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THE A·B·C·Ds OF SCHUBERT CALCULUS

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ABSTRACT. We collect *Atiyah–Bott Combinatorial Dreams* (A·B·C·Ds) in Schubert calculus. One result relates equivariant structure coefficients for two isotropic flag manifolds, with consequences to the thesis of C. Monical. We contextualize using work of N. Bergeron and F. Sottile, of S. Billey and M. Haiman, of P. Pragacz, and of T. Ikeda, L. Mihalcea, and I. Naruse. The relation complements a theorem of A. Kresch and H. Tamvakis in quantum cohomology. Results of A. Buch and V. Ravikumar rule out a similar correspondence in K-theory.

1. INTRODUCTION

1.1. **Conceptual framework.** Each generalized flag variety G/B has finitely many orbits under the left action of the (opposite) Borel subgroup B_- of a complex reductive Lie group G. They are indexed by elements w of the Weyl group $\mathcal{W} \cong N(\mathsf{T})/\mathsf{T}$, where $\mathsf{T} = \mathsf{B} \cap \mathsf{B}_-$ is a maximal torus. The *Schubert varieties* are closures X_w of these orbits. The Poincaré duals of the Schubert varieties $\{\sigma_w\}_{w\in\mathcal{W}}$ form a $\mathbb{Z}$-linear basis of the cohomology ring $H^\star(\mathsf{G}/\mathsf{B})$. The *Schubert structure coefficients* are nonnegative integers, defined by

$$\sigma_u \smile \sigma_v = \sum_{w\in\mathcal{W}} c_{u,v}^w \sigma_w.$$

Geometrically, $c_{u,v}^w \in \mathbb{Z}_{\geq 0}$ counts intersection points of generic translates of three Schubert varieties. The main problem of modern Schubert calculus is to combinatorially explain this positivity. For Grassmannians, this is achieved by the *Littlewood–Richardson rule* [13].

The title alludes to a principle, traceable to M. Atiyah and R. Bott [6], that *equivariant cohomology* is a lever on ordinary cohomology. In our case, each X_w is T-stable, so it admits a class ξ_w in $H^\star_{\mathsf{T}}(\mathsf{G}/\mathsf{B})$, the T-equivariant cohomology ring of G/B. These classes are a basis for $H^\star_{\mathsf{T}}(\mathsf{G}/\mathsf{B})$ as a module over the base ring $H^\star_{\mathsf{T}}(pt)$. If $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ are the simple roots of the root system $\Phi = \Phi^+ \cup \Phi^-$ associated to our pinning of G, $H^\star_{\mathsf{T}}(pt) \cong \mathbb{Z}[\alpha_1, \ldots, \alpha_r]$.

Define the *equivariant Schubert structure coefficient* $C_{u,v}^w \in H^\star_{\mathsf{T}}(pt)$ by

$$\xi_u \cdot \xi_v = \sum_{w\in\mathcal{W}} C_{u,v}^w \, \xi_w. \tag{1}$$

If $\ell(u) + \ell(v) = \ell(w)$, $C_{u,v}^w = c_{u,v}^w$. In fact, $C_{u,v}^w \in \mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_r]$; see (III) below. Thus we have a harder positivity problem, with $\#\Delta$-many parameters. Does the added complication make the problem simpler? We initiate our systematic exploration of the question.

The inclusion $(\mathsf{G}/\mathsf{B})^{\mathsf{T}} \hookrightarrow \mathsf{G}/\mathsf{B}$ induces an *injective* map

$$H^\star_{\mathsf{T}}(\mathsf{G}/\mathsf{B}) \hookrightarrow H^\star_{\mathsf{T}}((\mathsf{G}/\mathsf{B})^{\mathsf{T}}) \cong \bigoplus_{w\in\mathcal{W}} \mathbb{Z}[\alpha_1, \ldots, \alpha_r]. \tag{2}$$

Thus, each ξ_w is identified with a $\#\mathcal{W}$-size list of polynomials $(\xi_w|_v)_{v\in\mathcal{W}}$. Multiplication in $H^\star_{\mathsf{T}}(\mathsf{G}/\mathsf{B})$ is thereby pointwise multiplication of these lists. Moreover, there is a *positive* combinatorial formula for the *equivariant restriction* $\xi_w|_v$ due to H. Andersen. J. Jantzen, and W. Soergel [4], and rediscovered by S. Billey [8].[1] Let $I = s_{\underline{\alpha}_1} s_{\underline{\alpha}_2} \cdots s_{\underline{\alpha}_{\ell(v)}}$ be a reduced word for $v \in \mathcal{W}$, where $s_{\underline{\alpha}_i}$ is the reflection through the hyperplane perpendicular to $\underline{\alpha}_i := \alpha_j \in \Delta$ (for some j depending on i). Then,

$$\xi_w|_v = \sum_{J \subseteq I} \prod_I (\underline{\alpha}_i^{\langle i \in J \rangle} s_{\underline{\alpha}_i}) \cdot 1; \tag{3}$$

cf. [22, Theorem 1]. The sum is over subwords J that are reduced words for w. Also, $\underline{\alpha}_i^{\langle i \in J \rangle}$ means $\underline{\alpha}_i$ appears only if $i \in J$. Let us illustrate this formula.

Example 1.1. Fix the reduced word $I = s_1 s_2 s_3 s_2$ for $v = 2431 \in S_4$. Then for $w = 2314$,

$\xi_w|_v = \alpha_1 s_1 \alpha_2 s_2 s_3 s_2 \cdot 1 + \alpha_1 s_1 s_2 s_3 \alpha_2 s_2 \cdot 1 = \alpha_1 s_1 \alpha_2 + \alpha_1 s_1 s_2 s_3 \alpha_2 = \alpha_1(\alpha_1 + \alpha_2) + \alpha_1(\alpha_3)$. □

The combination of (3) and (2) provides the combinatorial definition of $H^\star_{\mathsf{T}}(\mathsf{G}/\mathsf{B})$ we use.

This description of $H^\star_{\mathsf{T}}(\mathsf{G}/\mathsf{B})$ permits a *non-positive* linear algebraic computation of $C^w_{u,v}$, see, e.g., [8, Section 6]. From this perspective, the solved combinatorics of equivariant restriction and the open problem of Schubert calculus seem far apart. However, we argue using an idealization that the concepts are closer than first supposed:

Atiyah–Bott Combinatorial Dream (A·B·C·D). *A combinatorial (positivity) statement true for equivariant restrictions also holds for Schubert structure coefficients.*

As with any maxim, this will not always hold, as illustrated in Section 2.3. However, the principle suggests new ideas. First we begin with retrospective examples to explore the connection between equivariant restrictions and Schubert structure coefficients:

(I) Sometimes combining (3) with *basic* Coxeter theory realizes an A·B·C·D. *Bruhat order* $\leq$ on $\mathcal{W}$ is geometrically defined by $w \leq v$ if $X_w \supseteq X_v$. Fix a reduced word I of v. The *subword property* of Bruhat order states that $w \leq v$ if and only if there exists a subword J of I that is a reduced word of w. Hence from (3),

$$\xi_w|_v = 0 \quad \text{unless } w \leq v. \tag{4}$$

Combining (4) and (2), we obtain

$$C^w_{u,v} = 0 \text{ unless } u \leq w \text{ and } v \leq w. \tag{5}$$

(II) The *converse* of the A·B·C·D, *i.e.*, a combinatorial (positivity) statement true for Schubert structure coefficients also holds for equivariant restrictions, is true. Following [22, Lemma 1], by (4) and (5),

$$\xi_v|_v \cdot \xi_w|_v = C^v_{v,w}\, \xi_v|_v.$$

By (3), $\xi_v|_v \neq 0$. Hence

$$C^v_{v,w} = \xi_w|_v. \tag{6}$$

This is a tantalizing clue about an eventual combinatorial rule for $C^v_{u,w}$. More concretely, in [25], (6) implies a recurrence that, with additional combinatorics, proves an equivariant Littlewood–Richardson rule *sans* symmetric functions.

[1] Equivariant restriction is part of *GKM-theory* [15], a subject of extensive investigation; see, e.g., J. Tymoczko's exposition [39], and the references therein, for an account germane to our discussion. However the case of Schubert varieties is found in work of B. Kostant and S. Kumar [26, 27].

(III) Here is a deep instance ([8], *cf.* [39, Section 2]). It is a textbook fact (see [18, Section 1.7]) that

$$\mathsf{Inv}(v^{-1}) := \{\alpha \in \Phi^+ : v^{-1}(\alpha) \in \Phi^-\} = \{s_{\underline{\alpha}_1} s_{\underline{\alpha}_2} \cdots s_{\underline{\alpha}_{k-1}} \underline{\alpha}_k : 1 \leq k \leq \ell(v)\}. \tag{7}$$

Since each positive root is a positive linear combination of simples, by (7) and (3),

$$\xi_w|_v \in \mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_r].$$

Indeed, D. Peterson conjectured, and W. Graham [16] geometrically proved that

$$C_{u,v}^w \in \mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_r]. \tag{8}$$

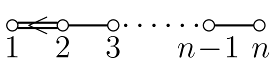
(IV) This is closely related to (III), but is folklore. Since $\ell(v) = \ell(v^{-1}) = \#\mathsf{Inv}(v^{-1})$, by (7), $s_{\underline{\alpha}_1} s_{\underline{\alpha}_2} \cdots s_{\underline{\alpha}_{k-1}} \underline{\alpha}_k \in \Phi^+$ are all distinct. Hence from (3), $\xi_w|_v$ is *square-free* when expressed in the positive roots. As A. Knutson (private communication) points out, the proof in [16] shows this to be true of $C_{u,v}^w$ as well.

(V) For any G/B, there is a recurrence, due to B. Kostant and S. Kumar to compute $\xi_w|_v$; it has an analogue to non-positively compute $C_{u,v}^w$, due to A. Knutson. See [23, Theorem 1] and [22, Section 1]. In turn, special cases of Knutson's recurrence give "descent cycling" relations on the *ordinary* Schubert structure constants [21]. We also point to recent work of R. Goldin and A. Knutson [14] which generalizes (3) to give another non-positive formula for $C_{u,v}^w$.

1.2. **Does the A·B·C·D suggest anything new?** Our main instance is of different flavor than (I)–(V). Using an instance of the A·B·C·D, we relate all structure coefficients of one isotropic flag variety to those of another; this has consequences. The results are neither explicit in the literature nor seem well-known. The correspondence generalizes, with a new proof, non-equivariant results of P. Pragacz [34] and of N. Bergeron and F. Sottile [7] (who rely on S. Billey and M. Haiman's work [9], which in turn generalizes [34]). We emphasize that the correspondence can also be derived from T. Ikeda, L. Mihalcea, and H. Naruse's paper [19]; see the discussion of Section 3.

Consider the classical groups $\mathsf{G} = \mathsf{SO}_{2n+1}$ and $\mathsf{G} = \mathsf{Sp}_{2n}$ of non-simply laced type. These are automorphism groups preserving a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$. In the former case it is a symmetric form on $W = \mathbb{C}^{2n+1}$ whereas in the latter case it is a skew symmetric form on $W = \mathbb{C}^{2n}$. A subspace $V \subseteq W$ is *isotropic* if, for all $v_1, v_2 \in V$, $\langle v_1, v_2 \rangle = 0$. The maximum dimension of an isotropic space is n. Any flag of isotropic subspaces $\langle 0 \rangle \subset F_1 \subset F_2 \subset \cdots \subset F_n$ extends to a complete flag in W by $\langle 0 \rangle \subset F_1 \subset F_2 \subset \cdots \subset F_n \subseteq F_n^\perp \subset F_{n-1}^\perp \subset \cdots \subset F_1^\perp \subset W$, where $F_k^\perp$ is the orthogonal complement of F_k. Then the flag manifolds $X = \mathsf{SO}_{2n+1}/\mathsf{B}$ and $Y = \mathsf{Sp}_{2n}/\mathsf{B}$ consist of complete flags of this form.

The root systems for SO_{2n+1} (type B_n) and Sp_{2n} (type C_n) are rank $r = n$. Let $\{\beta_1, \ldots, \beta_n\}$ and $\{\gamma_1, \ldots, \gamma_n\}$ be a choice of simple roots labeled by their respective Dynkin diagrams



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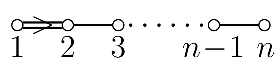


The two root systems share the *hyperoctahedral group* $\mathcal{B}_n$ as their common Weyl group. We represent $\mathcal{B}_n$ as *signed permutations* of $\{1, 2, \ldots, n\}$, e.g., $\underline{2}\,1\,\underline{3}$. Define

$$s(w) := \#\{1 \leq i \leq n : w(i) < 0\}.$$

Let $\overline{f} \in \mathbb{Z}[\beta_1, \beta_2, \ldots, \beta_n]$ be $f \in \mathbb{Z}[\gamma_1, \gamma_2, \ldots, \gamma_n]$ with $\gamma_1 \mapsto 2\beta_1$ and $\gamma_i \mapsto \beta_i$ for $1 < i \leq n$.

Theorem 1.2. $C_{u,v}^w(X) = 2^{s(w)-s(u)-s(v)} \overline{C_{u,v}^w(Y)}$.

Proof. This equivalence is from the definitions:

$$\xi_w(Y)|_x = \sum_{J\subseteq I}\prod_I(\underline{\gamma}_i^{\langle i\in J\rangle} s_{\underline{\alpha}_i})\cdot 1 \iff \overline{\xi_w(Y)|_x} = \sum_{J\subseteq I} 2^{\#\{1\in J\}}\prod_I(\underline{\beta}_i^{\langle i\in J\rangle} s_{\underline{\alpha}_i})\cdot 1.$$

The Coxeter combinatorics needed is merely this: since J is a reduced word for w, it is true that $\#\{1\in J\}=s(w)$. Therefore,

$$\overline{\xi_w(Y)|_x} = 2^{s(w)}\sum_{J\subseteq I}\prod_I(\underline{\beta}_i^{\langle i\in J\rangle} s_{\underline{\alpha}_i})\cdot 1 = 2^{s(w)}\xi_w(X)|_x; \tag{9}$$

i.e., a "power of two relationship" between the restrictions. Applying (1), (2) and (3) to Y,

$$\begin{aligned}
&\xi_u(Y)|_x\cdot\xi_v(Y)|_x = \sum_{w\in\mathcal{B}_n} C_{u,v}^w(Y)\,\xi_w(Y)|_x \quad \forall x\in\mathcal{B}_n\\
&\iff \overline{\xi_u(Y)|_x}\cdot\overline{\xi_v(Y)|_x} = \sum_{w\in\mathcal{B}_n}\overline{C_{u,v}^w(Y)}\;\overline{\xi_w(Y)|_x}\quad \forall x\in\mathcal{B}_n\\
&\iff (2^{-s(u)}\overline{\xi_u(Y)|_x})\cdot(2^{-s(v)}\overline{\xi_v(Y)|_x}) = \sum_{w\in\mathcal{B}_n}2^{s(w)-s(u)-s(v)}\overline{C_{u,v}^w(Y)}(2^{-s(w)}\overline{\xi_w(Y)|_x})\quad\forall x\in\mathcal{B}_n\\
&\iff \xi_u(X)|_x\cdot\xi_v(X)|_x = \sum_{w\in\mathcal{B}_n}2^{s(w)-s(u)-s(v)}\overline{C_{u,v}^w(Y)}\,\xi_w(X)|_x\quad\forall x\in\mathcal{B}_n\quad\text{[by (9)]}.
\end{aligned}$$

We are now done by (1), (2) and (3) applied to X, *i.e.*, uniqueness of the equivariant structure coefficients. □

Example 1.3. Consider $u = 3\,\underline{2}\,1, v=\underline{3}\,\underline{2}\,1$ and $w=\underline{2}\,\underline{3}\,1$ in $\mathcal{B}_3$. Then $s(w)-s(u)-s(v)=-1$ and $C_{u,v}^w(Y) = 2\gamma_1\gamma_2^2+2\gamma_1\gamma_2\gamma_3+4\gamma_2^3+6\gamma_2^2\gamma_3+2\gamma_2\gamma_3^2$, so

$$\overline{C_{u,v}^w(Y)} = 4\beta_1\beta_2^2+4\beta_1\beta_2\beta_3+4\beta_2^3+6\beta_2^2\beta_3+2\beta_2\beta_3^2.$$

We also have

$$C_{u,v}^w(X) = 2\beta_1\beta_2^2+2\beta_1\beta_2\beta_3+2\beta_2^3+3\beta_2^2\beta_3+\beta_2\beta_3^2.$$

Hence $C_{u,v}^w(X) = 2^{-1}\overline{C_{u,v}^w(Y)}$, in agreement with Theorem 1.2. □

Since the correspondence of Theorem 1.2 respects Graham-positivity, we obtain the following equivalence.

Corollary 1.4. *We have*

$$[\beta_1^{i_1}\cdots\beta_n^{i_n}]C_{u,v}^w(X)=0 \iff [\gamma_1^{i_1}\cdots\gamma_n^{i_n}]C_{u,v}^w(Y)=0.$$

In particular, $C_{u,v}^w(X)=0 \iff C_{u,v}^w(Y)=0$.

Let $X' = \mathsf{OG}(n,2n+1)$ be the *maximal orthogonal Grassmannian* of n-dimensional subspaces of $\mathbb{C}^{2n+1}$ that are isotropic with respect to a nondegenerate symmetric form. Also, let $Y'=\mathsf{LG}(n,2n)$ be the *Lagrangian Grassmannian* of n-dimensional subspaces of $\mathbb{C}^{2n}$ that are isotropic with respect to a nondegenerate skew symmetric form. A *strict partition* is an integer partition $\lambda=(\lambda_1>\lambda_2>\cdots>\lambda_\ell)$. The Schubert varieties and their (equivariant) cohomology classes are indexed by such λ with $\lambda_1\leq n$ and $\ell\leq n$. Let $\ell(\lambda)$ be the number of (nonzero) parts of a strict partition λ.

Corollary 1.5 (*cf.* Conjecture 5.1 of [32]). $C_{\lambda,\mu}^{\nu}(X') = 2^{\ell(\nu)-\ell(\lambda)-\ell(\mu)}\overline{C_{\lambda,\mu}^{\nu}(Y')}$.

Proof. The map $X \twoheadrightarrow X'$ that forgets all subspaces of a complete flag in X except the n-th induces $H_{\mathsf{T}}(X') \hookrightarrow H_{\mathsf{T}}(X)$ sending Schubert classes to Schubert classes. The image of $\xi_\lambda(X')$ is $\xi_{w_\lambda}(X)$ where $w_\lambda \in \mathcal{B}_n$ is the unique ascending signed permutation beginning as $-\lambda_1, -\lambda_2, \ldots, -\lambda_\ell$, followed by positive integers in increasing order. Therefore, $C_{\lambda,\mu}^{\nu}(X') = C_{w_\lambda,w_\mu}^{w_\nu}(X)$. Similarly, $C_{\lambda,\mu}^{\nu}(Y') = C_{w_\lambda,w_\mu}^{w_\nu}(Y)$. Hence, the result follows from Theorem 1.2 since by definition of w_λ, $\ell(\lambda) = s(w_\lambda)$. □

Example 1.6. Let $n = 3$ and $\lambda = (3,2), \mu = (2,1)$, and $\nu = (3,2,1)$.[2] Then $\ell(\nu) - \ell(\lambda) - \ell(\mu) = -1$. Now, $C_{\lambda,\mu}^{\nu}(Y') = 3\gamma_1^2 + 10\gamma_1\gamma_2 + 8\gamma_2^2 + 5\gamma_1\gamma_3 + 8\gamma_2\gamma_3 + 2\gamma_3^2$, so

$$\overline{C_{\lambda,\mu}^{\nu}(Y)} = 12\beta_1^2 + 20\beta_1\beta_2 + 8\beta_2^2 + 10\beta_1\beta_3 + 8\beta_2\beta_3 + 2\beta_3^2.$$

We also have

$$C_{\lambda,\mu}^{\nu}(X') = 6\beta_1^2 + 10\beta_1\beta_2 + 4\beta_2^2 + 5\beta_1\beta_3 + 4\beta_2\beta_3 + \beta_3^2,$$

so $C_{\lambda,\mu}^{\nu}(X') = 2^{-1}\overline{C_{\lambda,\mu}^{\nu}(Y')}$, agreeing with Corollary 1.5. □

Corollary 1.5 says that the open problems of giving (Graham positive) combinatorial rules to compute $C_{\lambda,\mu}^{\nu}(X')$ and $C_{\lambda,\mu}^{\nu}(Y')$ are equivalent.

Moreover, Corollary 1.5 makes exact a conjecture stated in the thesis of C. Monical [32, Conjecture 5.1]. In that thesis, one also finds [32, Conjecture 5.3], a conjectural recursive list of inequalities characterizing nonzeroness of $C_{\lambda,\mu}^{\nu}(X')$ (and implicitly, $C_{\lambda,\mu}^{\nu}(Y')$). That conjecture generalizes work of K. Purbhoo and F. Sottile [35]. It is an analogue of D. Anderson, E. Richmond, and A. Yong's result [3] that extends work of A. Klyachko [20] and of A. Knutson and T. Tao [24] on the eigenvalue problem for sums of Hermitian matrices. Corollary 1.5 implies the following fact.

Corollary 1.7. (*cf.* [32, Conjecture 5.3]) *C. Monical's inequalities characterize $C_{\lambda,\mu}^{\nu}(X') \neq 0$ if and only if they characterize $C_{\lambda,\mu}^{\nu}(Y') \neq 0$.*

Here is another consequence of Theorem 1.2. C. Li and V. Ravikumar [31] prove equivariant Pieri rules for (submaximal) isotropic Grassmannians of classical type B, C, D. Their type B and C rules are proved by separate geometric analyses. Theorem 1.2 immediately implies a Pieri rule for type C from the type B rule (or vice versa).

The "power of two" relationship between X' and Y' does not hold (in any obvious way) in the Grothendieck (K-theory) ring of algebraic vector bundles; see work of A. Buch and V. Ravikumar [11, Examples 4.9, 5.8]. On the other hand, Theorem 1.2 may be compared to the quantum cohomology result of A. Kresch and H. Tamvakis [29, Theorem 6].

2. More examples of realized A·B·C·Ds

2.1. **Inclusion of Dynkin diagrams.** Suppose we have an inclusion[3] of (finite) Dynkin diagrams $D \hookrightarrow E$ where the nodes $1, 2, \ldots, r(D)$ of D are sent to the nodes $1^\circ, 2^\circ, \ldots, r(D)^\circ$ of E, respectively. Let

$$\Delta(D) = \{\alpha_1, \ldots, \alpha_{r(D)}\} \text{ and } \Delta(E) = \{\beta_{1^\circ}, \ldots, \beta_{r(D)^\circ}, \beta_{(r(D)+1)^\circ}, \ldots \beta_{r(E)^\circ}\}.$$

[2] Hence $w_\lambda = \underline{3}\,\underline{2}\,1 = s_2s_1s_3s_2s_1$, $w_\mu = \underline{2}\,\underline{1}\,3 = s_1s_2s_1$ and $w_\nu = \underline{3}\,\underline{2}\,\underline{1} = s_1s_2s_1s_3s_2s_1$.

[3] a (multi)-graph theoretic injection that respects arrows

Given $w \in \mathcal{W}(D)$ we can unambiguously define $w^\circ \in \mathcal{W}(E)$ by taking a reduced word I for w and replacing s_{α_i} with $s_{\beta_{i^\circ}}$ to obtain a reduced word I° for w°. Let

$$\psi_{D,E} : \mathbb{Z}[\alpha_1, \ldots, \alpha_{r(D)}] \to \mathbb{Z}[\beta_{1^\circ}, \ldots, \beta_{r(D)^\circ}]$$

be defined by $\alpha_i \mapsto \beta_{i^\circ}$.

Theorem 2.1. $\psi_{D,E}(C_{u,v}^w(D)) = C_{u^\circ,v^\circ}^{w^\circ}(E)$.

Proof. We start with the restriction version of the statement, *i.e.*,

Claim 2.2. $\psi_{D,E}(\xi_w(D)|_v) = \xi_{w^\circ}(E)|_{v^\circ}$.

Proof of Claim 2.2: This is immediate from (3) using I and I° respectively in computing $\xi_w(D)|_v$ and $\xi_{w^\circ}(E)|_{v^\circ}$. This is since the inclusion of Dynkin diagrams induces a canonical isomorphism of the root system of D with a subroot system of E that maps α_i to β_{i°, and a canonical isomorphism of $\mathcal{W}(D)$ with the parabolic subgroup $\mathcal{W}(E)_D$ of $\mathcal{W}(E)$ generated by $s_{\beta_i^\circ}$ for $1 \leq i \leq r(D)$; see, *e.g.*, [18, Section 5.5]. □

Claim 2.3. *If $w \in \mathcal{W}(E) - \mathcal{W}(E)_D$ and $v \in \mathcal{W}(D)$ then $\xi_w(E)|_{v^\circ} = 0$.*

Proof of Claim 2.3: Since $w \in \mathcal{W}(E) - \mathcal{W}(E)_D$, by definition any reduced word for w involves a $s_{\beta_{t^\circ}}$ for some $t > r(D)$. Fix any reduced word I° of v°. Since $v^\circ \in \mathcal{W}(E)_D$, I° does not involve $s_{\beta_{t^\circ}}$. Hence no subword of I° can be a reduced word for w. Now the claim follows from (3). □

Combining Claims 2.2 and 2.3, we see that, for any $u, v, x \in \mathcal{W}(D)$,

$$\xi(E)_{u^\circ}|_{x^\circ} \cdot \xi(E)_{v^\circ}|_{x^\circ} = \sum_{w^\circ \in \mathcal{W}(E)_D} C_{u^\circ,v^\circ}^{w^\circ}(E)\, \xi(E)_{w^\circ}|_{x^\circ} \quad \forall x \in \mathcal{W}(D). \tag{10}$$

By Claim 2.2, for all $y \in \mathcal{W}(D)$,

$$\xi(E)_{y^\circ}|_{x^\circ} \in \mathbb{Z}_{\geq 0}[\beta_{1^\circ}, \ldots, \beta_{r(D)^\circ}].$$

Therefore by this nonnegativity and W. Graham's theorem (8), it must be that

$$C_{u^\circ,v^\circ}^{w^\circ}(E) \in \mathbb{Z}_{\geq 0}[\beta_{1^\circ}, \ldots, \beta_{r(D)^\circ}].$$

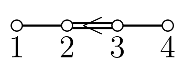
Therefore, it makes sense to apply $\psi_{D,E}^{-1}$ to both sides of (10) to obtain

$$\xi(D)_u|_x \cdot \xi(D)_v|_x = \sum_{w \in \mathcal{W}(D)} \psi_{D,E}^{-1}(C_{u^\circ,v^\circ}^{w^\circ}(E))\, \xi(D)_w|_x \quad \forall x \in \mathcal{W}(D). \tag{11}$$

We can now conclude as in the proof of Theorem 1.2. By uniqueness of the structure constants, (11) asserts

$$\psi_{D,E}^{-1}(C_{u^\circ,v^\circ}^{w^\circ}(E)) = C_{u,v}^w(D).$$

Apply $\psi_{D,E}$ to both sides to conclude the proof. □

Example 2.4. The Dynkin diagram for F_4 is 

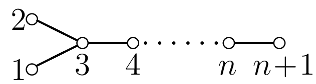
. Now, there is an embedding of $D = B_3$ into $E = F_4$ given by $1 \mapsto 1^\circ = 2, 2 \mapsto 2^\circ = 3, 3 \mapsto 3^\circ = 4$. One computes

$$C_{s_1s_2s_1,s_2s_3s_1}^{s_1s_2s_3s_1}(B_3) = 2\beta_1^2 + 3\beta_1\beta_2 + \beta_2^2 \text{ and } C_{s_2s_3s_2,s_3s_4s_2}^{s_2s_3s_4s_2}(F_4) = 2\zeta_2^2 + 3\zeta_2\zeta_3 + \zeta_3^2.$$

These are equal after $\beta_i \mapsto \zeta_{i+1}$ for $1 \leq i \leq 3$, in agreement with Theorem 2.1. □

Besides being computationally useful, Theorem 2.1 is a guiding property in the search for an eventual combinatorial rule for $C_{u,v}^w$. See [37, Section 5.2] for hints of this in the root-system uniform (non-equivariant) rule for the special case of minuscule flag varieties.

There are coincidences between types B_n and D_{n+1}, since the Dynkin diagram of the former is the "folding" of the Dynkin diagram for the latter:



Example 2.5. $C^{s_1s_2s_1}_{s_1s_2s_1,s_1s_2s_1}(B_2) = \beta_1(2\beta_1+\beta_2)(\beta_1+\beta_2)$. It is natural to compare $s_1s_2s_1 \in \mathcal{W}(B_2)$ with $s_1s_3s_2 \in \mathcal{W}(D_3)$. Indeed,

$$C^{s_1s_3s_2}_{s_1s_3s_2,s_1s_3s_2}(D_3) = \delta_1(\delta_1+\delta_2+\delta_3)(\delta_1+\delta_3)$$

equals $C^{s_1s_2s_1}_{s_1s_2s_1,s_1s_2s_1}(B_2)$ under the "folding substitution" $\delta_1,\delta_2 \mapsto \beta_1$ and $\delta_3 \mapsto \beta_2$. $\square$

Such a substitution gives a correspondence between $\mathsf{OG}(n, 2n+1)$ restrictions and a subset of restrictions of $\mathsf{OG}(n+1, 2n+2)$ (the maximal isotropic Grassmannian of type D_{n+1}); see [17, Remark 5.7] and the references therein. By using a similar argument as in Theorem 1.2, one obtains a correspondence of structure coefficients. Unfortunately, this correspondence is not true in general, even for restrictions.

Example 2.6. One calculates that

$$\xi_{s_2s_1s_2s_3}|_{s_2s_1s_2s_3}(B_3) = 4\beta_1^3\beta_2 + 10\beta_1^2\beta_2^2 + 2\beta_1^2\beta_2\beta_3 + 8\beta_1\beta_2^3 + 3\beta_1\beta_2^2\beta_3 + 2\beta_2^4 + \beta_2^3\beta_3.$$

By direct search, there is no $\xi_v|_v(D_4)$ which, after the folding substitution $\delta_1,\delta_2 \mapsto \beta_1, \delta_3 \mapsto \beta_2, \delta_4 \mapsto \beta_3$, has even the same monomial support as $\xi_{s_2s_1s_2s_3}|_{s_2s_1s_2s_3}(B_3)$. $\square$

2.2. **Nonvanishing.** The result is known, *cf.* [8, Corollary 4.5] which credits [26]. We include a proof to be self-contained.

Proposition 2.7. $\xi_w|_v \neq 0$ *for all* $w \leq v \leq w_0$.

Proof. Suppose $v \leq v'$ and fix a reduced word I' for v'. By the subword property of Bruhat order, there is a subword I of I' which is reduced for v. Any subword J of I that is a reduced word for w is also a subword of I'. Thus, by (3), any monomial appearing in $\xi_w|_v$ associated to J corresponds to a maybe different monomial (in the positive roots) in $\xi_w|_{v'}$. Now use that (3) says $\xi_w|_w$ is a nonzero monomial. $\square$

Conjecture 2.8 (Applying the A·B·C·D to Proposition 2.7)**.** *Assume* $C_{u,v}^w \neq 0$.

(I) $C^w_{u,s_\alpha v} \neq 0$ *when* $v < s_\alpha v \leq w$ *and* $\alpha \in \Delta$.
(II) *If* $\ell(w) < \ell(u) + \ell(v)$ *then there exists* s_α *(*$\alpha \in \Delta$*) with* $s_\alpha v < v$ *such that* $C^w_{u,s_\alpha v} \neq 0$.

Example 2.9. In Conjecture 2.8, the existential quantification in (II) is needed. In type B_3,

$$C^{s_2s_1s_3}_{s_2s_3,s_1s_3} = \beta_2+\beta_3, \text{ but } C^{s_2s_1s_3}_{s_2s_3,s_1(s_1s_3)} = 0.$$

Now, $C^{s_2s_1s_3}_{s_2s_3,s_3(s_1s_3)} = 1$, as predicted. $\square$

We exhaustively checked Conjecture 2.8 for A_4, B_3 and G_2 and for many examples in A_5, B_4 and F_4. Conjecture 2.8 holds for Grassmannians, where it plays a key role in [3], which connects [12] to the equivariant structure coefficients. C. Monical's extension, discussed in Section 1, motivates this conjecture.

Example 2.10. There is no "righthand version" of either part of Conjecture 2.8. For (I),

$$C^{s_1s_2s_1}_{s_1s_2,s_1}(A_2) = 1 \text{ but } C^{s_1s_2s_1}_{s_1s_2,(s_1)s_2}(A_2) = 0.$$

Whereas for (II), $C^{s_1s_2s_1}_{s_1s_2,s_2s_1}(A_2) = \alpha_1 + \alpha_2$ yet $C^{s_1s_2s_1}_{s_1s_2,(s_2s_1)s_1}(A_2) = 0$. □

Proposition 2.7 implies that, for the classical types, the decision problem `Restriction` "$\xi_w|_v \neq 0$?" is in the class P of polynomial time problems.[4] This is since there is a polynomial time *tableau criterion* for deciding if $w \leq v$ for corresponding Weyl groups; see [10, Chapters 2, 8] (here the input size is bounded by a polynomial in r). Applying the A·B·C·D to this claim concerns the decision problem `Nonvanishing`: "$C^w_{u,v} \neq 0$?" given input $u, v, w \in \mathcal{W}$ (in one line notation).

Conjecture 2.11. *For each classical Lie type,* `Nonvanishing` $\in$ P.

Conjecture 2.11 is highly speculative. That said, it holds for Grassmannians [2]. In our opinion, this conjecture is related to the (testable) Conjecture 2.17 given below.

2.3. **Unsuccessful A·B·C·Ds.** It is interesting to study situations where the A·B·C·D is (seemingly) not realized. For instance, here is a true statement for restrictions.

Theorem 2.12 (Monotonicity). *If $w \leq v \leq v'$ then $\xi_w|_{v'} - \xi_w|_v \in \mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_r]$.*

Proof. It suffices to prove this when v' covers v. Then

$$\xi_w \cdot \xi_v = C^v_{w,v}\,\xi_v + C^{v'}_{w,v}\,\xi_{v'} + \sum_{\tilde{v} \geq v, \tilde{v} \neq v'} C^{\tilde{v}}_{w,v}\,\xi_{\tilde{v}}.$$

Restricting the above equation at v', using (5) and the fact (6) that $C^v_{w,v} = \xi_w|_v$, we get

$$\xi_w|_{v'} \cdot \xi_v|_{v'} = \xi_w|_v \cdot \xi_v|_{v'} + C^{v'}_{w,v}\xi_{v'}|_{v'}. \tag{12}$$

By Proposition 2.7, $\xi_v|_{v'} \neq 0$, therefore,

$$\xi_w|_{v'} - \xi_w|_v = \frac{\xi_{v'}|_{v'}}{\xi_v|_{v'}} C^{v'}_{w,v}. \tag{13}$$

Fix a reduced word $s_{i_1}s_{i_2}\cdots s_{i_m}$ for v'. By the *strong exchange property* of Bruhat order [18, Section 5.8], there exists a *unique* $1 \leq k \leq m$ such that $s_{i_1}\cdots s_{i_{k-1}}s_{i_{k+1}}\cdots s_{i_m}$ (s_{i_k} omitted) is a reduced word of v. Therefore by (3),

$$\frac{\xi_{v'}|_{v'}}{\xi_v|_{v'}} = s_{i_1}\cdots s_{i_{k-1}} \cdot \alpha_{i_k} \in \Phi^+.$$

Hence, by (8), $\xi_w|_{v'} - \xi_w|_v = (s_{i_1}\cdots s_{i_{k-1}} \cdot \alpha_{i_k})C^{v'}_{w,v} \in \mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_r]$, as desired. □

Example 2.13 (Monotonicity counterexample). Thus, it is tempting to conjecture that if $u, v, w \in \mathcal{W}$ and s_α is a simple reflection such that $u \leq us_\alpha := u'$ and $w \leq ws_\alpha := w'$ then $C^{w'}_{u',v} - C^w_{u,v} \in \mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_r]$. In particular, this would imply $c^{w'}_{u',v} \geq c^w_{u,v}$. However, that is false in general. For instance in A_5 if $u = 351624, v = 214356, w = 631524$ and $s = s_3$ $c^w_{u,v} = c^{631524}_{351624,214356} = 1$ but $c^{w'}_{u',v} = c^{635124}_{356124,214356} = 0$. □

[4]For complexity considerations, the exceptional types are ignored since they are finite in number.

A theorem of A. Arabia [5] states:

$$\alpha \text{ divides } \xi_w|_{s_\alpha v} - \xi_w|_v.$$

(In general, this is the condition of [15] that describes the image of (2).)

Example 2.14 (Divisibility counterexample). Does α divide $C_{s_\alpha u,v}^{s_\alpha w} - C_{u,v}^w$? This is false in general. In type A_3 let $u = s_3$, $v = s_2s_3s_1$ and $w = s_2s_3s_1$. Then $C_{u,v}^w = \alpha_2 + \alpha_3$. Let $s_\alpha = s_1$ and hence $s_\alpha u = s_1 u = s_1 s_3, s_\alpha w = s_1 w = s_1s_2s_3s_1$. Now $C_{s_\alpha u,v}^{s_\alpha w} = \alpha_1 + \alpha_2$, and thus $C_{s_\alpha u,v}^{s_\alpha w} - C_{u,v}^w = \alpha_1 - \alpha_3$ is neither α-positive nor divisible by α_1. □

A number of other simple variations on monotonicity and divisibility are false as well. Can the A·B·C·Ds for monotonicity/divisibility be realized, under a hypothesis?

2.4. **Newton polytopes.** The *Newton polytope* of

$$f = \sum_{(n_1,\ldots,n_r)\in\mathbb{Z}_{\geq 0}^r} c_{n_1,\ldots,n_r} \prod_{j=1}^r \alpha_j^{n_j} \in \mathbb{R}[\alpha_1,\ldots,\alpha_r]$$

is $\mathsf{Newton}(f) := \mathsf{conv}\{(n_1,\ldots,n_r) : c_{n_1,\ldots,n_r} \neq 0\} \subseteq \mathbb{R}^r$.

Proposition 2.15. *Let $w \in \mathcal{W}$ and $w \leq v \leq v'$. Then $\mathsf{Newton}(\xi_w|_v) \subseteq \mathsf{Newton}(\xi_w|_{v'})$.*

Proof. This is immediate from Theorem 2.12. □

f has *saturated Newton polytope* (SNP) [33] if $c_{n_1,\ldots,n_r} \neq 0 \iff (n_1,\ldots,n_r) \in \mathsf{Newton}(f)$.

Conjecture 2.16. *Let $v, w \in \mathcal{W}$, then $\xi_w|_v$ has SNP.*

Conjecture 2.17 (A·B·C·D applied to Conjecture 2.16)**.** *Let $u, v, w \in \mathcal{W}$, then $C_{u,v}^w$ has SNP.*

We exhaustively checked these conjectures for A_4, B_3, D_4, G_2 and many examples in A_6 and B_4. A proof of either conjecture for Grassmannians would be interesting.

SNP is connected to computational complexity in [1, Section 1]. We suspect the concrete SNP claim of Conjecture 2.17 to be the combinatorial harbinger of the P assertion of Conjecture 2.11. Let `Schubert` be the decision problem "$(n_1,\ldots,n_r) \in \mathsf{Newton}(C_{u,v}^w)$?", given input $u, v, w \in \mathcal{W}$ and $(n_1,\ldots,n_r) \in \mathbb{Z}_{\geq 0}^r$. It is reasonable to conjecture existence of:

- a combinatorial rule for $C_{u,v}^w$ that moreover implies counting $C_{u,v}^w$ is a problem in the counting complexity class #P, and
- a halfspace description of $\mathsf{Newton}(C_{u,v}^w)$ where each *individual* inequality can be checked in polynomial time (even if there are exponentially many inequalities).

Conjecture 2.17 would then imply `Schubert` $\in$ NP $\cap$ coNP. *Often* problems in NP $\cap$ coNP are in fact in P (see [1, Section 1.2] for a discussion). `Schubert` $\in$ P implies the important case of Conjecture 2.11 for the non-equivariant $c_{u,v}^w$ to be true.

3. Comparisons to Schubert polynomial theory

The theory of *Schubert polynomials*, introduced by A. Lascoux and M.-P. Schützenberger [30], is influential in the conversation of positivity in Schubert calculus.

These polynomials "lift" the Schur polynomials from the ring of symmetric polynomials to the ring of all polynomials. The study of Schur polynomials is backed by an extensive literature on Young tableaux, from which one obtains the Littlewood–Richardson rule. Thus one might hope for an analogous theory for Schubert polynomials; this remains unrealized. For the purposes of our discussion, let us call this the "lifting dream".

Theorem 1.2 generalizes the identity

$$c_{u,v}^{w}(X) = 2^{s(w)-s(u)-s(v)} c_{u,v}^{w}(Y).$$

This seems to have been first stated in [7, (3.2)], who rely on the Schubert polynomials for classical groups of S. Billey and M. Haiman [9]. Similarly, Corollary 1.5 generalizes the equality

$$c_{\lambda,\mu}^{\nu}(X') = 2^{\ell(\nu)-\ell(\lambda)-\ell(\mu)} c_{\lambda,\mu}^{\nu}(Y'), \tag{14}$$

which is a consequence of P. Pragacz [34, Theorem 6.17] on the Schubert calculus interpretation of the Schur Q- and P-functions. Theorem 1.2 can also be deduced from T. Ikeda, L. Mihalcea, and H. Naruse's paper [19] who give an equivariant generalization of the polynomials of [9]. Specifically one refers to Section 1.6, and the identity $\mathfrak{B}_w = 2^{s(w)}\mathfrak{C}_w$, where $\mathfrak{B}_w$ and $\mathfrak{C}_w$ are their double Schubert polynomials for types B_n and C_n respectively. Note however that this identity does not prove Theorem 1.2 without the appropriate change of variables that we describe. Specifically, it is *not* true that $C_{u,v}^{w}(X) = 2^{s(w)-s(u)-s(v)} C_{u,v}^{w}(Y)$. This subtlety seems to not be explicitly addressed in [19].

Over the past three decades, within algebraic combinatorics, the emphasis has been on the Schubert polynomial rather than the list of many restrictions.[5] Our proof replaces the effort of the polynomial constructions [34, 9, 19] with the general geometric result (2). This work suggests the A·B·C·D as an alternative to the "lifting dream" and one that opens up some new and testable possibilities.

Is there concrete evidence for preferring one approach to the other? For example, can one give a proof using the A·B·C·D of S. Robinson's equivariant Pieri rule for GL_n/B [36]? Can one give a Schubert polynomial (in this case, factorial Schur polynomial) proof of one or more of the combinatorial rules [25, 28, 38] by giving an equivariant version of Schensted insertion? Based on earlier conversation of the third author with H. Thomas, this latter question seems quite nontrivial.

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[5]Not that the two viewpoints are unrelated: in type A for example, one can compute the restrictions as certain specializations of the double Schubert polynomials; see, e.g., [8].

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