Promotion of Kreweras words

joint work with Sam Hopkins

9.9.2020

How many ways are there . . .

for $2n$ people to cast votes for Alice and Bob, such that

- Alice and Bob both receive n votes, and
- Alice never trails Bob?

$$\text{AB}$$

$$\text{AABB}, \text{ABAB}$$

$$\text{AAABBB}, \text{AABABB}, \text{AABBAB}, \text{ABAABB}, \text{ABABAB}$$

$$\ldots$$

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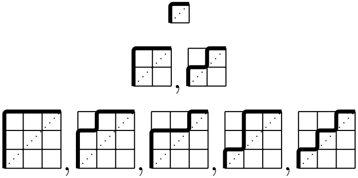

$$\dots$$

$$\frac{1}{n+1}\binom{2n}{n}$$

Promotion of Dyck words

The *promotion* $\operatorname{pr} w$ of a Dyck word $w = w_1 \dots w_{2n}$ is:

- remove the first letter w_1
- replace the first letter w_i such that $w_2 \dots w_i$ has more B's than A's with an A
- append B.

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Remarks:

- Dyck words of length $2n$ are linear extensions of the poset $[2] \times [n]$
- promotion generalizes to linear extensions of arbitrary posets
- promotion is invertible

Promotion of Dyck words

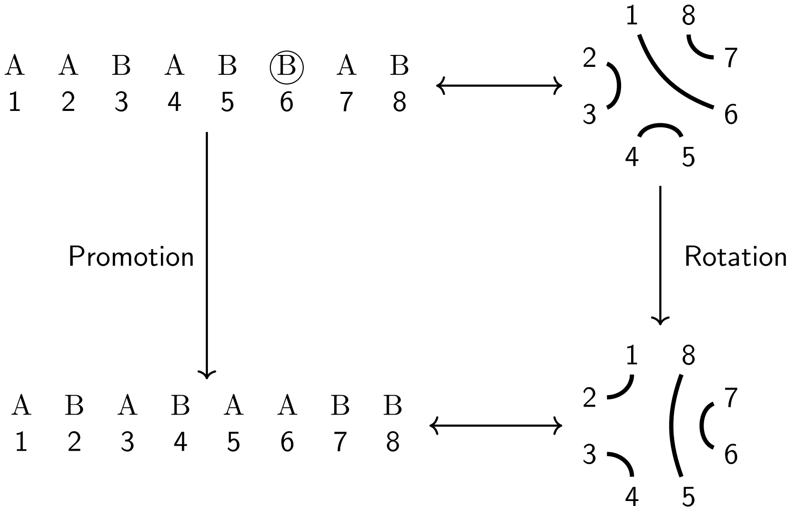
Theorem (folklore)

Promotion on Dyck words has order $2n$: $\mathrm{pr}^{2n} w = w$.
Its character is

$$\frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q,$$

where $[n]_q = \frac{1-q^n}{1-q}$, $[n]_q! = [n]_q \cdots [1]_q$ *and* $\begin{bmatrix}n\\ m\end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q}$.

Promotion of Dyck words is rotation of noncrossing matchings



How many ways are there . . .

for $3n$ people to cast votes for Alice, Bob and Charlie, such that

- Alice, Bob and Charlie all receive n votes, and
- Alice never trails either Bob or Charlie?

$$\mathrm{ABC}, \mathrm{ACB}$$

$$\mathrm{AABBCC},\mathrm{AABCBC},\mathrm{AABCCB},\mathrm{ABABCC},\mathrm{ABACBC},\mathrm{ABACCB},\mathrm{ABCABC},\mathrm{ABCACB},\ldots$$

$$\ldots$$

How many ways are there ...

for $3n$ people to cast votes for Alice, Bob and Charlie, such that

- Alice, Bob and Charlie all receive n votes, and
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$$\text{ABC}, \text{ACB}$$

AABBCC,AABCBC,AABCCB,ABABCC,ABACBC,ABACCB,ABCABC,ABCACB,...

$$\cdots$$

$$\frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$$

Promotion of Kreweras words

The *promotion* $\operatorname{pr} w$ of a Kreweras word $w = w_1 \dots w_{3n}$ is:

- remove the first letter w_1
- replace the first letter w_i such that $w_2 \dots w_i$ has more B's than A's or more C's than A's with an A
- append w_i.

Promotion of Kreweras words

The *promotion* $\operatorname{pr} w$ of a Kreweras word $w = w_1 \dots w_{3n}$ is:

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- append w_i.

Remark:

- Kreweras words of length $3n$ are linear extensions of the poset $V \times [n]$

Promotion of Kreweras words

Theorem

Promotion on Kreweras words almost has order $3n$: $\mathrm{pr}^{3n} w$ is obtained from w by switching all B's and C's.

Conjecture

Its character is

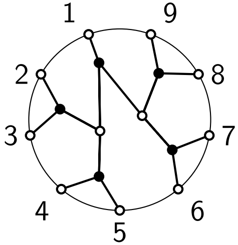
$$\frac{[2]_q^{2n}[3n]_{q^2}!}{[n+1]_{q^2}![2n+1]_q!}.$$

Remark:

$$\frac{4^n}{(n+1)(2n+1)}\binom{3n}{n} = \frac{2^{2n}(3n)!}{(n+1)!(2n+1)!}$$

Promotion of Kreweras words is rotation of webs

A	A	B	B	C	A	C	C	B
1	2	3	4	5	6	7	8	9



Promotion

Rotation

A	B	A	C	A	C	C	B	B
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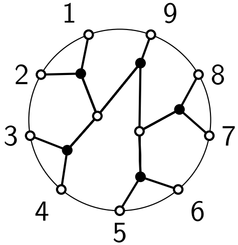
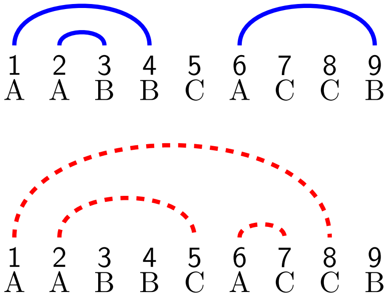


Step 1: Kreweras words to permutations

blue and crimson noncrossing matchings



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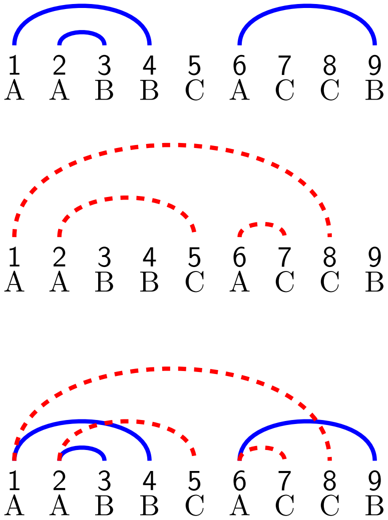
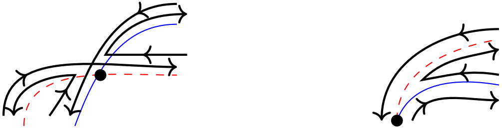
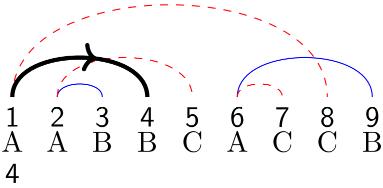


Step 1: Kreweras words to permutations

rules of the road





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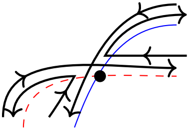

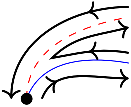

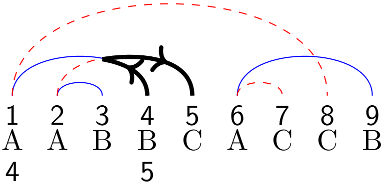

Lemma

$$\sigma_{\mathrm{pr}\,w} = \mathrm{rot}(\sigma_w)$$

$$\varepsilon_{\mathrm{pr}(w)} = [\varepsilon_w(2), \varepsilon_w(3), \ldots, \varepsilon_w(3n), -\varepsilon_w(1)]$$

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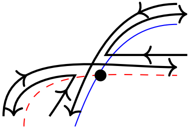

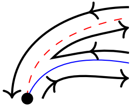

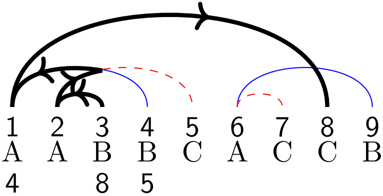

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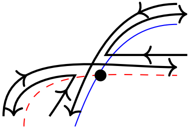

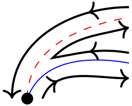

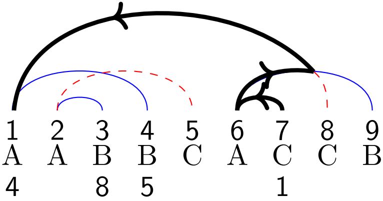

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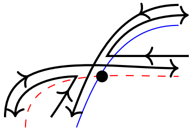 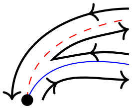

1	2	3	4	5	6	7	8	9
A	A	B	B	C	A	C	C	B
4	3	8	5	2	7	1	9	6
B	B	C	C	B	C	B	B	C

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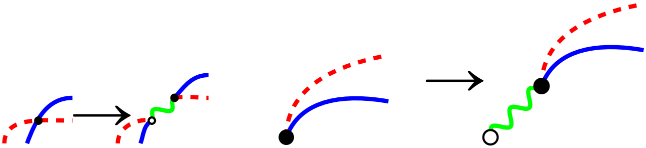
Corollary

$\mathsf{pr}^{3n}(w)$ *is obtained from* w *be swapping* B*'s and* C*'s.*

Step 2: Kreweras words to Kuperberg webs



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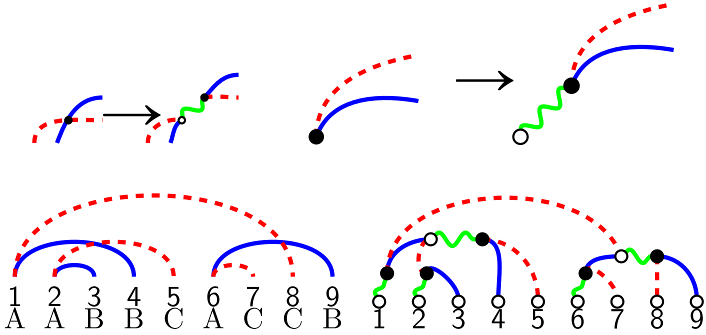

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