_2, s_3, s_4, respectively, for brevity. Thus, for example, w is represented by 21342. The interval $[e, w]$ contains the broken rhombus (s_2s_3, s_2, s_1s_2) since there is no $y \leq w$ such that $s_2s_3 \rightarrow y \leftarrow s_1s_2$ although $s_2s_3 \rightarrow s_1s_2s_3 \leftarrow s_1s_2$. Moreover note that $\mathrm{Bg}_{S(D_4)}(w)$ is not regular. Hence, X_w is not smooth for $w = s_2s_1s_3s_4s_2 \in W(D_4)$.

The following result is proved in [13] as a direct application of Theorem 2.9. It can also be deduced using the main result of Dyer [11].

Theorem 2.14. *The Schubert variety X_w is rationally smooth if and only if $[e, w]$ contains no broken rhombus.*

Corollary 2.15. *Suppose W is simply laced and let $w \in W$. Then, the Schubert variety X_w is smooth if and only if $[e, w]$ contains no broken rhombus.*

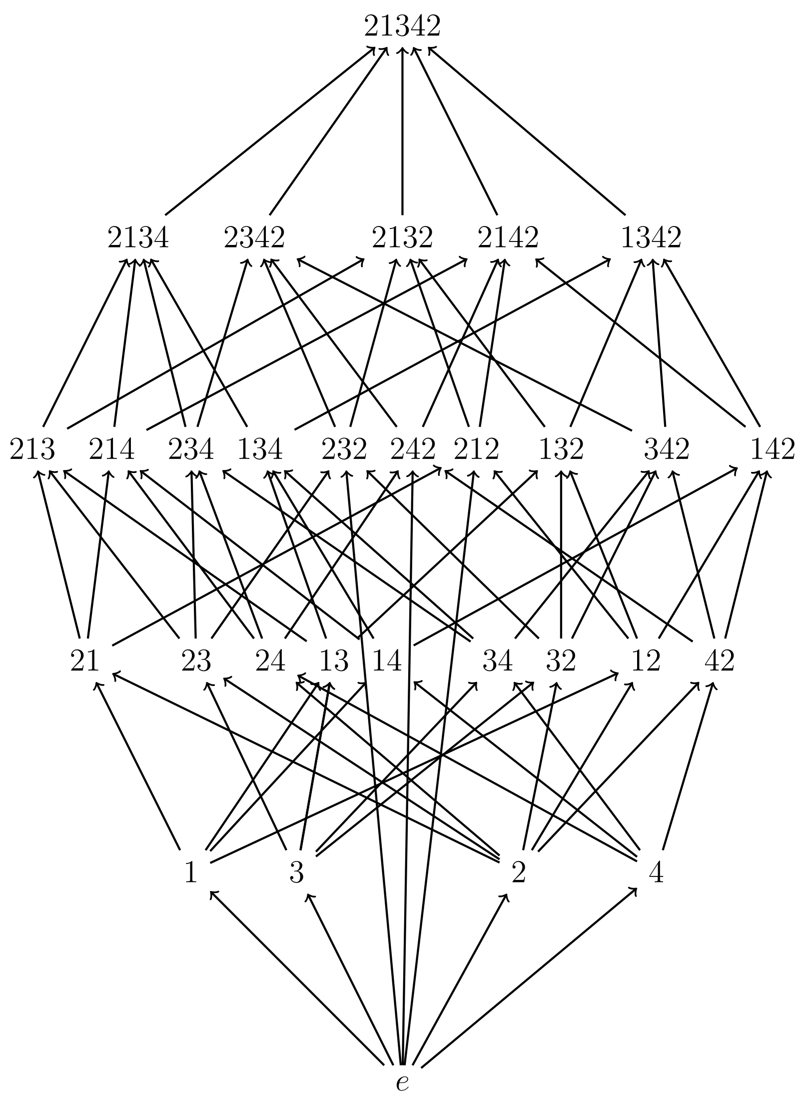

FIGURE 2. The Bruhat graph of $s_2 s_1 s_3 s_4 s_2 \in W(D_4)$.

3. Schubert varieties indexed by involutions

In this section, which contains the main result, we consider Schubert varieties indexed by involutions of finite simply laced groups.

Again let (W, S) be an arbitrary Coxeter system. A *standard parabolic subgroup* of W is a subgroup of the form $W_J = \langle J \rangle$ for $J \subseteq S$. If W_J is finite its longest element will be denoted by $w_0(J)$.

For $v \in W$ define $S(v) := \{s \in S : s \leq v\}$. Then $W_{S(v)}$ is the minimal standard parabolic subgroup of W which contains v.

Theorem 3.1. *Suppose (W, S) is finite and simply laced and let $v \in W$ be an involution. Then the Schubert variety X_v is smooth if and only if $v = w_0(J)$ for some $J \subseteq S$.*

Proof. The "if" assertion is obvious: $X_{w_0(J)}$ is a (smooth) flag variety. For the "only if" direction, let v be an involution which is not the longest element of any standard parabolic subgroup W_J of W. Since $v \neq w_0(S(v))$, there exists $s \in S(v)$ such that $v < sv$. If $\deg_v(e) \neq \ell(v)$, $\mathrm{Bg}_S(v)$ is not regular and, by Corollary 2.11, X_v is not smooth. Thus, we may assume $\deg_v(e) = \ell(v)$. Since $\deg_{sv}(e)$ is the number of $t \in T$ such that $t \leq sv$ and that degree is at least $\ell(sv)$, there exists a reflection $t \leq sv$ such that $t \not\leq v$. Since $t \leq sv$, by Lemma 2.2, $t^{-1} \leq v^{-1}s$, which implies that $t \leq vs$. Moreover we must have $st < t$. To see this we use Lemma 2.3 in the following way: since $s \in D_L(sv)$, if we would have $t < st$ then by using the lifting property this would imply that $t \leqslant ssv = v$ which is a contradiction. Now since $st < t$, by Lemma 2.2 we see that $ts < t$. Because $ts < t$ and $t \leq vs$, we have $ts < vs$. Since $s \notin D_L(st)$, we have $s \in D_L(sv) \setminus D_L(st)$ and then by Lemma 2.3 we get $st \leq v$. Consider the dihedral subgroup $D = \langle s, sts \rangle$ of W. Since W is finite and simply laced, then, by Lemma 2.6, D is also simply laced. By Theorem 2.7, $\mathrm{Bg}_S(D)$ equals the Bruhat graph of (D, X), where X is as in Theorem 2.5. Since, moreover, $st \to t$, D is not Abelian and hence $|D| = 6$. Therefore, $D = \{e, s, sts, ts, st, t\}$, and $\mathrm{Bg}_S(D)$ is as shown in Figure 3.

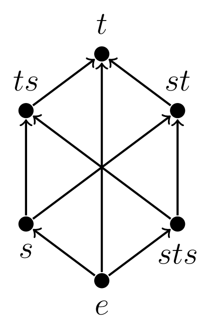

Figure 3. The Bruhat graph of D.

Now $ts, st \leq v$ but $t \not\leq v$. From Figure 3, (st, sts, ts) is a broken rhombus of $[e, v]$ because there is no $x \leq v$ such that there are directed edges from st and ts

to x. To see this, suppose that $st \to x \leftarrow ts$. So there exist $t', t'' \in T$ with $t' \neq t''$ such that $stt' = tst''$. Then $t't'' = tsts \neq e$. By Lemma 2.4 we therefore have a dihedral subgroup $W' = \langle sts, t, t', t'' \rangle$ of W, and W' is simply laced since W is (by Lemma 2.6). Clearly $D \subseteq W'$. Since W' has no more than six elements, $W' = D$. So $stt' = x \in D$. Since there is a directed edge from st to x, $x = t \not\leq v$. Since (st, sts, ts) is a broken rhombus of $[e, v]$, by Corollary 2.15, X_v is not smooth. $\square$

When W is not simply laced, there may exist an involution $w \in W$ which is not the longest element of any standard parabolic subgroup of W but for which X_w is smooth. For example, in type C_2 there are two involutions of length three. One of them indexes a smooth Schubert variety (and, as was mentioned above, the other one indexes a rationally smooth but not smooth Schubert variety). This example shows that Theorem 3.1 cannot be extended to multiply laced types. We do, however, not know what happens in infinite simply laced types.

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Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden
Email address: axel.hultman@liu.se
Email address: vincent.umutabazi@liu.se