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ON THE POSITIVITY CONJECTURE FOR FINITE ABELIAN p-GROUPS

C. P. ANIL KUMAR

ABSTRACT. For a partition $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k})$ and its associated finite $\mathcal{R}$-module $\mathcal{A}_{\boldsymbol{\lambda}} = \bigoplus_{i=1}^{k} (\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i}$, where $\mathcal{R}$ is a discrete valuation ring, with maximal ideal generated by a uniformizing element π, having finite residue field $\mathbf{k} = \mathcal{R}/\pi\mathcal{R} \cong \mathbb{F}_q$, with at least three elements, the number of orbits of pairs

$$n_{\boldsymbol{\lambda}}(q) = |\mathcal{G}_{\boldsymbol{\lambda}} \backslash (\mathcal{A}_{\boldsymbol{\lambda}} \times \mathcal{A}_{\boldsymbol{\lambda}})|$$

for the diagonal action of the automorphism group $\mathcal{G}_{\boldsymbol{\lambda}} = \mathrm{Aut}(\mathcal{A}_{\boldsymbol{\lambda}})$, is a polynomial in q with integer coefficients. The positivity conjecture states that these coefficients are in fact non-negative. In this article, we prove this conjecture.

1. Introduction

The automorphism orbits in finite Abelian groups have been understood quite well for over a hundred years (see [7, 3]). The combinatorics of automorphism orbits in Abelian groups is also studied by K. Dutta and A. Prasad [5] and by C. P. Anil Kumar [1, 2]. Some other authors who have worked on this subject are S. Delsarte [4], M. Schwachhöfer and M. Stroppel [8], B. L. Kerby and E. Rode [6]. In this article we focus on a conjecture for finite Abelian p-groups called the positivity conjecture about the total number of orbits of pairs. This conjecture is stated as Conjecture 1.1 below in the context of finite modules over discrete valuation rings where a finite Abelian p-group is naturally a finite module over the ring of p-adic integers. This was initially jointly conjectured by C. P. Anil Kumar and Amritanshu Prasad in [1].

Let Λ_0 denote the set of all sequences of the form

$$\boldsymbol{\lambda} = (\lambda_1^{\rho_1}, \lambda_2^{\rho_2}, \ldots, \lambda_k^{\rho_k}), \tag{1.1}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ is a strictly decreasing sequence of positive integers and $\rho_1, \rho_2, \ldots, \rho_k$ are positive integers. We allow the case where $k = 0$, resulting in the empty sequence, which we denote by $\emptyset$. Let $\mathcal{R}$ be a discrete valuation ring with maximal ideal generated by a uniformizing element π having finite

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residue field $\mathbf{k} = \mathcal{R}/\pi\mathcal{R}$. Every finite $\mathcal{R}$-module is, up to an isomorphism, of the form

$$\mathcal{A}_{\lambda} = \bigoplus_{i=1}^{k} \left(\mathcal{R}/\pi^{\lambda_i}\mathcal{R}\right)^{\rho_i} \tag{1.2}$$

for a unique $\boldsymbol{\lambda} \in \Lambda_0$.

Now we state the positivity conjecture which originally appeared in [1].

Conjecture 1.1. *Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ be a partition and $\mathcal{R}$ be a discrete valuation ring with maximal ideal generated by a uniformizing element π having finite residue field $\mathbf{k} = \mathcal{R}/\pi\mathcal{R} \cong \mathbb{F}_q$ with at least three elements. Let $\mathcal{A}_{\lambda} = \bigoplus_{i=1}^{k} \left(\mathcal{R}/\pi^{\lambda_i}\mathcal{R}\right)^{\rho_i}$ be its associated finite $\mathcal{R}$-module and $\mathcal{G}_{\lambda}$ be its automorphism group. Let $n_{\lambda}(q)$ be the number of orbits of pairs in $\mathcal{A}_{\lambda} \times \mathcal{A}_{\lambda}$ for the diagonal action of $\mathcal{G}_{\lambda}$ on $\mathcal{A}_{\lambda} \times \mathcal{A}_{\lambda}$. Then $n_{\lambda}(q)$ is a polynomial of degree λ_1 with non-negative integer coefficients.*

1.1. Results Known about the Positivity Conjecture.

We already know the following:

(1) The number $n_{\lambda}(q)$ is a polynomial in q with integer coefficients (cf. [1, Thm. 5.11]).
(2) The degree of $n_{\lambda}(q)$ is λ_1, the highest part of $\boldsymbol{\lambda}$ (cf. [1, Thm. 5.11]).
(3) The polynomial $n_{\lambda}(q)$ satisfies $n_{\lambda}(q) = n_{\lambda^2}(q)$, where

$$\boldsymbol{\lambda}^2 = (\lambda_1^{\min(\rho_1,2)} > \lambda_2^{\min(\rho_2,2)} > \cdots > \lambda_k^{\min(\rho_k,2)})$$

(cf. [1, Cor. 4.5]).

So, for the computation of the polynomial $n_{\lambda}(q)$, any part of the partition which repeats more than twice can be reduced to two. Here in this article we prove the positivity conjecture.

1.2. Strategy of the Proof of Conjecture 1.1.

We say an element $a \in \mathcal{A}_{\lambda}$ is of height zero if the equation $a = px$ has no solution for $x \in \mathcal{A}_{\lambda}$. The set of height zero elements in the group $\mathcal{A}_{\lambda}$ is invariant under the action of $\mathcal{G}_{\lambda}$. We show in Section 4 that, for a nonempty partition $\boldsymbol{\lambda}$,

$$n^0_{\lambda}(q) = \left|\mathcal{G}_{\lambda}\backslash\left(\{\text{Height Zero Elements}\} \times \mathcal{A}_{\lambda}\right)\right| \tag{1.3}$$

is also a polynomial in q with integer coefficients. The strategy of the proof of the conjecture is as follows. We establish "coupled" recurrence relations between the polynomials $n_{\lambda}(q)$ and $n^0_{\lambda}(q)$, expressing these in terms of the $n_{\mu}(q)$ and $n^0_{\mu}(q)$ for $\mu \subsetneq \lambda$ (containment of Young diagrams), with coefficients in $\mathbb{Z}_{\geq 0}[q]$. The initial conditions of $n^0_\lambda(q)$ for $\lambda = \emptyset$ being defined as 1 and for $\lambda = (1)$ it is q then establish the required positivity of $n_\lambda(q)$ by induction (Theorem 4.17).

For a partition $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ such that $\rho_{i_0} = 2$, that is, λ_{i_0} is repeated for some $1 \leq i_0 \leq 2$, using the lattice decomposition $\mathcal{J}(P_{\boldsymbol{\lambda}})$ given in Theorem 3.4 into two disjoint sublattices $\mathcal{J}(P_{\boldsymbol{\mu}})$ and $\mathcal{J}(P_{\boldsymbol{\nu}})$, we associate two particular smaller partitions $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. Then Theorem Δ for the repeated part case establishes the relation $n_{\boldsymbol{\lambda}}(q) = n_{\boldsymbol{\mu}}(q) + n_{\boldsymbol{\nu}}(q)$. Analogously, Theorem 4.5 establishes the following relation among the height zero polynomials: if $\lambda_{i_0} > 1$ then $n^0_{\boldsymbol{\lambda}}(q) = n^0_{\boldsymbol{\mu}}(q) + n^0_{\boldsymbol{\nu}}(q)$ and if $\lambda_{i_0} = 1$, that is, $i_0 = k, \lambda_k = 1$ then $n^0_{\boldsymbol{\lambda}}(q) = n^0_{\boldsymbol{\mu}}(q) + n_{\boldsymbol{\nu}}(q)$.

In Theorem 4.2, for a partition $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$, we establish $n_{\boldsymbol{\lambda}}(q) = n^0_{\boldsymbol{\lambda}}(q) + n_{\boldsymbol{\mu}}(q) + n^0_{\boldsymbol{\mu}}(q)$ if $\lambda_k > 1$, where $\boldsymbol{\mu} = \big((\lambda_1 - 1)^{\rho_1} > \cdots > (\lambda_k - 1)^{\rho_k}\big)$ and $n_{\boldsymbol{\lambda}}(q) = n^0_{\boldsymbol{\lambda}}(q) + 2n_{\boldsymbol{\mu}}(q)$ if $\lambda_k = 1$, where $\boldsymbol{\mu} = \big((\lambda_1 - 1)^{\rho_1} > \cdots > (\lambda_{k-1} - 1)^{\rho_{k-1}}\big)$. In particular if $\boldsymbol{\lambda}$ has distinct parts then $\boldsymbol{\mu}$ also has distinct parts in both cases.

In Theorem 4.9, we express $n^0_{\boldsymbol{\lambda}}(q)$ for a partition $\boldsymbol{\lambda}$ which has all its parts distinct as a polynomial combination of $n^0_{\boldsymbol{\mu}}(q)$ for partitions $\boldsymbol{\mu}$ which has all its parts distinct such that $\boldsymbol{\mu} \subsetneq \boldsymbol{\lambda}$ with coefficients of the combination being polynomials in q with nonnegative integer coefficients.

Combining the recurrences obtained in Theorems Δ, 4.2, 4.5, 4.9 we prove Conjecture 1.1 in Theorem 4.17.

Remark 1.2. The recurrences obtained in Theorems Δ, 4.2, 4.5, 4.9 allow a rapid computation of the polynomials $n_{\boldsymbol{\lambda}}(q), n^0_{\boldsymbol{\lambda}}(q)$ for $\boldsymbol{\lambda} \in \Lambda_0$.

2. Preliminaries

In this section we present some preliminaries about finite $\mathcal{R}$-modules.

2.1. **Orbits of Elements.** For the present purposes, the combinatorial description of orbits due to K. Dutta and A. Prasad [5] is useful. We recall the notation and some of the results in [5] here for the case of finite $\mathcal{R}$-modules.

It turns out in the case of finite $\mathcal{R}$-modules, that, for any module $\mathcal{A}_\lambda$, the $\mathcal{G}_\lambda$-orbits in $\mathcal{A}_\lambda$ are in bijective correspondence with certain classes of ideals in a poset P, which we call the fundamental poset. As a set

$$P = \{(v, l) \mid v \in \mathbb{N} \cup \{0\}, l \in \mathbb{N}, 0 \leq v < l\}. \tag{2.1}$$

The partial order on P is defined by setting

$$(v, l) \leq (v', l') \text{ if and only if } v \geq v' \text{ and } l - v \leq l' - v'.$$

Let $\mathcal{J}(P)$ denote the lattice of order ideals in P. A typical element of $\mathcal{A}_\lambda$ is a column vector of the form $x = (x_{\lambda_i, r_i})$, where i runs over the set $\{1, \ldots, k\}$, and, for each i, r_i runs over the set $\{1, \ldots, \rho_i\}$. To $x \in \mathcal{A}_\lambda$, we associate the order ideal $I(x) \in \mathcal{J}(P)$ generated by the elements $(v(x_{\lambda_i, r_i}), \lambda_i)$ for all pairs (i, r_i) such that $x_{\lambda_i, r_i} \neq 0$ in $\mathcal{R}/\pi^{\lambda_i}\mathcal{R}$. Here for any $x \in \mathcal{A}_\lambda, v(x)$ denotes the largest l for which $x \in \pi^l \mathcal{A}_\lambda$ (in particular, $v(0) = \infty$).

A key observation is the following theorem.

Theorem 2.1. *Let $\mathcal{A}_\lambda, \mathcal{A}_\mu$ be two finite $\mathcal{R}$-modules. An element $y \in \mathcal{A}_\mu$ is a homomorphic image of $x \in \mathcal{A}_\lambda$ if and only if $I(y) \subseteq I(x)$. Moreover for $x, y \in \mathcal{A}_\lambda$, they lie in the same $\mathcal{G}_\lambda$-orbit if and only if $I(x) = I(y)$.*

Note that the orbit of 0 corresponds to the empty ideal.

For $\lambda \in \Lambda_0$, let $\mathcal{J}(P_\lambda)$ denote the sublattice of $\mathcal{J}(P)$ consisting of ideals such that $\max I$ is contained in the set

$$P_\lambda = \{(v, l) \mid 0 \leq v < l, l = \lambda_i, 1 \leq i \leq k\}. \tag{2.2}$$

Then the $\mathcal{G}_\lambda$-orbits in $\mathcal{A}_\lambda$ are in bijective correspondence with this set $\mathcal{J}(P_\lambda)$ of order ideals. The lattice $\mathcal{J}(P_\lambda)$ is isomorphic to the lattice $\mathcal{J}(P_\lambda)$ of order ideals in the induced subposet P_λ. For each order ideal $I \in \mathcal{J}(P_\lambda)$, we use the notation

$$(\mathcal{A}_\lambda)_I^* = \{x \in \mathcal{A}_\lambda \mid I(x) = I\}$$

for the orbit corresponding to I.

A convenient way to think about the ideals in P is in terms of what we call boundaries: for a positive integer l define the boundary valuation of I at l to be

$$\partial_l I = \min\{v \mid \min(v, l) \in I\}.$$

We denote the sequence $\{\partial_l I\}$ of boundary valuations ∂I and call it the boundary of I.

For an order ideal $I \subset P$ let $\max I$ denote the set of maximal elements in I. The ideal I is completely determined by $\max I$: in fact, taking I to $\max I$ gives a bijection from the lattice $\mathcal{J}(P_\lambda)$ to the set of antichains in P_λ.

An alternative description for $(\mathcal{A}_\lambda)_I^*$ is

$$\begin{aligned}(\mathcal{A}_\lambda)_I^* = \{x = (x_{\lambda_i, r_i}) \in \mathcal{A}_\lambda \mid v(x_{\lambda_i, r_i}) \geq \partial_{\lambda_i} I \text{ for all } \lambda_i \text{ and } r_i \text{ and for} \\ (\partial_{\lambda_i} I, \lambda_i) \in \max I \text{ we must have } \max_{r_i}\{v(x_{\lambda_i, r_i}) \mid \text{ for all } r_i\} = \partial_{\lambda_i} I\}.\end{aligned}$$

For an ideal $I \in \mathcal{J}(P_\lambda)$ with

$$\max I = \{(v_1, l_1), \ldots, (v_s, l_s)\}$$

define an element e_I of $\mathcal{A}_\lambda$ a column vector whose co-ordinates are given by

$$x_{\lambda_i,r_i} = \begin{cases} \pi^{v_j}, & \text{if } \lambda_i = l_j \text{ and } r_j = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

For the $\mathcal{R}$-module $\mathcal{A}_\lambda$ the functions $x \longrightarrow I(x)$ and $I \longrightarrow e_I$ induce mutually inverse bijections between the set of $\mathcal{G}_\lambda$-orbits in $\mathcal{A}_\lambda$ and the set of order ideals in $\mathcal{J}(P_\lambda)$.

For an ideal $I \in \mathcal{J}(P_\lambda)$, define

$$(\mathcal{A}_\lambda)_I = \bigsqcup_{J \subseteq I} (\mathcal{A}_\lambda)^*_J.$$

This is a submodule of $\mathcal{A}_\lambda$ which is $\mathcal{G}_\lambda$-invariant. The description of $(\mathcal{A}_\lambda)_I$ in terms of valuations of co-ordinates and boundary valuations is as follows:

$$(\mathcal{A}_\lambda)_I = \{x = (x_{\lambda_i,r_i}) \mid v(x_{\lambda_i,r_i}) \geq \partial_{\lambda_i} I\}.$$

Such a module $(\mathcal{A}_\lambda)_I$ given by an ideal $I \in \mathcal{J}(P_\lambda)$ is called a characteristic submodule. The $\mathcal{G}_\lambda$ orbits in $\mathcal{A}_\lambda$ are parametrized by the finite distributive lattice $\mathcal{J}(P_\lambda)$. Moreover, each order ideal $I \in \mathcal{J}(P_\lambda)$ gives rise to a characteristic submodule $(\mathcal{A}_\lambda)_I$ of $\mathcal{A}_\lambda$. The lattice structure of $\mathcal{J}(P_\lambda)$ gets reflected in the poset structure of the characteristic submodules $(\mathcal{A}_\lambda)_I$ when they are partially ordered by inclusion. For ideals $I, J \in \mathcal{J}(P_\lambda)$, we have

$$(\mathcal{A}_\lambda)_{I\cup J} = (\mathcal{A}_\lambda)_I + (\mathcal{A}_\lambda)_J \text{ and } (\mathcal{A}_\lambda)_{I\cap J} = (\mathcal{A}_\lambda)_I \cap (\mathcal{A}_\lambda)_J.$$

The map $I \rightarrow (\mathcal{A}_\lambda)_I$ gives an isomorphism from $\mathcal{J}(P_\lambda)$ to the poset of $\mathcal{G}_\lambda$-invariant characteristic submodules. In fact, when the residue field $\mathbf{k}$ of $\mathcal{R}$ has at least three elements then every $\mathcal{G}_\lambda$-invariant submodule is a characteristic submodule. Therefore $\mathcal{J}(P_\lambda)$ is isomorphic to the lattice of $\mathcal{G}_\lambda$-invariant submodules (cf. [6]).

2.2. **Stabilizer of e_I.** By the description of $\mathcal{G}_\lambda$-orbits in $\mathcal{A}_\lambda$, every $\mathcal{G}_\lambda$-orbit of pairs of elements $(x_1, x_2) \in \mathcal{A}_\lambda \times \mathcal{A}_\lambda$ contains a pair of the form (e_I, x) for some $I \in \mathcal{J}(P_\lambda)$ and $x \in \mathcal{A}_\lambda$. Then the $\mathcal{G}_\lambda$-orbits of pairs in $\mathcal{A}_\lambda \times \mathcal{A}_\lambda$ which contain an element of the form (e_I, x) for some $x \in \mathcal{A}_\lambda$ are in bijective correspondence with $(\mathcal{G}_\lambda)_I$-orbits in $\mathcal{A}_\lambda$. Here we give a description of $(\mathcal{G}_\lambda)_I$ which facilitates the classification of $(\mathcal{G}_\lambda)_I$-orbits in $\mathcal{A}_\lambda$.

The main idea here is to decompose $\mathcal{A}_\lambda$ into a direct sum of two $\mathcal{R}$-modules (this decomposition depends on I):

$$\mathcal{A}_\lambda = \mathcal{A}_{\lambda'} \oplus \mathcal{A}_{\lambda''}, \tag{2.4}$$

where $\mathcal{A}_{\lambda'}$ consists of those cyclic summands in the decomposition 1.2 of $\mathcal{A}_\lambda$ where e_I has non-zero co-ordinates, and $\mathcal{A}_{\lambda''}$ consists of remaining cyclic summands. With respect to this decomposition we have

$$e_I = (e'_I, 0).$$

The reason for introducing this decomposition is that the description of stabilizer of e'_I in the automorphism group $\mathcal{G}_{\lambda'}$ of $\mathcal{A}_{\lambda'}$ is quite nice (see [1, Lemma 4.2]). The stabilizer of e'_I in $\mathcal{G}_{\lambda'}$ is

$$(\mathcal{G}_{\lambda'})_I = \{id_{\mathcal{A}_{\lambda'}} + u \mid u \in \mathrm{Hom}(\mathcal{A}_{\lambda'}, \mathcal{A}_{\lambda'}) \text{ satisfies } u(e'_I) = 0\}.$$

Every endomorphism of $\mathcal{A}_\lambda$ can be written as a matrix of the form $\left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{smallmatrix}\right)$ where $g_{11} : \mathcal{A}_{\lambda'} \to \mathcal{A}_{\lambda'}, g_{22} : \mathcal{A}_{\lambda''} \to \mathcal{A}_{\lambda''}, g_{12} : \mathcal{A}_{\lambda''} \to \mathcal{A}_{\lambda'}, g_{21} : \mathcal{A}_{\lambda'} \to \mathcal{A}_{\lambda''}$. The stabilizer of e_I in $\mathcal{G}_\lambda$ consists of matrices of the form

$$\begin{pmatrix} id_{\mathcal{A}_{\lambda'}} + u & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

where $u \in \mathrm{Hom}(\mathcal{A}_{\lambda'}, \mathcal{A}_{\lambda'})$ satisfies $u(e'_I) = 0, g_{12} \in \mathrm{Hom}(\mathcal{A}_{\lambda''}, \mathcal{A}_{\lambda'})$ is arbitrary, $g_{21} \in \mathrm{Hom}(\mathcal{A}_{\lambda'}, \mathcal{A}_{\lambda''})$ satisfies $g_{21}(e'_I) = 0$ and $g_{22} \in \mathcal{G}_{\lambda''} = \mathrm{Aut}(\mathcal{A}_{\lambda''})$ is invertible. For this see [1, Thm. 4.4].

2.3. **The Stabilizer Orbit of an Element.** Let $(\mathcal{G}_\lambda)_I$ denote the stabilizer of $e_I \in \mathcal{A}_\lambda$. Write each element $x \in \mathcal{A}_\lambda$ as $x = (x', x'')$ with respect to the decomposition 2.4 of $\mathcal{A}_\lambda$. Furthermore, for any $x' \in \mathcal{A}_{\lambda'}$ let $\overline{x}'$ denote the image of x' in $\mathcal{A}_{\lambda'/I} = \mathcal{A}_{\lambda'} \backslash \mathcal{R}e'_I$. The partition λ'/I is completely determined by the partition λ' and the ideal $I \cap P_{\lambda'} \in J(\mathcal{P})_{\lambda'}$ (see [1, Lemma 6.2]).

Now we describe the stabilizer $(\mathcal{G}_\lambda)_I$-orbit of $x \in \mathcal{A}_\lambda$.

Theorem 2.2. *Given $x, y \in \mathcal{A}_\lambda, y = (y', y'')$ lies in the $(\mathcal{G}_\lambda)_I$-orbit of $x \in \mathcal{A}_\lambda$ if and only if the following conditions hold:*

$$y' \in x' + (\mathcal{A}_{\lambda'})_{I(\overline{x}') \cup I(x'')} \tag{2.5}$$

and

$$y'' \in (\mathcal{A}_{\lambda''})^*_{I(x'')} + (\mathcal{A}_{\lambda''})_{I(\overline{x}')} \tag{2.6}$$

The size of the orbit is $\left|(\mathcal{A}_{\lambda'})_{I(\overline{x}') \cup I(x'')}\right| \left|(\mathcal{A}_{\lambda''})^*_{I(x'')} + (\mathcal{A}_{\lambda''})_{I(\overline{x}')}\right|$.

For the proof of this result see [1, Thm. 5.1].

2.4. **Overview of Section 3.** In this section we state and prove in Theorem Δ, the case of a repeated part and how to express $n_{\boldsymbol{\lambda}}(q) = n_{\boldsymbol{\mu}}(q) + n_{\boldsymbol{\nu}}(q)$ for suitable partitions $\boldsymbol{\mu}, \boldsymbol{\nu} \subseteq \boldsymbol{\lambda}$. The advantage of this theorem is that, by using this repeatedly, $n_{\boldsymbol{\lambda}}(q)$ can be expressible as the sum of polynomials $n_{\boldsymbol{\mu}}(q)$ corresponding to partitions $\boldsymbol{\mu} \subseteq \boldsymbol{\lambda}$ which have all its parts distinct. This is mentioned in Theorem 3.2. The important ingredient in proving Theorem Δ is the decomposition Theorem 3.4, where the lattice $\mathcal{J}(P_{\boldsymbol{\lambda}})$ with a part λ_{i_0} in $\boldsymbol{\lambda}$ is decomposed in two parts $\mathcal{L} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} = \emptyset\}$ and $\mathcal{M} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. Here $\mathcal{L}$ turns out to be isomorphic to $\mathcal{J}(P_{\boldsymbol{\mu}})$ and $\mathcal{M}$ turns out to be isomorphic to $\mathcal{J}(P_{\boldsymbol{\nu}})$ for the same suitable partitions $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. Now Theorem Δ follows by proving three lattice identities in Equations (3.1), (3.2), (3.3) given in Theorems 3.6, 3.7, 3.8, respectively. Theorems 3.6, 3.7 follow directly and their proofs are not long. The proof of Theorem 3.8 turns out to be a bit technical and long.

2.4.1. **Overview of Proof of Theorem 3.8.** First we illustrate Example 3.9 for $\boldsymbol{\lambda} = (\lambda_1 = 4^1 > \lambda_2^2 = \lambda_k^2 = 2^2)$ with repeated part $\lambda_{i_0} = \lambda_k = 2$ for $i_0 = k$ supporting the identity in Equation (3.3) given in Theorem 3.8. Then we prove required Propositions 3.10, 3.11, 3.12, 3.13, 3.16, 3.18, 3.19, 3.20 which are useful in the proof of Theorem 3.8 especially in establishing the identity in Equation (3.3).

Proposition 3.10 describes for an $I \in \mathcal{M}, I_1 = \psi(I) \in \mathcal{J}(P_{\boldsymbol{\nu}})$, where ψ is as described in Theorem 3.4(2), the partition $\boldsymbol{\lambda}'/I$ and the partition $\boldsymbol{\nu}'/I_1$ and the relation between them. It also describes some properties with respect to the parts $\lambda_{i_0} \pm 1$ which are adjacent to the repeated part λ_{i_0} in $\boldsymbol{\lambda}$. It is an observation that λ_{i_0} is never a part of $\boldsymbol{\lambda}'/I$.

Proposition 3.11 introduces a new partition $\boldsymbol{\lambda}'''$ which is obtained by inserting λ_{i_0} into the partition $\boldsymbol{\lambda}'/I$. Then it considers the subset $\mathcal{N} = \{\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''}) \mid \max(\tilde{J}) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$ and describes a lattice homomorphism $\phi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ by exactly describing $\max(\phi(\tilde{J}))$. This homomorphism ϕ called the lower lattice homomorphism.

Proposition 3.12 describes another lattice homomorphism called the upper lattice homomorphism $\xi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$. Actually this proposition introduces new partitions $\boldsymbol{\lambda}''''$ obtained by inserting both $\lambda_{i_0} \pm 1$ into $\boldsymbol{\lambda}'''$ (if either of them is not a part of $\boldsymbol{\lambda}'''$) and $\boldsymbol{\nu}'''$ obtained by inserting $\lambda_{i_0} - 1$ into $\boldsymbol{\nu}'/I_1$ (if it is not a part of $\boldsymbol{\nu}'/I_1$). Then the map ξ is described as a composition of certain maps in Proposition 3.12.

Proposition 3.13 makes it clear as to why the lattice homomorphisms ϕ, ξ are lower and upper lattice homomorphisms as in Remark 3.14. It proves that for any $\tilde{J} \in \mathcal{N}$, $\max(\phi(\tilde{J})) \subseteq \max(\xi(\tilde{J}))$.

Proposition 3.16 describes a lattice homomorphism $\zeta : \mathcal{J}(P_{\nu'/I_1}) \longrightarrow \mathcal{N}$ in the reverse direction which again is obtained as a composition of certain maps. As in Remark 3.17 if $\lambda_{i_0} - 1$ is a part of ν'/I_1, that is, $\nu'/I_1 = \nu'''$ then the maps $\xi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\nu'/I_1})$ and $\zeta : \mathcal{J}(P_{\nu'/I_1}) \longrightarrow \mathcal{N}$ are inverses of each other.

Proposition 3.18 explores the relationship between the following two ideals:

- The ideal $L \in \mathcal{M}$ which can be written as $L = [J \cup K]_{\lambda}$ for a $\tilde{J} \in \mathcal{N}$ with $\max(J) = \max(\tilde{J}) \backslash P_{(\lambda_{i_0})}$ and
$$K \in \mathcal{R} = \{T \in \mathcal{J}(P_{\lambda''}) \mid \max(T) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$$
with $[\tilde{J}]_{\lambda_{i_0}} = [K]_{\lambda_{i_0}}$.
- The ideal $\psi(L) = L_1 \in \mathcal{J}(P_{\nu})$.

For $K_1 = \psi(K) \in \mathcal{J}(P_{\nu''})$, where $\psi : \mathcal{R} \longrightarrow \mathcal{J}(P_{\nu''})$ similarly defined with respect to λ'' and ν'', for $J_1 = \phi(\tilde{J}), J_3 = \xi(\tilde{J})$, Proposition 3.18 proves that $[K_1]_{\nu} \cup [J_1]_{\nu} = [K_1]_{\nu} \cup [J_3]_{\nu} = L_1 = \psi(L) = \psi([J \cup K]_{\lambda})$ among many consequences. In this sense the map ψ is nicely behaved.

Proposition 3.19 is similar to previous proposition. But here we start with ideals $J_4 \in \mathcal{J}(P_{\nu'/I_1})$ and $K_2 \in \mathcal{J}(P_{\nu''})$ with $L_1 = [J_4]_{\nu} \cup [K_2]_{\nu} \in \mathcal{J}(P_{\nu})$. The proposition gives the unique ideal $L \in \mathcal{M}$ such that $\psi(L) = L_1$ and constructs ideal $K \in \mathcal{R}$ as $K = \psi^{-1}(K_2) \cup [\langle\{(u, \lambda_{i_0})\}\rangle]_{\lambda''}$ with $K_1 = \psi(K) \in \mathcal{J}(P_{\nu''})$. Now, if $J_5 = \zeta(J_4)$ with $\tilde{J} = J_5 \cup [\langle\{(u, \lambda_{i_0})\}\rangle]_{\lambda'''}$, then we actually find $\tilde{J} \in \mathcal{N}$ and if $J_1 = \phi(\tilde{J}), J_3 = \xi(\tilde{J})$ then the proposition proves $[K_1]_{\nu} \cup [J_1]_{\nu} = [K_1]_{\nu} \cup [J_3]_{\nu} = L_1 = [J_4]_{\nu} \cup [K_2]_{\nu}$ among many consequences.

Proposition 3.20 compares the cardinalities of $\frac{|\mathcal{A}_{\lambda'}|}{|\mathcal{A}_{\lambda'/I}|}$ and $\frac{|\mathcal{A}_{\nu'}|}{|\mathcal{A}_{\nu'/I_1}|}$ and also compares the cardinalities $|(\mathcal{A}_{\lambda})_L|$ and $|(\mathcal{A}_{\nu})_{L_1}|$.

Now all Propositions 3.10, 3.11, 3.12, 3.13, 3.16, 3.18, 3.19, 3.20 lead to a nice proof of the identity in Equation (3.3) given in Theorem 3.8. In order to prove Equation (3.3), the main identity we prove is Equation (3.21). In order to prove Equation (3.21) the key ingredient that we prove is Equation (3.16). Note that Equation (3.16) links the expressions involving partitions $\lambda, \lambda', \lambda'/I, \lambda'', \lambda'''$ and ideals $I, \tilde{J}, K$ in its left-hand side to expressions involving partitions $\nu, \nu', \nu'/I_1, \nu''$ and ideals $I_1, J_1, J_3, J_4, K_2, K_1$. Once we prove Equation (3.16) using the previous propositions, Equations (3.21), (3.3) follow.

2.5. **Overview of Section 4.** Section 4 introduces the height zero polynomial defined in Equation (1.3) and proves a number of "coupled" recurrences involving $n_{\lambda}(q)$ and $n^0_{\lambda}(q)$. These finally lead to the proof of positivity conjecture as outlined in Section 1.2. The recurrences are given in Theorems Δ, 4.2, 4.5, 4.9

which we have briefly described in Section 1.2. The details of Theorem 4.9 is not described which we describe now.

2.5.1. **Overview of the Proof of Theorem 4.9.** First we illustrate Example 4.10 supporting Theorem 4.9. Then we prove Theorems 4.11, 4.12, 4.13, 4.14, 4.15 to prove Theorem 4.9. All these theorems prove lattice identities with various conditions in order establish Theorem 4.9. These lattice identities present themselves when we are estimating the sum

$$n^0_{\lambda}(q) = \sum_{I \in \mathcal{J}^0(P_{\lambda})} \left(\sum_{J \in \mathcal{J}(P_{\lambda'/I}),\, K \in \mathcal{J}(P_{\lambda''})} \frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}} \right).$$

This sum splits into two parts as given in Equations (4.7), (4.16). Theorem 4.11 proves Equation (4.7). Equation (4.16) further splits into two sums further giving identities in Equation (4.17) and Equation (4.18). These identities are proved by proving Equations (4.8), (4.9), (4.10), (4.11) in Theorems 4.12, 4.13, 4.14, 4.15, respectively.

3. Statement and Proof of the Theorem in the case of Partitions with a Repeated Part

We state the theorem.

Theorem Δ (The Repeated Part Case). *Let*

$$\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$$

be such that $\rho_{i_0} = 2$ *for some* i_0 *with* $1 \leq i_0 \leq k$. *Let*

$$\boldsymbol{\mu} = \left(\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \cdots > \lambda_{i_0-1}^{\rho_{i_0-1}} > \lambda_{i_0}^{\rho_{i_0}-1} = \lambda_{i_0}^1 > \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\right).$$

Let

$$\boldsymbol{\nu} = \big((\lambda_1 - 2)^{\rho_1} > (\lambda_2 - 2)^{\rho_2} > \cdots > (\lambda_{i_0-1} - 2)^{\rho_{i_0-1}} \geq (\lambda_{i_0} - 1)^{\rho_{i_0}} \\ \geq \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\big).$$

Let

$$n_{\boldsymbol{\lambda}}(q) = \left|\mathcal{G}_{\boldsymbol{\lambda}} \backslash \mathcal{A}_{\boldsymbol{\lambda}} \times \mathcal{A}_{\boldsymbol{\lambda}}\right|, n_{\boldsymbol{\mu}}(q) = \left|\mathcal{G}_{\boldsymbol{\mu}} \backslash \mathcal{A}_{\boldsymbol{\mu}} \times \mathcal{A}_{\boldsymbol{\mu}}\right|, n_{\boldsymbol{\nu}}(q) = \left|\mathcal{G}_{\boldsymbol{\nu}} \backslash \mathcal{A}_{\boldsymbol{\nu}} \times \mathcal{A}_{\boldsymbol{\nu}}\right|.$$

Then we have

$$n_{\boldsymbol{\lambda}}(q) = n_{\boldsymbol{\mu}}(q) + n_{\boldsymbol{\nu}}(q).$$

Example 3.1.
- If $\boldsymbol{\lambda} = (\lambda_1^2) = (1^2)$ then $i_0 = 1, \boldsymbol{\mu} = (1), \boldsymbol{\nu} = \emptyset$.
- If $\boldsymbol{\lambda} = (\lambda_1 = 2 > \lambda_2^2 = 1^2)$ then $i_0 = 2, \boldsymbol{\mu} = (2 > 1), \boldsymbol{\nu} = \emptyset$.
- If $\boldsymbol{\lambda} = (\lambda_1^2 = 2^2 > \lambda_2 = 1)$ then $i_0 = 1, \boldsymbol{\mu} = (2, 1), \boldsymbol{\nu} = (1)$.

- If $\boldsymbol{\lambda} = (\lambda_1^2 = 2^2 > \lambda_2^2 = 1^2)$ and $i_0 = 1$ then $\boldsymbol{\mu} = (2 > 1^2), \boldsymbol{\nu} = (1^2)$.
- If $\boldsymbol{\lambda} = (\lambda_1^2 = 2^2 > \lambda_2^2 = 1^2)$ and $i_0 = 2$ then $\boldsymbol{\mu} = (2^2 > 1), \boldsymbol{\nu} = \emptyset$.
- If $\boldsymbol{\lambda} = (\lambda_1^2 = 4^2 > \lambda_2^2 = 3^2 > \lambda_3^2 = 2^2)$ and $i_0 = 2$ then $\boldsymbol{\mu} = (4^2 > 3 > 2^2), \boldsymbol{\nu} = (2^2 \geq 2^2 \geq 2^2) = (2^6)$. Alternatively, we can choose $\boldsymbol{\nu}^2 = (2^{\min(6,2)}) = (2^2)$ for the computation of the polynomial $n_{\boldsymbol{\nu}}(q) = n_{\boldsymbol{\nu}^2}(q)$, as any part of the partition which repeats more than twice can be reduced to two.

As a consequence of Theorem Δ we have the following theorem.

Theorem 3.2. *Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ be a partition. Let $n_{\boldsymbol{\lambda}}(q) = |\mathcal{G}_{\boldsymbol{\lambda}} \backslash \mathcal{A}_{\boldsymbol{\lambda}} \times \mathcal{A}_{\boldsymbol{\lambda}}|$. Then we have that $n_{\boldsymbol{\lambda}}(q)$ is expressible as the sum of polynomials $n_{\boldsymbol{\mu}}(q)$ corresponding to partitions $\boldsymbol{\mu}$ which have all its parts distinct. As a consequence if $n_{\boldsymbol{\mu}}(q)$ is a polynomial with nonnegative integer coefficients for any partition $\boldsymbol{\mu}$ which has all its parts distinct then $n_{\boldsymbol{\lambda}}(q)$ is a polynomial with nonnegative integer coefficients for any partition $\boldsymbol{\lambda} \in \Lambda_0$ in general.*

Proof. This theorem follows by applying repeatedly Theorem Δ and reducing the number of repetitions of any part in any partition, which occurs more than twice to two (cf. [1, Cor. 4.5]). ■

Example 3.3. Let us illustrate Theorem 3.2 with an example where $\boldsymbol{\lambda} = (3^2 > 2^2 > 1^2)$. First we have

- $n_{(3>2^2>1^2)}(q) = n_{(3>2>1^2)}(q) + n_{(1^2)}(q)$.
- $n_{(3>2>1^2)}(q) = n_{(3>2>1)}(q) + n_{(1)}(q)$.
- $n_{(2^2>1^2)}(q) = n_{(2>1^2)}(q) + n_{(1^2)}(q)$.
- $n_{(2>1^2)}(q) = n_{(2>1)}(q) + n_{\emptyset}(q)$.
- $n_{(1^2)}(q) = n_{(1)}(q) + n_{\emptyset}(q)$.

So

$$\begin{aligned}
n_{(3^2>2^2>1^2)}(q) &= n_{(3>2^2>1^2)}(q) + n_{(2^2>1^2)}(q) \\
&= n_{(3>2>1^2)}(q) + n_{(1^2)}(q) + n_{(2>1^2)}(q) + n_{(1^2)}(q) \\
&= n_{(3>2>1^2)}(q) + n_{(2>1^2)}(q) + 2n_{(1^2)}(q) \\
&= n_{(3>2>1)}(q) + n_{(1)}(q) + n_{(2>1)}(q) + n_{\emptyset}(q) + 2(n_{(1)}(q) + n_{\emptyset}(q)) \\
&= n_{(3>2>1)}(q) + n_{(2>1)}(q) + 3n_{(1)}(q) + 3n_{\emptyset}(q).
\end{aligned}$$

We prove another important theorem and state a few other required theorems before proving Theorem Δ.

Theorem 3.4. *Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ and λ_{i_0} be a part of $\boldsymbol{\lambda}$ for some $1 \leq i_0 \leq k$. Let $\boldsymbol{\mu} = \left(\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \cdots > \lambda_{i_0-1}^{\rho_{i_0-1}} > \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\right)$ and let*

$$\nu = \big((\lambda_1-2)^{\rho_1} > (\lambda_2-2)^{\rho_2} > \cdots > (\lambda_{i_0-1}-2)^{\rho_{i_0-1}} \geq (\lambda_{i_0}-1)^{\rho_{i_0}} \\ \geq \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\big).$$

(1) *Consider the subset $\mathcal{L} \subset \mathcal{J}(P_{\boldsymbol{\lambda}})$ of ideals given by*

$$\mathcal{L} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} = \emptyset\}.$$

Then $\mathcal{L}$ is a sublattice. It is isomorphic to the lattice $\mathcal{J}(P_{\boldsymbol{\mu}})$ with the lattice isomorphism being

$$I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \longrightarrow I \cap P_{\boldsymbol{\mu}} \in \mathcal{J}(P_{\boldsymbol{\mu}}).$$

(2) *Consider the subset $\mathcal{M} \subset \mathcal{J}(P_{\boldsymbol{\lambda}})$ of ideals given by*

$$\mathcal{M} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\} = \mathcal{J}(P_{\boldsymbol{\lambda}})\backslash\mathcal{L}.$$

Then $\mathcal{M}$ is a sublattice. It is isomorphic to the lattice $\mathcal{J}(P_{\boldsymbol{\nu}})$ with the lattice isomorphism being

$$\psi : I \in \mathcal{M} \longrightarrow J = \langle \max(J) \rangle \in \mathcal{J}(P_{\boldsymbol{\nu}}),$$

where $\max(J)$ *is given as follows. Let*

$$K = \{(v-1, \lambda-2) \mid (v,\lambda) \in \max(I), \lambda > \lambda_{i_0}\} \\ \cup \{(v,\lambda) \mid (v,\lambda) \in \max(I), \lambda < \lambda_{i_0}\} \subset P_{\boldsymbol{\nu}}.$$

(Note that $v-1 < \lambda-2$ for $(v,\lambda) \in \max(I), \lambda > \lambda_{i_0}$.) Let $\max(I) \cap P_{(\lambda_{i_0})} = \{(u, \lambda_{i_0})\}$. Then

$$\max(J) \\ = \begin{cases} K \cup \{(u, \lambda_{i_0}-1)\}, & \text{if } K \cup \{(u,\lambda_{i_0}-1)\} \text{ is an antichain,} \\ K, & \text{if } u = \lambda_{i_0}-1 \text{ or if } K \cup \{(u,\lambda_{i_0}-1)\} \text{ is not an antichain.} \end{cases}$$

Proof. Assertion (1) is clear and immediate. We prove assertion (2). We first show that K is an antichain. Let $(v,\lambda),(w,\mu) \in \max(I), \lambda > \lambda_{i_0}, \mu < \lambda_{i_0}$. We know that $(u,\lambda_{i_0}) \in \max(I)$. So we have $v > u > w, \lambda - v > \lambda_{i_0} - u > \mu - w$, which implies $v-1 > w, (\lambda-2)-(v-1) = \lambda - v - 1 > \mu - w$. Hence $(v-1, \lambda-2), (w,\mu)$ are not comparable with each other. Therefore K is an antichain.

Given an ideal $I \in \mathcal{M}$, we obtain $J = \psi(I)$ directly as follows:

$$J = \{(v-1,\lambda-2) \mid (v,\lambda) \in I, \lambda > \lambda_{i_0}, v+1 < \lambda\} \\ \cup \{(v, \lambda_{i_0}-1) \mid (v,\lambda_{i_0}) \in I, v \neq \lambda_{i_0}-1\} \\ \cup \{(v,\lambda) \mid (v,\lambda) \in I, \lambda < \lambda_{i_0}\}.$$

For this direct definition of $J = \psi(I)$, the set $\max(J)$ is precisely given as in the statement of the theorem. This is because of the following facts. Let

$\max([I]_{(\lambda_i)}) = \{(a_i, \lambda_i)\}$ if it is nonempty and define $a_i = \lambda_i$ if $\max([I]_{(\lambda_i)})$ is empty for $1 \leq i \leq k$. We have $a_{i_0} = u$ and it follows that since $(u, \lambda_{i_0}) \in \max(I)$, we have

$$\max([I]_{(\lambda_{i_0}+1)}) = \{(u+1, \lambda_{i_0}+1)\}, \max([I]_{(\lambda_{i_0}-1)}) = \{(u, \lambda_{i_0}-1)\}$$

and

$$a_{i_0-1} - 1 \geq a_{i_0} = u, \lambda_{i_0} - a_{i_0} - 1 \geq \lambda_{i_0+1} - a_{i_0+1}.$$

Using this and from the definition of J, we observe that

$$\max([J]_{(\lambda_i-2)}) = \{(a_i - 1, \lambda_i - 2)\}$$

for all $\lambda_i > \lambda_{i_0}$, $\max([J]_{(\lambda_i)}) = \{(a_i, \lambda_i)\}$ for all $\lambda_i < \lambda_{i_0}$ and

$$\max([J]_{(\lambda_{i_0}-1)}) = \{(a_{i_0}, \lambda_{i_0} - 1)\}.$$

Since I is an ideal in $\mathcal{J}(P_{\boldsymbol{\lambda}})$ we have

$$a_1 \geq a_2 \geq \cdots \geq a_{i_0-1} \geq a_{i_0} \geq a_{i_0+1} \geq \cdots \geq a_{k-1} \geq a_k,$$
$$\lambda_1 - a_1 \geq \cdots \geq \lambda_{i_0-1} - a_{i_0-1} \geq \lambda_{i_0} - a_{i_0} \geq \lambda_{i_0+1} - a_{i_0+1} \geq \cdots \geq \lambda_k - a_k.$$

Now we observe that

$$a_1 - 1 \geq a_2 - 1 \geq \cdots \geq a_{i_0-1} - 1 \geq a_{i_0} \geq a_{i_0+1} \geq \cdots \geq a_{k-1} \geq a_k,$$
$$\lambda_1 - a_1 - 1 \geq \cdots \geq \lambda_{i_0-1} - a_{i_0-1} - 1 \geq \lambda_{i_0} - a_{i_0} - 1$$
$$\geq \lambda_{i_0+1} - a_{i_0+1} \geq \cdots \geq \lambda_k - a_k.$$

So J is indeed an ideal in $\mathcal{J}(P_{\boldsymbol{\nu}})$. Now it is easy to check that $\max(J)$ is precisely given as in the statement of the theorem. Moreover we observe that $(a_{i_0}, \lambda_{i_0} - 1) = (u, \lambda_{i_0} - 1) \in \max(J)$ if and only if

$$a_{i_0-1} - 1 > a_{i_0} = u, \lambda_{i_0} - a_{i_0} - 1 > \lambda_{i_0+1} - a_{i_0+1}.$$

Now the bijection $\psi : \mathcal{M} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}})$ is clear as we can define J from I and define I from J. It is also clear that ψ is a lattice isomorphism. This proves the theorem. ∎

Example 3.5. We give three examples here to Theorem 3.4(2). Let $\boldsymbol{\lambda} = (\lambda_1^1 = 4^1 > \lambda_2^1 = \lambda_k^1 = 2^1)$. Then $\boldsymbol{\nu} = ((\lambda_1 - 2)^1 = 2^1 > (\lambda_k - 1)^1 = 1^1)$. Then the bijection is given as

$$\begin{aligned} \max(I), I \in \mathcal{J}(P_{\boldsymbol{\lambda}}), \max(I) \cap P_{(\lambda_k)} \neq \emptyset &\overset{\psi}{\longleftrightarrow} \max(J), J \in \mathcal{J}(P_{\boldsymbol{\nu}}) \\ \{(1,4),(0,2)\} &\longleftrightarrow \{(0,2)\} \\ \{(0,2)\} &\longleftrightarrow \{(0,1)\} \\ \{(2,4),(1,2)\} &\longleftrightarrow \{(1,2)\} \\ \{(1,2)\} &\longleftrightarrow \emptyset. \end{aligned}$$

Let $\boldsymbol{\lambda} = (\lambda_1^1 = 5^1 > \lambda_2^1 = \lambda_k^1 = 2^1)$. Then $\boldsymbol{\nu} = ((\lambda_1 - 2)^1 = 3^1 > (\lambda_k - 1)^1 = 1^1)$. Then the bijection is given as

$$\begin{aligned}
\max(I), I \in \mathcal{J}(P_{\boldsymbol{\lambda}}), \max(I) \cap P_{(\lambda_k)} \neq \emptyset &\overset{\psi}{\longleftrightarrow} \max(J), J \in \mathcal{J}(P_{\boldsymbol{\nu}}) \\
\{(1,5),(0,2)\} &\longleftrightarrow \{(0,3)\} \\
\{(2,5),(0,2)\} &\longleftrightarrow \{(1,3),(0,1)\} \\
\{(0,2)\} &\longleftrightarrow \{(0,1)\} \\
\{(2,5),(1,2)\} &\longleftrightarrow \{(1,3)\} \\
\{(3,5),(1,2)\} &\longleftrightarrow \{(2,3)\} \\
\{(1,2)\} &\longleftrightarrow \emptyset.
\end{aligned}$$

Let $\boldsymbol{\lambda} = (\lambda_1^1 = 3^1 > \lambda_2^1 = \lambda_k^1 = 2^1)$. Then $\boldsymbol{\nu} = ((\lambda_1 - 2)^1 = 1^1 \geq (\lambda_k - 1)^1 = 1^1) = (1^2)$. Then the bijection is given as

$$\begin{aligned}
\max(I), I \in \mathcal{J}(P_{\boldsymbol{\lambda}}), \max(I) \cap P_{(\lambda_k)} \neq \emptyset &\overset{\psi}{\longleftrightarrow} \max(J), J \in \mathcal{J}(P_{\boldsymbol{\nu}}) \\
\{(0,2)\} &\longleftrightarrow \{(0,1)\} \\
\{(1,2)\} &\longleftrightarrow \emptyset.
\end{aligned}$$

Now we introduce some notation in order to state three important lattice identities.

(1) Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ be such that $\rho_{i_0} = 2$ for some $1 \leq i_0 \leq k$.

(2) Let $\boldsymbol{\mu} = \left(\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \cdots > \lambda_{i_0-1}^{\rho_{i_0-1}} > \lambda_{i_0}^{\rho_{i_0}-1} = \lambda_{i_0}^1 > \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\right)$.

(3) Let $\boldsymbol{\nu} = \big((\lambda_1 - 2)^{\rho_1} > (\lambda_2 - 2)^{\rho_2} > \cdots > (\lambda_{i_0-1} - 2)^{\rho_{i_0-1}} \geq (\lambda_{i_0} - 1)^{\rho_{i_0}} \geq \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\big)$.

(4) For $I \in \mathcal{J}(P_{\boldsymbol{\lambda}}), J \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I}), K \in \mathcal{J}(P_{\boldsymbol{\lambda}''})$, let

$$X^{\boldsymbol{\lambda}}_{I,J,K} = \{(x', x'') \in \mathcal{A}_{\boldsymbol{\lambda}} \mid I(\overline{x}') = J, I(x'') = K\}$$

and

$$\alpha^{\boldsymbol{\lambda}}_{I,J,K} = \left|(\mathcal{A}_{\boldsymbol{\lambda}'})_{J \cup K} \oplus \left((\mathcal{A}_{\boldsymbol{\lambda}''})^*_K + (\mathcal{A}_{\boldsymbol{\lambda}''})_J\right)\right|.$$

(5) For $I_1 \in \mathcal{J}(P_{\boldsymbol{\mu}}), J_1 \in \mathcal{J}(P_{\boldsymbol{\mu}'/I_1}), K_1 \in \mathcal{J}(P_{\boldsymbol{\mu}''})$ let

$$X^{\boldsymbol{\mu}}_{I_1,J_1,K_1} = \{(x', x'') \in \mathcal{A}_{\boldsymbol{\mu}} \mid I(\overline{x}') = J_1, I(x'') = K_1\}$$

and

$$\alpha^{\boldsymbol{\mu}}_{I_1,J_1,K_1} = \left|(\mathcal{A}_{\boldsymbol{\mu}'})_{J_1 \cup K_1} \oplus \left((\mathcal{A}_{\boldsymbol{\mu}''})^*_{K_1} + (\mathcal{A}_{\boldsymbol{\mu}''})_{J_1}\right)\right|.$$

(6) Similarly, for $I_1 \in \mathcal{J}(P_\nu), J_1 \in \mathcal{J}(P_{\nu'/I_1}), K_1 \in \mathcal{J}(P_{\nu''})$, let
$$X^\nu_{I_1,J_1,K_1} = \{(x', x'') \in \mathcal{A}_\nu \mid I(\overline{x}') = J_1, I(x'') = K_1\}$$
and
$$\alpha^\nu_{I_1,J_1,K_1} = \left|(\mathcal{A}_{\nu'})_{J_1 \cup K_1} \oplus ((\mathcal{A}_{\nu''})^*_{K_1} + (\mathcal{A}_{\nu''})_{J_1})\right|.$$
(7) Depending on the context we will know to which of the two items (5) and (6) the ideals I_1, J_1, K_1 belong.

Now we partition the set $\mathcal{J}(P_\lambda)$ into two parts as: $\mathcal{J}(P_\lambda) = \mathcal{L} \sqcup \mathcal{M}$ according to Theorem 3.4, where
$$\mathcal{L} = \{I \in \mathcal{J}(P_\lambda), \max(I) \cap P_{(\lambda_{i_0})} = \emptyset\}$$
and
$$\mathcal{M} = \{I \in \mathcal{J}(P_\lambda), \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\}.$$
For $I \in \mathcal{M}$ we partition the set $\mathcal{J}(P_{\lambda'/I}) \times \mathcal{J}(P_{\lambda''})$ into two parts as
$$\mathcal{J}(P_{\lambda'/I}) \times \mathcal{J}(P_{\lambda''}) = \mathcal{M}^1_I \sqcup \mathcal{M}^2_I,$$
, where
$$\mathcal{M}^1_I = \Big\{(J,K) \in \mathcal{J}(P_{\lambda'/I}) \times \mathcal{J}(P_{\lambda''}) \mid \max([J]_{(\lambda_{i_0})}) \geq \max([K]_{(\lambda_{i_0})}) \text{ or }$$
$$\max([J]_{(\lambda_{i_0})}) < \max([K]_{(\lambda_{i_0})}), \max(K) \cap P_{(\lambda_{i_0})} = \emptyset\Big\}$$
and
$$\mathcal{M}^2_I = \Big\{(J,K) \in \mathcal{J}(P_{\lambda'/I}) \times \mathcal{J}(P_{\lambda''}) \mid \max([J]_{(\lambda_{i_0})}) < \max([K]_{(\lambda_{i_0})}) \text{ and }$$
$$\max(K) \cap P_{(\lambda_{i_0})} \neq \emptyset\Big\}.$$

Similarly we partition the set $\mathcal{J}(P_\mu)$ into two parts as: $\mathcal{J}(P_\mu) = \mathcal{S} \sqcup \mathcal{T}$, where
$$\mathcal{S} = \{I \in \mathcal{J}(P_\mu), \max(I) \cap P_{(\lambda_{i_0})} = \emptyset\}$$
and
$$\mathcal{T} = \{I \in \mathcal{J}(P_\mu), \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\}.$$

Now we state the three lattice identities in the following three theorems.

Theorem 3.6. *Using the notation as above we have*

(3.1)
$$\sum_{I \in \mathcal{L}} \left(\sum_{J \in \mathcal{J}(P_{\lambda'/I}), K \in \mathcal{J}(P_{\lambda''})} \frac{\left|X^\lambda_{I,J,K}\right|}{\alpha^\lambda_{I,J,K}} \right) = \sum_{I_1 \in \mathcal{S}} \left(\sum_{J_1 \in \mathcal{J}(P_{\mu'/I_1}), K_1 \in \mathcal{J}(P_{\mu''})} \frac{\left|X^\mu_{I_1,J_1,K_1}\right|}{\alpha^\mu_{I_1,J_1,K_1}} \right).$$

Theorem 3.7. *Using the notation as above we have*

$$\sum_{I\in\mathcal{M}}\left(\sum_{(J,K)\in\mathcal{M}^1_I}\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}\right)=\sum_{I_1\in\mathcal{T}}\left(\sum_{J_1\in\mathcal{J}(P_{\mu'/I_1}),K_1\in\mathcal{J}(P_{\mu''})}\frac{\left|X^{\mu}_{I_1,J_1,K_1}\right|}{\alpha^{\mu}_{I_1,J_1,K_1}}\right). \tag{3.2}$$

Theorem 3.8. *Using the notation as above we have*

$$\sum_{I\in\mathcal{M}}\left(\sum_{(J,K)\in\mathcal{M}^2_I}\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}\right)=\sum_{I_1\in\mathcal{J}(P_\nu)}\left(\sum_{J_1\in\mathcal{J}(P_{\nu'/I_1}),K_1\in\mathcal{J}(P_{\nu''})}\frac{\left|X^{\nu}_{I_1,J_1,K_1}\right|}{\alpha^{\nu}_{I_1,J_1,K_1}}\right) \tag{3.3}$$

We prove Theorem Δ first and then prove Theorems 3.6, 3.7, 3.8 subsequently.

Proof of Theorem Δ. First we observe that for an ideal $I\in\mathcal{J}(P_\lambda)=\mathcal{J}(P_\mu)$ we have the following data. The partitions λ',λ'', the canonical form $e_I=(e'_I,0)$ with $e'_I\in\mathcal{A}_{\lambda'}$, the partition λ'/I associated to $\mathcal{A}_{\lambda'/I}=\mathcal{A}_{\lambda'}/\mathcal{R}e'_I$, the stabilizer subgroup $(\mathcal{G}_\lambda)_I\subseteq\mathcal{G}_\lambda$ of the canonical form e_I.

Let $x=(x',x'')\in\mathcal{A}_{\lambda'}\oplus\mathcal{A}_{\lambda''}=\mathcal{A}_\lambda$. For $\overline{x}'\in\mathcal{A}_{\lambda'/I}$, let $I(\overline{x}')=J\in\mathcal{J}(P_{\lambda'/I})$ and $K=I(x'')\in\mathcal{J}(P_{\lambda''})$. Let $\alpha^{\lambda}_{I,J,K}$ be the size of the $(\mathcal{G}_\lambda)_I$ orbit of $x=(x',x'')$ which is the cardinality of the valuative set

$$(\mathcal{A}_{\lambda'})_{J\cup K}\oplus\left((\mathcal{A}_{\lambda''})^*_K+(\mathcal{A}_{\lambda''})_J\right)\subseteq\mathcal{A}_{\lambda'}\oplus\mathcal{A}_{\lambda''}=\mathcal{A}_\lambda.$$

Let $X^{\lambda}_{I,J,K}=\{(x',x'')\in\mathcal{A}_\lambda\mid I(\overline{x}')=J,I(x'')=K\}$. Then the number of $(\mathcal{G}_\lambda)_I$ orbits in $\mathcal{A}_\lambda$ is a polynomial in q with integer coefficients and is given by

$$n_{\lambda,I}(q)=|(\mathcal{G}_\lambda)_I\backslash\mathcal{A}_\lambda|=\sum_{J\in\mathcal{J}(P_{\lambda'/I})}\sum_{K\in\mathcal{J}(P_{\lambda''})}\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}.$$

We also have

$$n_\lambda(q)=\sum_{I\in\mathcal{J}(P_\lambda)}\left(\sum_{J\in\mathcal{J}(P_{\lambda'/I}),K\in\mathcal{J}(P_{\lambda''})}\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}\right).$$

Similarly the number of $(\mathcal{G}_\mu)_I$ orbits in $\mathcal{A}_\mu$ is a polynomial in q with integer coefficients and is given by

$$n_{\mu,I}(q)=\left|(\mathcal{G}_\mu)_I\backslash\mathcal{A}_\mu\right|=\sum_{J\in\mathcal{J}(P_{\mu'/I})}\sum_{K\in\mathcal{J}(P_{\mu''})}\frac{\left|X^{\mu}_{I,J,K}\right|}{\alpha^{\mu}_{I,J,K}}.$$

We also have

$$n_\mu(q)=\sum_{I\in\mathcal{J}(P_\mu)}\left(\sum_{J\in\mathcal{J}(P_{\mu'/I}),K\in\mathcal{J}(P_{\mu''})}\frac{\left|X^{\mu}_{I,J,K}\right|}{\alpha^{\mu}_{I,J,K}}\right).$$

For an ideal $I \in \mathcal{J}(P_\nu)$, the number of $(\mathcal{G}_\nu)_I$ orbits in $\mathcal{A}_\nu$ is a polynomial in q with integer coefficients and is given by

$$n_{\nu,I}(q) = |(\mathcal{G}_\nu)_I \backslash \mathcal{A}_\nu| = \sum_{J \in \mathcal{J}(P_{\nu'/I})} \sum_{K \in \mathcal{J}(P_{\nu''})} \frac{\left|X^{\nu}_{I,J,K}\right|}{\alpha^{\nu}_{I,J,K}}.$$

We also have

$$n_\nu(q) = \sum_{I \in \mathcal{J}(P_\nu)} \left(\sum_{J \in \mathcal{J}(P_{\nu'/I}), K \in \mathcal{J}(P_{\nu''})} \frac{\left|X^{\nu}_{I,J,K}\right|}{\alpha^{\nu}_{I,J,K}} \right).$$

Theorem Δ follows if we establish the identity in Equation (3.1) and the following identity:

(3.4)

$$\sum_{I \in \mathcal{M}} \left(\sum_{J \in \mathcal{J}(P_{\lambda'/I}), K \in \mathcal{J}(P_{\lambda''})} \frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}} \right)$$

$$= \sum_{I \in \mathcal{T}} \left(\sum_{J \in \mathcal{J}(P_{\mu'/I}), K \in \mathcal{J}(P_{\mu''})} \frac{\left|X^{\mu}_{I,J,K}\right|}{\alpha^{\mu}_{I,J,K}} \right) + \sum_{I \in \mathcal{J}(P_\nu)} \left(\sum_{J \in \mathcal{J}(P_{\nu'/I}), K \in \mathcal{J}(P_{\nu''})} \frac{\left|X^{\nu}_{I,J,K}\right|}{\alpha^{\nu}_{I,J,K}} \right).$$

Theorem 3.6 proves the identity in Equation (3.1). The identity in Equation (3.4) is obtained by summing the identities in Equations (3.2) and (3.3). Theorem 3.7 proves the identity in Equation (3.2) and Theorem 3.8 proves the identity in Equation (3.3). Hence Theorem Δ follows. ■

3.1. Proofs of Theorems 3.6 and 3.7.

3.1.1. Proof of Theorem 3.6.

Proof. We consider the case of ideals $I \in \mathcal{J}(P_\lambda) = \mathcal{J}(P_\mu)$ such that $\max(I) \cap P_{(\lambda_{i_0})} = \emptyset$. For such an ideal I, we have $\boldsymbol{\lambda}' = \boldsymbol{\mu}', \boldsymbol{\lambda}''$ differs from $\boldsymbol{\mu}''$ by an extra part λ_{i_0}, that is, the isotypic part of $\boldsymbol{\lambda}''$ corresponding to λ_{i_0} is $\lambda_{i_0}^{\rho_{i_0}} = \lambda_{i_0}^2$, whereas the isotypic part of $\boldsymbol{\mu}''$ corresponding to λ_{i_0} is $\lambda_{i_0}^{\rho_{i_0}-1} = \lambda_{i_0}^1$. We have $\mathcal{J}(P_{\lambda''}) = \mathcal{J}(P_{\mu''})$. We also have $\boldsymbol{\lambda}'/I = \boldsymbol{\mu}'/I, \mathcal{A}_{\lambda'/I} = \mathcal{A}_{\lambda'}/\mathcal{R}e'_I = \mathcal{A}_\mu/\mathcal{R}e'_I = \mathcal{A}_{\mu'/I}$.

Let $J \in \mathcal{J}(P_{\lambda'/I}) = \mathcal{J}(P_{\mu'/I})$ and $K \in \mathcal{J}(P_{\lambda''}) = \mathcal{J}(P_{\mu''})$ be such that

$$\max([J]_{(\lambda_{i_0})}) \geq \max([K]_{(\lambda_{i_0})})$$

that is, either $\max([K]_{(\lambda_{i_0})}) = \emptyset$ or if $\max([K]_{(\lambda_{i_0})}) = \{(u, \lambda_{i_0})\}$ and $\max([J]_{(\lambda_{i_0})}) = \{(v, \lambda_{i_0})\}$ then $v \leq u$. Choose $u = \lambda_{i_0}$ if $\max([K]_{(\lambda_{i_0})}) = \emptyset$ and $v = \lambda_{i_0}$ if $\max([J]_{(\lambda_{i_0})}) = \max([K]_{(\lambda_{i_0})}) = \emptyset$. Then we have

$$(\mathcal{A}_{\lambda'})_{J\cup K} = (\mathcal{A}_{\mu'})_{J\cup K}, \left|(\mathcal{A}_{\lambda''})^*_K + (\mathcal{A}_{\lambda''})_J\right| = q^{\lambda_{i_0}-v}\left|(\mathcal{A}_{\mu''})^*_K + (\mathcal{A}_{\mu''})_J\right|.$$

Hence

$$\alpha^{\lambda}_{I,J,K} = \left(q^{\lambda_{i_0}-v}\right)\alpha^{\mu}_{I,J,K}.$$

Now, if $\max([J]_{(\lambda_{i_0})}) = \max([K]_{(\lambda_{i_0})})$, that is, $v = u$ and if $\max(K)\cap P_{(\lambda_{i_0})} = \emptyset$, that is, $(u,\lambda_{i_0}) \notin \max(K)$ then the isotypic component of $(\mathcal{A}_{\lambda''})^*_K$ corresponding to λ_{i_0} is $\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})^{\rho_{i_0}}$ and the isotypic component of $(\mathcal{A}_{\mu''})^*_K$ corresponding to λ_{i_0} is $\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})^{\rho_{i_0}-1}$. Hence we have

$$\left|X^{\lambda}_{I,J,K}\right| = \left(q^{\lambda_{i_0}-u}\right)\left|X^{\mu}_{I,J,K}\right| = \left(q^{\lambda_{i_0}-v}\right)\left|X^{\mu}_{I,J,K}\right|.$$

In this case, we therefore have

$$\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}} = \frac{\left|X^{\mu}_{I,J,K}\right|}{\alpha^{\mu}_{I,J,K}}.$$

If $\max([J]_{(\lambda_{i_0})}) = \max([K]_{(\lambda_{i_0})})$, that is, $v = u$ and if $\max(K)\cap P_{(\lambda_{i_0})} \neq \emptyset$, that is, $(u,\lambda_{i_0}) \in \max(K)$ then the isotypic component of $(\mathcal{A}_{\lambda''})^*_K$ corresponding to λ_{i_0} is $\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})^{\rho_{i_0}} - \pi^{u+1}(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})^{\rho_{i_0}}$ and the isotypic component of $(\mathcal{A}_{\mu''})^*_K$ corresponding to λ_{i_0} is $\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})^{\rho_{i_0}-1} - \pi^{u+1}(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})^{\rho_{i_0}-1}$. Let $K_1 \in \mathcal{J}(P_{\lambda''})$ be such that

$$\max(K_1) = \max(K)\backslash\{(u,\lambda_{i_0})\}.$$

Here, we consider ideals $L \in \mathcal{J}(P_{\lambda''})$ such that

$$K_1 \subseteq L \subseteq K.$$

Now $(\mathcal{A}_{\lambda''})^*_{K_1} \subseteq (\mathcal{A}_{\lambda''})^*_L \subseteq (\mathcal{A}_{\lambda''})^*_K$ and $\max(K) = \max(K_1)\sqcup\{(u,\lambda_{i_0})\}$ implies that $\max(K_1) \subseteq \max(L)$.

Conversely, given any $L \in \mathcal{J}(P_{\lambda''})$ such that $\max([J]_{(\lambda_{i_0})}) > \max([L]_{(\lambda_{i_0})})$, there is exactly one $K \in \mathcal{J}(P_{\lambda''})$ such that

$$\{(v,\lambda_{i_0})\} = \max([J]_{(\lambda_{i_0})}) = \max([K]_{(\lambda_{i_0})}) = \{(u,\lambda_{i_0})\},$$

and $K_1 \subseteq L \subsetneq K$ where $K_1 \in \mathcal{J}(P_{\lambda''})$ is such that $\max(K_1) = \max(K)\backslash\{(u,\lambda_{i_0})\}$. The ideals K and K_1 are obtained from L in a unique manner as follows. The set $\max(K_1)$ is obtained from $\max(L)$ by excluding those elements from $\max(L)$ which are comparable with (u,λ_{i_0}). The set $\max(K) = \max(K_1)\sqcup\{(u,\lambda_{i_0})\}$ is an antichain and the ideal K is generated by this antichain. Clearly we have $K_1 \subseteq L \subsetneq K$.

Now, for any such L with $K_1 \subseteq L \subseteq K$ we have $J \cup L = J \cup K$ in the big fundamental poset P as defined in 2.1, because $(u, \lambda_{i_0}) \in J, \max(K_1) \subseteq \max(K) \cap \max(L)$. Hence

$$(\mathcal{A}_{\lambda'})_{J\cup L} = (\mathcal{A}_{\lambda'})_{J\cup K}.$$

We also have in this scenario that, $\max(K)\backslash[J]_{\lambda''} = \max(L)\backslash[J]_{\lambda''}$ implying

$$\left|(\mathcal{A}_{\lambda''})^*_L + (\mathcal{A}_{\lambda''})_J\right| = \left|(\mathcal{A}_{\lambda''})^*_K + (\mathcal{A}_{\lambda''})_J\right|.$$

Therefore

$$\alpha^{\lambda}_{I,J,L} = \alpha^{\lambda}_{I,J,K} \text{ for all } K_1 \subseteq L \subseteq K.$$

Similarly we have

$$(\mathcal{A}_{\mu'})_{J\cup L} = (\mathcal{A}_{\mu'})_{J\cup K}, \left|(\mathcal{A}_{\mu''})^*_L + (\mathcal{A}_{\mu''})_J\right| = \left|(\mathcal{A}_{\mu''})^*_K + (\mathcal{A}_{\mu''})_J\right|.$$

Hence

$$\alpha^{\mu}_{I,J,L} = \alpha^{\mu}_{I,J,K} \text{ for all } K_1 \subseteq L \subseteq K.$$

Moreover we have

$$\alpha^{\lambda}_{I,J,L} = \alpha^{\lambda}_{I,J,K} = \left(q^{\lambda_{i_0}-u}\right)\alpha^{\mu}_{I,J,K} = \left(q^{\lambda_{i_0}-u}\right)\alpha^{\mu}_{I,J,L}. \tag{3.5}$$

Now we observe that

$$\bigsqcup_{K_1\subseteq L\subseteq K} (\mathcal{A}_{\lambda''})^*_L = \text{Product of two sets } = T_1 \times T_2$$

and

$$\bigsqcup_{K_1\subseteq L\subseteq K} (\mathcal{A}_{\mu''})^*_L = \text{Product of two sets } = S_1 \times S_2,$$

where $S_1 = T_1$ and T_2, S_2 are products of sets of the form $\pi^j(\mathcal{R}/\pi^{\lambda}\mathcal{R})^{\rho}$ such that the isotypic component of T_2 corresponding to λ_{i_0} is $\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})^{\rho_{i_0}}$, the isotypic component of S_2 corresponding to λ_{i_0} is $\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})^{\rho_{i_0}-1}$ and the remaining isotypic components of T_2 matches with the remaining isotypic components of S_2. Hence we have $|T_2| = \left(q^{\lambda_{i_0}-u}\right)|S_2|$.

So

$$\left|\bigsqcup_{K_1\subseteq L\subseteq K} X^{\lambda}_{I,J,L}\right| = \left(q^{\lambda_{i_0}-u}\right)\left|\bigsqcup_{K_1\subseteq L\subseteq K} X^{\mu}_{I,J,L}\right|. \tag{3.6}$$

So we obtain by combining Equations (3.5) and (3.6),

$$\sum_{K_1\subseteq L\subseteq K} \frac{\left|X^{\lambda}_{I,J,L}\right|}{\alpha^{\lambda}_{I,J,L}} = \frac{\left|\bigsqcup\limits_{K_1\subseteq L\subseteq K} X^{\lambda}_{I,J,L}\right|}{\alpha^{\lambda}_{I,J,K}} = \frac{\left|\bigsqcup\limits_{K_1\subseteq L\subseteq K} X^{\mu}_{I,J,L}\right|}{\alpha^{\mu}_{I,J,K}} = \sum_{K_1\subseteq L\subseteq K} \frac{\left|X^{\mu}_{I,J,L}\right|}{\alpha^{\mu}_{I,J,L}}.$$

Now we consider the case $J \in \mathcal{J}(P_{\lambda'/I}) = \mathcal{J}(P_{\mu'/I})$ and $K \in \mathcal{J}(P_{\lambda''}) = \mathcal{J}(P_{\mu''})$ such that

$$\max([J]_{(\lambda_{i_0})}) < \max([K]_{(\lambda_{i_0})}),$$

that is, if $\max([J]_{(\lambda_{i_0})}) = \{(v, \lambda_{i_0})\}$ and $\max([K]_{(\lambda_{i_0})}) = \{(u, \lambda_{i_0})\}$ then $v > u$ or $\max([J]_{(\lambda_{i_0})}) = \emptyset$ and $\max([K]_{(\lambda_{i_0})}) = \{(u, \lambda_{i_0})\} \neq \emptyset$. Here we have

$$(\mathcal{A}_{\lambda'})_{J\cup K} = (\mathcal{A}_{\mu'})_{J\cup K}$$

and if $(u, \lambda_{i_0}) \notin \max(K)$ then

$$\left|(\mathcal{A}_{\lambda''})^*_K + (\mathcal{A}_{\lambda''})_J\right| = q^{\lambda_{i_0}-u}\left|(\mathcal{A}_{\mu''})^*_K + (\mathcal{A}_{\mu''})_J\right|, \left|(\mathcal{A}_{\lambda''})^*_K\right| = q^{\lambda_{i_0}-u}\left|(\mathcal{A}_{\mu''})^*_K\right|.$$

So we get

$$\alpha^{\lambda}_{I,J,K} = \left(q^{\lambda_{i_0}-u}\right)\alpha^{\mu}_{I,J,K}, \left|X^{\lambda}_{I,J,K}\right| = \left(q^{\lambda_{i_0}-u}\right)\left|X^{\mu}_{I,J,K}\right|.$$

Therefore

$$\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}} = \frac{\left|X^{\mu}_{I,J,K}\right|}{\alpha^{\mu}_{I,J,K}}.$$

Now, if $(u, \lambda_{i_0}) \in \max(K)$, then

$$\left|(\mathcal{A}_{\lambda''})^*_K + (\mathcal{A}_{\lambda''})_J\right| = \frac{\left(q^{(\lambda_{i_0}-u)\rho_{i_0}} - q^{(\lambda_{i_0}-u-1)\rho_{i_0}}\right)}{\left(q^{(\lambda_{i_0}-u)(\rho_{i_0}-1)} - q^{(\lambda_{i_0}-u-1)(\rho_{i_0}-1)}\right)}\left|(\mathcal{A}_{\mu''})^*_K + (\mathcal{A}_{\mu''})_J\right|,$$

$$\left|(\mathcal{A}_{\lambda''})^*_K\right| = \frac{\left(q^{(\lambda_{i_0}-u)\rho_{i_0}} - q^{(\lambda_{i_0}-u-1)\rho_{i_0}}\right)}{\left(q^{(\lambda_{i_0}-u)(\rho_{i_0}-1)} - q^{(\lambda_{i_0}-u-1)(\rho_{i_0}-1)}\right)}\left|(\mathcal{A}_{\mu''})^*_K\right|.$$

So we get

$$\alpha^{\lambda}_{I,J,K} = \frac{\left(q^{(\lambda_{i_0}-u)\rho_{i_0}} - q^{(\lambda_{i_0}-u-1)\rho_{i_0}}\right)}{\left(q^{(\lambda_{i_0}-u)(\rho_{i_0}-1)} - q^{(\lambda_{i_0}-u-1)(\rho_{i_0}-1)}\right)}\alpha^{\mu}_{I,J,K},$$

$$\left|X^{\lambda}_{I,J,K}\right| = \frac{\left(q^{(\lambda_{i_0}-u)\rho_{i_0}} - q^{(\lambda_{i_0}-u-1)\rho_{i_0}}\right)}{\left(q^{(\lambda_{i_0}-u)(\rho_{i_0}-1)} - q^{(\lambda_{i_0}-u-1)(\rho_{i_0}-1)}\right)}\left|X^{\mu}_{I,J,K}\right|.$$

Therefore again

$$\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}} = \frac{\left|X^{\mu}_{I,J,K}\right|}{\alpha^{\mu}_{I,J,K}}.$$

Hence we have, for a fixed $I \in \mathcal{J}(P_{\lambda}) = \mathcal{J}(P_{\mu})$ such that $\max(I) \cap P_{(\lambda_{i_0})} = \emptyset$,

$$\left(\sum_{J\in\mathcal{J}(P_{\lambda'/I}),K\in\mathcal{J}(P_{\lambda''})}\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}\right) = \left(\sum_{J\in\mathcal{J}(P_{\mu'/I}),K\in\mathcal{J}(P_{\mu''})}\frac{\left|X^{\mu}_{I,J,K}\right|}{\alpha^{\mu}_{I,J,K}}\right).$$

Upon summation over $I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) = \mathcal{J}(P_{\boldsymbol{\mu}})$ such that $\max(I) \cap P_{(\lambda_{i_0})} = \emptyset$, the identity in Equation (3.1) follows. ∎

3.1.2. **Proof of Theorem 3.7.**

Proof. Now we consider the case when $\max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset, I \in \mathcal{J}(P_{\boldsymbol{\lambda}})$. We have to prove the identity in Equation (3.2).

For an ideal $I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) = \mathcal{J}(P_{\boldsymbol{\mu}}), \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset$ we have $\boldsymbol{\lambda}' = \boldsymbol{\mu}'$ with the isotypic part corresponding to λ_{i_0} being $\lambda_{i_0}^1$. We also have $\boldsymbol{\lambda}''$ differs from $\boldsymbol{\mu}''$ by a part λ_{i_0}, that is, the isotypic part of $\boldsymbol{\lambda}''$ corresponding to λ_{i_0} being $\lambda_{i_0}^{\rho_{i_0}-1} = \lambda_{i_0}^1$ and there is no isotypic part of $\boldsymbol{\mu}''$ corresponding to λ_{i_0}. We also have $\boldsymbol{\lambda}'/I = \boldsymbol{\mu}'/I$ and $\mathcal{A}_{\boldsymbol{\mu}'/I} = \mathcal{A}_{\boldsymbol{\mu}'}/\mathcal{R}e'_I = \mathcal{A}_{\boldsymbol{\lambda}'}/\mathcal{R}e'_I = \mathcal{A}_{\boldsymbol{\lambda}'/I}$.

First we observe that ideals $K \in \mathcal{J}(P_{\boldsymbol{\lambda}''})$ such that $\max(K) \cap P_{(\lambda_{i_0})} = \emptyset$ are in bijection with ideals $L \in \mathcal{J}(P_{\boldsymbol{\mu}''})$ using Theorem 3.4(1). The bijection being

$$K \longrightarrow L = K \cap P_{\boldsymbol{\mu}''} \text{ with } \max(K) = \max(L).$$

Let $J \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I}) = \mathcal{J}(P_{\boldsymbol{\mu}'/I}), K \in \mathcal{J}(P_{\boldsymbol{\lambda}''})$. We define two ideals now, that arise from K. Let $K_1 \in \mathcal{J}(P_{\boldsymbol{\lambda}''})$ be such that $\max(K_1) = \max(K) \backslash P_{(\lambda_{i_0})} \subset P_{\boldsymbol{\mu}''}$. Let $K_2 = K \cap P_{\boldsymbol{\mu}''} \in \mathcal{J}(P_{\boldsymbol{\mu}''})$. If $\max(K) \cap P_{(\lambda_{i_0})} = \emptyset$ then $\max(K_1) = \max(K_2) = \max(K)$ and $K_1 = K, K_1 \cap P_{\boldsymbol{\mu}''} = K_2$. If $\max([K]_{(\lambda_{i_0})}) = \{(u, \lambda_{i_0})\}$ and $\max(K) \cap P_{(\lambda_{i_0})} \neq \emptyset$ then we have $\max(K) = \max(K_1) \sqcup \{(u, \lambda_{i_0})\}$ and $\max(K_1) \subseteq \max(K_2)$. In general $\max(K_1)$ need not be equal to $\max(K_2)$.

Suppose

$$\emptyset \neq \{(v, \lambda_{i_0})\} = \max([J]_{(\lambda_{i_0})}) \geq \max([K]_{(\lambda_{i_0})}) = \{(u, \lambda_{i_0})\} \neq \emptyset,$$

that is, $v \leq u$. Choose $v = \lambda_{i_0}$ if $\max([J]_{(\lambda_{i_0})}) = \emptyset$ and $u = \lambda_{i_0}$ if $\max([K]_{(\lambda_{i_0})}) = \emptyset$. Then $v \leq u$ continues to hold. Let $K' \in \mathcal{J}(P_{\boldsymbol{\lambda}''}), L' \in \mathcal{J}(P_{\boldsymbol{\mu}''})$ be such that $K_1 \subseteq K' \subseteq K$ and $K_1 \cap P_{\boldsymbol{\mu}''} = [K_1]_{\boldsymbol{\mu}''} \subseteq L' \subseteq K_2$. Then we observe that the elements in the sets $\max(K') \backslash \max(K_1), \max(L') \backslash \max(K_1)$ are comparable and less than or equal to (u, λ_{i_0}) if the sets are nonempty. Hence these elements are less than or equal to (v, λ_{i_0}). As a consequence we have $\max(K') \backslash \max(K_1) \subseteq [J]_{\boldsymbol{\lambda}''}, \max(L') \backslash \max(K_1) \subseteq [J]_{\boldsymbol{\mu}''}$ implying $\max(K') \backslash [J]_{\boldsymbol{\lambda}''} = \max(K_1) \backslash [J]_{\boldsymbol{\lambda}''} = \max(K) \backslash [J]_{\boldsymbol{\lambda}''}$ and $\max(L') \backslash [J]_{\boldsymbol{\mu}''} = \max(K_1) \backslash [J]_{\boldsymbol{\mu}''} = \max(K_2) \backslash [J]_{\boldsymbol{\lambda}''}$. So

$$(\mathcal{A}_{\boldsymbol{\lambda}''})^*_{K'} + (\mathcal{A}_{\boldsymbol{\lambda}''})_J = (\mathcal{A}_{\boldsymbol{\lambda}''})^*_K + (\mathcal{A}_{\boldsymbol{\lambda}''})_J = (\mathcal{A}_{\boldsymbol{\lambda}''})^*_{K_1} + (\mathcal{A}_{\boldsymbol{\lambda}''})_J,$$
$$(\mathcal{A}_{\boldsymbol{\mu}''})^*_{L'} + (\mathcal{A}_{\boldsymbol{\mu}''})_J = (\mathcal{A}_{\boldsymbol{\mu}''})^*_{K_2} + (\mathcal{A}_{\boldsymbol{\mu}''})_J = (\mathcal{A}_{\boldsymbol{\mu}''})^*_{K_1} + (\mathcal{A}_{\boldsymbol{\mu}''})_J.$$

We also have

$$\begin{aligned} \left|(\mathcal{A}_{\boldsymbol{\lambda}''})^*_{K'} + (\mathcal{A}_{\boldsymbol{\lambda}''})_J\right| &= \left|(\mathcal{A}_{\boldsymbol{\lambda}''})^*_K + (\mathcal{A}_{\boldsymbol{\lambda}''})_J\right| \\ &= q^{\lambda_{i_0}-v}\left|(\mathcal{A}_{\boldsymbol{\mu}''})^*_{K_2} + (\mathcal{A}_{\boldsymbol{\mu}''})_J\right| = q^{\lambda_{i_0}-v}\left|(\mathcal{A}_{\boldsymbol{\mu}''})^*_{L'} + (\mathcal{A}_{\boldsymbol{\mu}''})_J\right|. \end{aligned}$$

We observe that $J\cup K, J\cup K_1, J\cup K_2, J\cup K', J\cup L'$ all generate the same ideal in the big fundamental poset P defined in Equation (2.1). Hence we have

$$(\mathcal{A}_{\lambda'})_{J\cup K'}=(\mathcal{A}_{\lambda'})_{J\cup K}=(\mathcal{A}_{\lambda'})_{J\cup K_1}=(\mathcal{A}_{\mu'})_{J\cup K_1}=(\mathcal{A}_{\mu'})_{J\cup K_2}=(\mathcal{A}_{\mu'})_{J\cup L'}.$$

So

(3.7)
$$\alpha^{\lambda}_{I,J,K'}=\alpha^{\lambda}_{I,J,K}=\alpha^{\lambda}_{I,J,K_1}=\left(q^{\lambda_{i_0}-v}\right)\alpha^{\mu}_{I,J,K_1}=\left(q^{\lambda_{i_0}-v}\right)\alpha^{\mu}_{I,J,K_2}=\left(q^{\lambda_{i_0}-v}\right)\alpha^{\mu}_{I,J,L'}.$$

Now, if $\max([J]_{(\lambda_{i_0})})=\max([K]_{(\lambda_{i_0})})$, that is, $v=u$ and if $\max(K)\cap P_{(\lambda_{i_0})}=\emptyset$, then the isotypic component of $(\mathcal{A}_{\lambda''})^*_K$ corresponding to λ_{i_0} is $\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})$ and $(\mathcal{A}_{\lambda''})^*_K=(\mathcal{A}_{\mu''})^*_{K_2}\oplus\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})$. Hence we have

$$\left|X^{\lambda}_{I,J,K}\right|=\left(q^{\lambda_{i_0}-u}\right)\left|X^{\mu}_{I,J,K_2}\right|=\left(q^{\lambda_{i_0}-v}\right)\left|X^{\mu}_{I,J,K_2}\right|.$$

In this case, we therefore have from Equation (3.7)

$$\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}=\frac{\left|X^{\mu}_{I,J,K_2}\right|}{\alpha^{\mu}_{I,J,K_2}}.$$

Now, if $\max([J]_{(\lambda_{i_0})})=\max([K]_{(\lambda_{i_0})})$, that is, $v=u$ and if $\max(K)\cap P_{(\lambda_{i_0})}\neq\emptyset$, that is, $(u,\lambda_{i_0})\in\max(K)$ then we consider ideals $K'\in\mathcal{J}(P_{\lambda''})$ such that

$$K_1\subseteq K'\subseteq K.$$

Note that

$$K_1\subseteq K'\subseteq K, K'\neq K \text{ if and only if } K_1\subseteq K'\subseteq K, \max([J]_{(\lambda_{i_0})})>\max([K']_{(\lambda_{i_0})}).$$

We also consider ideals $L'\in\mathcal{J}(P_{\mu''})$ such that

$$K_1\cap P_{\mu''}=[K_1]_{\mu''}\subseteq L'\subseteq K_2=[K]_{\mu''}=K\cap P_{\mu''}.$$

Note that for such ideals we have $\max([J]_{(\lambda_{i_0})})>\max([L']_{(\lambda_{i_0})})$.

Given any $K'\in\mathcal{J}(P_{\lambda''})$ such that $\max([J]_{(\lambda_{i_0})})>\max([K']_{(\lambda_{i_0})})$, there is exactly one $K\in\mathcal{J}(P_{\lambda''})$ such that $\{(v,\lambda_{i_0})\}=\max([J]_{(\lambda_{i_0})})=\max([K]_{(\lambda_{i_0})})=\{(u,\lambda_{i_0})\}$, and $K_1\subseteq K'\subsetneq K$ where $K_1\in\mathcal{J}(P_{\lambda''})$ is such that $\max(K_1)=\max(K)\backslash\{(u,\lambda_{i_0})\}$. The ideals K and K_1 are obtained from K' in a unique manner as follows. The set $\max(K_1)$ is obtained from $\max(K')$ by excluding those elements from $\max(K')$ which are comparable with (u,λ_{i_0}). The set $\max(K)=\max(K_1)\sqcup\{(u,\lambda_{i_0})\}$ is an antichain and the ideal K is generated by this antichain. Clearly we have $K_1\subseteq K'\subsetneq K$ and K,K_1 are uniquely determined by K'.

Similarly given any $L'\in\mathcal{J}(P_{\mu''})$ such that $\max([J]_{(\lambda_{i_0})})>\max([L']_{(\lambda_{i_0})})$, there is exactly one $K\in\mathcal{J}(P_{\lambda''})$ such that $\{(v,\lambda_{i_0})\}=\max([J]_{(\lambda_{i_0})})=\max([K]_{(\lambda_{i_0})})=$

$\{(u,\lambda_{i_0})\}$ and $[K_1]_{\mu''} \subseteq L' \subseteq K_2$ where $K_1 \in \mathcal{J}(P_{\lambda''})$ is such that $\max(K_1) = \max(K)\backslash\{(u,\lambda_{i_0})\}$ and $K_2 = K \cap P_{\mu''}$. The ideals K and K_1 are obtained from L' as follows. The set $\max(K_1)$ is obtained from $\max(L')$ by excluding those elements from $\max(L')$ which are comparable with (u,λ_{i_0}). The set $\max(K) = \max(K_1) \sqcup \{(u,\lambda_{i_0})\}$ is an antichain and the ideal K is generated by this antichain. Clearly we have $[K_1]_{\mu''} \subseteq L' \subseteq K_2 = K \cap P_{\mu''}$ and K, K_1, K_2 are uniquely determined by L'.

Now we observe that

$$\bigsqcup_{K_1 \subseteq K' \subseteq K} (\mathcal{A}_{\lambda''})^*_{K'} = \text{Product of two sets } = T_1 \times T_2$$

and

$$\bigsqcup_{[K_1]_{\mu''} \subseteq L' \subseteq K_2} (\mathcal{A}_{\mu''})^*_{L'} = \text{Product of two sets } = S_1 \times S_2,$$

where $S_1 = T_1$ and T_2, S_2 are products of sets of the form $\pi^j(\mathcal{R}/\pi^{\lambda}\mathcal{R})^{\rho}$ such that the isotypic component of T_2 corresponding to λ_{i_0} is $\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})$ and the other isotypic components of T_2 matches with the isotypic components of S_2. Hence we have $|T_2| = \left(q^{\lambda_{i_0}-u}\right)|S_2| = \left(q^{\lambda_{i_0}-v}\right)|S_2|$.

So

$$\left| \bigsqcup_{K_1 \subseteq K' \subseteq K} X^{\lambda}_{I,J,K'} \right| = \left(q^{\lambda_{i_0}-v}\right) \left| \bigsqcup_{[K_1]_{\mu''} \subseteq L' \subseteq K_2} X^{\mu}_{I,J,L'} \right|. \tag{3.8}$$

So we obtain by combining Equations (3.7) and (3.8),

$$\sum_{K_1 \subseteq K' \subseteq K} \frac{\left|X^{\lambda}_{I,J,K'}\right|}{\alpha^{\lambda}_{I,J,K'}} = \frac{\left|\bigsqcup_{K_1 \subseteq K' \subseteq K} X^{\lambda}_{I,J,K'}\right|}{\alpha^{\lambda}_{I,J,K}} = \frac{\left|\bigsqcup_{[K_1]_{\mu''} \subseteq L' \subseteq K_2} X^{\mu}_{I,J,L'}\right|}{\alpha^{\mu}_{I,J,K_2}} = \sum_{[K_1]_{\mu''} \subseteq L' \subseteq K_2} \frac{\left|X^{\mu}_{I,J,L'}\right|}{\alpha^{\mu}_{I,J,L'}}.$$

Suppose

$$\emptyset \neq \{(v,\lambda_{i_0})\} = \max([J]_{(\lambda_{i_0})}) < \max([K]_{(\lambda_{i_0})}) = \{(u,\lambda_{i_0})\}, \text{ that is, } v > u.$$

Choose $v = \lambda_{i_0}$ if $\max([J]_{(\lambda_{i_0})}) = \emptyset$. Furthermore, let $K \in \mathcal{J}(P_{\lambda''})$ be such that $\max(K) \cap P_{(\lambda_{i_0})} = \emptyset, L = K \cap P_{\mu''} \in \mathcal{J}(P_{\mu''})$. Then we have $\max([J]_{(\lambda_{i_0})}) < \max([L]_{(\lambda_{i_0})}) = \{(u,\lambda_{i_0})\}$. Moreover, the sets $J \cup K, J \cup L$ generate the same ideal in the big fundamental poset P defined in Equation (2.1). Hence we have

$$(\mathcal{A}_{\lambda'})_{J\cup K} = (\mathcal{A}_{\mu'})_{J\cup L}.$$

Since $\max(K) \cap P_{(\lambda_{i_0})} = \emptyset$, that is, $(u, \lambda_{i_0}) \notin \max(K)$ we have $(\mathcal{A}_{\boldsymbol{\lambda}''})^*_K = (\mathcal{A}_{\boldsymbol{\mu}''})^*_L \oplus \pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})$. So $|(\mathcal{A}_{\boldsymbol{\lambda}''})^*_K| = \left(q^{\lambda_{i_0}-u}\right)\left|(\mathcal{A}_{\boldsymbol{\mu}''})^*_L\right|$. Hence

$$\left|X^{\boldsymbol{\lambda}}_{I,J,K}\right| = q^{\lambda_{i_0}-u}\left|X^{\boldsymbol{\mu}}_{I,J,L}\right|.$$

Then

$$\left|((\mathcal{A}_{\boldsymbol{\lambda}''})^*_K + (\mathcal{A}_{\boldsymbol{\lambda}''})_J)\right| = q^{\lambda_{i_0}-u}\left|((\mathcal{A}_{\boldsymbol{\mu}''})^*_L + (\mathcal{A}_{\boldsymbol{\mu}''})_J)\right|,$$

which implies

$$\alpha^{\boldsymbol{\lambda}}_{I,J,K} = q^{\lambda_{i_0}-u}\alpha^{\boldsymbol{\mu}}_{I,J,L}.$$

Therefore

$$\frac{\left|X^{\boldsymbol{\lambda}}_{I,J,K}\right|}{\alpha^{\boldsymbol{\lambda}}_{I,J,K}} = \frac{\left|X^{\boldsymbol{\mu}}_{I,J,L}\right|}{\alpha^{\boldsymbol{\mu}}_{I,J,L}}.$$

So we have proved the identity in Equation (3.2). Hence Theorem 3.7 follows. ∎

3.2. **Proof of Theorem 3.8.** The proof of this theorem is technical and long. So we discuss an example in the following subsection and prove some required propositions in the next subsection before proving the theorem.

3.2.1. **An Example.** Here we discuss an example which supports Equation (3.3) in Theorem 3.8.

Example 3.9. Let $\boldsymbol{\lambda} = (\lambda_1 = 4^1 > \lambda_2^2 = \lambda_k^2 = 2^2), \boldsymbol{\mu} = (4^1 > 2^1), \boldsymbol{\nu} = ((\lambda_1 - 2)^1 = 2^1 > (\lambda_k - 1)^2 = 1^2)$. We have $n_{\boldsymbol{\lambda}}(q) = q^4 + 5q^3 + 13q^2 + 16q + 10$, $n_{\boldsymbol{\mu}}(q) = q^4 + 5q^3 + 12q^2 + 11q + 4$ and $n_{\boldsymbol{\nu}}(q) = q^2 + 5q + 6$. We have $\{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_k)} \neq \emptyset\} = \{I^1 = \langle\{(1,4),(0,2)\}\rangle > I^2 = \langle\{(0,2)\}\rangle > I^3 = \langle\{(2,4),(1,2)\}\rangle > I^4 = \langle\{(1,2)\}\rangle\}$. We also have $\mathcal{J}(P_{\boldsymbol{\nu}}) = \{I_1 = \langle\{(0,2)\}\rangle > I_2 = \langle\{(0,1)\}\rangle > I_3 = \langle\{(1,2)\}\rangle > I_4 = \emptyset\}$. Then map $I^i \longrightarrow I_i$ from the set $\{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_k)} \neq \emptyset\}$ to $\mathcal{J}(P_{\boldsymbol{\nu}})$ is an isomorphism.

Ideal	$J \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I})$ $\max(J)$	$K \in \mathcal{J}(P_{\boldsymbol{\lambda}''})$ $\max(K) \cap P_{(\lambda_k)} \neq \emptyset$	$\max([J]_{(\lambda_k)})$ $< \max([K]_{(\lambda_k)})$	$\alpha^{\boldsymbol{\lambda}}_{I,J,K}$	$\left\lvert X^{\boldsymbol{\lambda}}_{I,J,K}\right\rvert$	Ratio $\frac{\lvert X^{\boldsymbol{\lambda}}_{I,J,K}\rvert}{\alpha^{\boldsymbol{\lambda}}_{I,J,K}}$
$I^1, \boldsymbol{\lambda}' = (4 > 2), \boldsymbol{\lambda}'' = (2), \boldsymbol{\lambda}'/I^1 = (3)$	$\{(1,3)\}$	$\{(0,2)\}$	$(1,2) < (0,2)$	$q^5(q-1)$	$q^5(q-1)^2$	$(q-1)$
	$\{(2,3)\}$	$\{(0,2)\}$	$\emptyset < (0,2)$	$q^5(q-1)$	$q^4(q-1)^2$	$\frac{q-1}{q}$
	$\emptyset$	$\{(0,2)\}$	$\emptyset < (0,2)$	$q^5(q-1)$	$q^4(q-1)$	$\frac{1}{q}$
	$\{(2,3)\}$	$\{(1,2)\}$	$\emptyset < (1,2)$	$q^2(q-1)$	$q^3(q-1)^2$	$q(q-1)$
	$\emptyset$	$\{(1,2)\}$	$\emptyset < (1,2)$	$q^2(q-1)$	$q^3(q-1)$	q

Ideal	$J \in \mathcal{J}(P_{\nu'/I})$ $\max(J)$	$K \in \mathcal{J}(P_{\nu''})$ $\max(K)$	$\alpha^{\nu}_{I,J,K}$	$\lvert X^{\nu}_{I,J,K} \rvert$	Ratio $\frac{\lvert X^{\nu}_{I,J,K} \rvert}{\alpha^{\nu}_{I,J,K}}$
$I_1, \nu' = (2), \nu'' = (1,1), \nu'/I_1 = \emptyset$	$\emptyset$	$\{(0,1)\}$	$q(q^2-1)$	$q^2(q^2-1)$	q
	$\emptyset$	$\emptyset$	1	q^2	q^2

Ideal	$J \in \mathcal{J}(P_{\lambda'/I})$ $\max(J)$	$K \in \mathcal{J}(P_{\lambda''})$ $\max(K) \cap P_{(\lambda_k)} \neq \emptyset$	$\max([J]_{(\lambda_k)})$ $< \max([K]_{(\lambda_k)})$	$\alpha^{\lambda}_{I,J,K}$	$\lvert X^{\lambda}_{I,J,K} \rvert$	Ratio $\frac{\lvert X^{\lambda}_{I,J,K} \rvert}{\alpha^{\lambda}_{I,J,K}}$
$I^2, \lambda' = (2), \lambda'' = (4 > 2), \lambda'/I^2 = \emptyset$	$\emptyset$	$\{(1,4),(0,2)\}$	$\emptyset < (0,2)$	$q^5(q-1)^2$	$q^5(q-1)^2$	1
	$\emptyset$	$\{(0,2)\}$	$\emptyset < (0,2)$	$q^5(q-1)$	$q^5(q-1)$	1
	$\emptyset$	$\{(2,4),(1,2)\}$	$\emptyset < (1,2)$	$q^2(q-1)^2$	$q^3(q-1)^2$	q
	$\emptyset$	$\{(1,2)\}$	$\emptyset < (1,2)$	$q^2(q-1)$	$q^3(q-1)$	q

Ideal	$J \in \mathcal{J}(P_{\nu'/I})$ $\max(J)$	$K \in \mathcal{J}(P_{\nu''})$ $\max(K)$	$\alpha^{\nu}_{I,J,K}$	$\lvert X^{\nu}_{I,J,K} \rvert$	Ratio $\frac{\lvert X^{\nu}_{I,J,K} \rvert}{\alpha^{\nu}_{I,J,K}}$
$I_2, \nu' = (1), \nu'' = (2,1), \nu'/I_2 = \emptyset$	$\emptyset$	$\{(0,2)\}$	$q^3(q-1)$	$q^3(q-1)$	1
	$\emptyset$	$\{(0,1)\}$	$q^2(q-1)$	$q^2(q-1)$	1
	$\emptyset$	$\{(1,2)\}$	$(q-1)$	$q(q-1)$	q
	$\emptyset$	$\emptyset$	1	q	q

Ideal	$J \in \mathcal{J}(P_{\lambda'/I})$ $\max(J)$	$K \in \mathcal{J}(P_{\lambda''})$ $\max(K) \cap P_{(\lambda_k)} \neq \emptyset$	$\max([J]_{(\lambda_k)})$ $< \max([K]_{(\lambda_k)})$	$\alpha^{\lambda}_{I,J,K}$	$\lvert X^{\lambda}_{I,J,K} \rvert$	Ratio $\frac{\lvert X^{\lambda}_{I,J,K} \rvert}{\alpha^{\lambda}_{I,J,K}}$
$I^3, \lambda' = (4 > 2), \lambda'' = (2), \lambda'/I^3 = (3 > 1)$	$\{(1,3),(0,1)\}$	$\{(0,2)\}$	$(1,2) < (0,2)$	$q^5(q-1)$	$q^4(q-1)^3$	$\frac{(q-1)^2}{q}$
	$\{(0,1)\}$	$\{(0,2)\}$	$(1,2) < (0,2)$	$q^5(q-1)$	$q^4(q-1)^2$	$\frac{q-1}{q}$
	$\{(1,3)\}$	$\{(0,2)\}$	$(1,2) < (0,2)$	$q^5(q-1)$	$q^4(q-1)^2$	$\frac{q-1}{q}$
	$\{(2,3)\}$	$\{(0,2)\}$	$\emptyset < (0,2)$	$q^5(q-1)$	$q^3(q-1)^2$	$\frac{q-1}{q^2}$
	$\emptyset$	$\{(0,2)\}$	$\emptyset < (0,2)$	$q^5(q-1)$	$q^3(q-1)$	$\frac{1}{q^2}$
	$(2,3)$	$\{(1,2)\}$	$\emptyset < (1,2)$	$q^2(q-1)$	$q^2(q-1)^2$	$q-1$
	$\emptyset$	$\{(1,2)\}$	$\emptyset < (1,2)$	$q^2(q-1)$	$q^2(q-1)$	1

Ideal	$J \in \mathcal{J}(P_{\nu'/I})$ $\max(J)$	$K \in \mathcal{J}(P_{\nu''})$ $\max(K)$	$\alpha^{\nu}_{I,J,K}$	$\lvert X^{\nu}_{I,J,K} \rvert$	Ratio $\frac{\lvert X^{\nu}_{I,J,K} \rvert}{\alpha^{\nu}_{I,J,K}}$
$I_3, \nu' = (2), \nu'' = (1,1), \nu'/I_3 = (1)$	$\{(0,1)\}$	$\{(0,1)\}$	q^3	$q(q-1)^2(q+1)$	$q - 1 - \frac{1}{q} + \frac{1}{q^2}$
	$\emptyset$	$\{(0,1)\}$	$q(q-1)(q+1)$	$q(q-1)(q+1)$	1
	$\{(0,1)\}$	$\emptyset$	q^3	$q(q-1)$	$\frac{(q-1)}{q^2}$
	$\emptyset$	$\emptyset$	1	q	q

Ideal	$J \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I})$ $\max(J)$	$K \in \mathcal{J}(P_{\boldsymbol{\lambda}''})$ $\max(K) \cap P_{(\lambda_k)} \neq \emptyset$	$\max([J]_{(\lambda_k)})$ $< \max([K]_{(\lambda_k)})$	$\alpha^{\boldsymbol{\lambda}}_{I,J,K}$	$\lvert X^{\boldsymbol{\lambda}}_{I,J,K}\rvert$	Ratio $\frac{\lvert X^{\boldsymbol{\lambda}}_{I,J,K}\rvert}{\alpha^{\boldsymbol{\lambda}}_{I,J,K}}$
$I^4, \boldsymbol{\lambda}' = (2), \boldsymbol{\lambda}'' = (4 > 2), \boldsymbol{\lambda}'/I^4 = (1)$	$\{(0,1)\}$	$\{(1,4),(0,2)\}$	$(1,2) < (0,2)$	$q^5(q-1)^2$	$q^4(q-1)^3$	$\frac{q-1}{q}$
	$\emptyset$	$\{(1,4),(0,2)\}$	$\emptyset < (0,2)$	$q^5(q-1)^2$	$q^4(q-1)^2$	$\frac{1}{q}$
	$\{(0,1)\}$	$\{(0,2)\}$	$(1,2) < (0,2)$	$q^5(q-1)$	$q^4(q-1)^2$	$\frac{q-1}{q}$
	$\emptyset$	$\{(0,2)\}$	$\emptyset < (0,2)$	$q^5(q-1)$	$q^4(q-1)$	$\frac{1}{q}$
	$\emptyset$	$\{(2,4),(1,2)\}$	$\emptyset < (1,2)$	$q^2(q-1)^2$	$q^2(q-1)^2$	1
	$\emptyset$	$\{(1,2)\}$	$\emptyset < (1,2)$	$q^2(q-1)$	$q^2(q-1)$	1

Ideal	$J \in \mathcal{J}(P_{\boldsymbol{\nu}'/I})$ $\max(J)$	$K \in \mathcal{J}(P_{\boldsymbol{\nu}''})$ $\max(K)$	$\alpha^{\boldsymbol{\nu}}_{I,J,K}$	$\lvert X^{\boldsymbol{\nu}}_{I,J,K}\rvert$	Ratio $\frac{\lvert X^{\boldsymbol{\nu}}_{I,J,K}\rvert}{\alpha^{\boldsymbol{\nu}}_{I,J,K}}$
$I_4, \boldsymbol{\nu}' = \emptyset, \boldsymbol{\nu}'' = (2 > 1^2), \boldsymbol{\nu}'/I_4 = \emptyset$	$\emptyset$	$\{(0,2)\}$	$q^3(q-1)$	$q^3(q-1)$	1
	$\emptyset$	$\{(0,1)\}$	$q(q-1)(q+1)$	$q(q-1)(q+1)$	1
	$\emptyset$	$\{(1,2)\}$	$(q-1)$	$(q-1)$	1
	$\emptyset$	$\emptyset$	1	1	1

We have that the sum of ratios $\frac{\lvert X^{\boldsymbol{\lambda}}_{I,J,K}\rvert}{\alpha^{\boldsymbol{\lambda}}_{I,J,K}}$ equals

$$(q-1) + \frac{q-1}{q} + \frac{1}{q} + q(q-1) + q + 1 + 1 + q + q + \frac{(q-1)^2}{q} + \frac{q-1}{q}$$
$$+ \frac{q-1}{q} + \frac{q-1}{q^2} + \frac{1}{q^2} + (q-1) + 1 + \frac{q-1}{q} + \frac{1}{q} + \frac{q-1}{q} + \frac{1}{q} + 1 + 1$$
$$= q^2 + 5q + 6 = n_{\boldsymbol{\nu}}(q).$$

Hence Example 3.9 agrees with Equation (3.3).

3.2.2. **Some Technical Propositions on Lattice of Characteristic Submodules.** Propositions 3.10, 3.11, 3.12, 3.13, 3.16, 3.18, 3.19, 3.20 below are useful in the proof of Theorem 3.8, especially in establishing the identity in Equation (3.3). They may appear slightly unmotivated and technical because they appear before the proof of Theorem 3.8. The proofs of these propositions use lattice theory, especially the theory of lattice of characteristic submodules $\mathcal{J}(P_{\boldsymbol{\lambda}})$ of any finite $\mathcal{R}$-module $\mathcal{A}_{\boldsymbol{\lambda}}$.

Proposition 3.10. *Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ and λ_{i_0} be a part of $\boldsymbol{\lambda}$ for some $1 \leq i_0 \leq k$. Let*

$$\boldsymbol{\nu} = ((\lambda_1 - 2)^{\rho_1} > (\lambda_2 - 2)^{\rho_2} > \cdots > (\lambda_{i_0-1} - 2)^{\rho_{i_0-1}} \geq (\lambda_{i_0} - 1)^{\rho_{i_0}}$$
$$\geq \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}).$$

Let $\mathcal{M} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\} \subseteq \mathcal{J}(P_{\boldsymbol{\lambda}})$. *For* $I \in \mathcal{M}, I_1 = \psi(I)$, *where* ψ *is as defined in Theorem 3.4(2),*

- *if* $\mu - 2$ *is a part of* $\boldsymbol{\nu}'/I_1$ *and* $\mu > \lambda_{i_0} + 1$ *then* μ *is a part of* $\boldsymbol{\lambda}'/I$. *Moreover, if* $\mu < \lambda_{i_0} - 1$ *and* μ *is a part of* $\boldsymbol{\nu}'/I_1$ *then* μ *is a part of* $\boldsymbol{\lambda}'/I$.
- *If* λ *is a part of* $\boldsymbol{\lambda}'/I$ *and* $\lambda > \lambda_{i_0} + 1$ *then* $\lambda - 2$ *is a part of* $\boldsymbol{\nu}'/I_1$. *If* λ *is a part of* $\boldsymbol{\lambda}'/I$ *and* $\lambda < \lambda_{i_0} - 1$ *then* λ *is a part of* $\boldsymbol{\nu}'/I_1$. *If* λ *is a part of* $\boldsymbol{\lambda}'/I$ *and* $\lambda = \lambda_{i_0} + 1$ *then* $\lambda - 2$ *need not be a part of* $\boldsymbol{\nu}'/I_1$. *If* λ *is a part of* $\boldsymbol{\lambda}'/I$ *and* $\lambda = \lambda_{i_0} - 1$ *then* λ *need not be a part of* $\boldsymbol{\nu}'/I_1$.
- *If* $\max(I_1) \cap P_{(\lambda_0-1)} \neq \emptyset$ *then both* $\lambda_{i_0} \pm 1$ *are not parts of* $\boldsymbol{\lambda}'/I$. *Moreover,* $\lambda_{i_0} - 1$ *is not a part of* $\boldsymbol{\nu}'/I_1$.
- *If* $\max(I_1) \cap P_{(\lambda_0-1)} = \emptyset$ *then either* $\lambda_{i_0} + 1$ *is a part of* $\boldsymbol{\lambda}'/I$ *or* $\lambda_{i_0} - 1$ *is a part of* $\boldsymbol{\lambda}'/I$. *Both* $\lambda_{i_0} \pm 1$ *are parts of* $\boldsymbol{\lambda}'/I$ *if and only if* $\lambda_{i_0} - 1$ *is a part of* $\boldsymbol{\nu}'/I$.
- *(More obvious conclusion:)* λ_{i_0} *is a never a part of* $\boldsymbol{\lambda}'/I$ *for any* $I \in \mathcal{M}$.

Proof. Let

$$\begin{aligned}\max(I) = \{&(v_1, t_1), (v_2, t_2), \cdots, (v_{i-1}, t_{i-1}), (v_i, t_i) = (v_i, \lambda_{i_0}), (v_{i+1}, t_{i+1}) \\ &\cdots, (v_s, t_s) \mid t_1 > t_2 > \cdots > t_{i-1} > t_i = \lambda_{i_0} > t_{i+1} > \cdots > t_s\}.\end{aligned}$$

Then $t_{i-1} > \lambda_{i_0} + 1 \geq s - i + 2$, $v_{i-1} \geq v_i + 1 \geq s - i + 1$, $\lambda_{i_0} - 1 > t_{i+1} \geq s - i$, $v_{i+1} \geq s - i - 1$ and

$$\begin{aligned}v_1 > v_2 > \cdots > v_{i-1} > v_i > v_{i+1} > \cdots > v_{s-1} > v_s \geq 0,\\ t_1 - v_1 > t_2 - v_2 > \cdots > t_{i-1} - v_{i-1}\\ > t_i - v_i > t_{i+1} - v_{i+1} > \cdots > t_{s-1} - v_{s-1} > t_s - v_s \geq 1.\end{aligned}$$

Furthermore,

$$\boldsymbol{\lambda}' = (t_1 > t_2 > \cdots > t_{i-1} > t_i = \lambda_{i_0} > t_{i+1} > \cdots > t_{s-1} > t_s),$$

$$\begin{aligned}\boldsymbol{\lambda}'/I = \big(&(v_1 + t_2 - v_2) > (v_2 + t_3 - v_3) > \cdots > (v_{i-2} + t_{i-1} - v_{i-1}) \\ &> (v_{i-1} + t_i - v_i = v_{i-1} + \lambda_{i_0} - v_i) > (v_i + t_{i+1} - v_{i+1}) > \cdots \\ &> (v_{s-2} + t_{s-1} - v_{s-1}) > (v_{s-1} + t_s - v_s) > v_s\big).\end{aligned} \tag{3.9}$$

Note that $v_{i-1} + t_i - v_i > t_i = \lambda_{i_0} > v_i + t_{i+1} - v_{i+1}$. In case $i = s$ then we have $v_{s-1} + t_s - v_s > t_s = \lambda_{i_0} > v_s$. So λ_{i_0} is never a part of $\boldsymbol{\lambda}'/I$ for any $I \in \mathcal{M}$. The parts of $\boldsymbol{\lambda}'$ interleave the parts of $\boldsymbol{\lambda}'/I$.

In the first case, if

$$\begin{aligned}\max(I_1) = \{&(v_1 - 1, t_1 - 2), (v_2 - 1, t_2 - 2), \cdots, (v_{i-1} - 1, t_{i-1} - 2), \\ &(v_i, t_i - 1) = (v_i, \lambda_{i_0} - 1), (v_{i+1}, t_{i+1}), \cdots, (v_{s-1}, t_{s-1}), (v_s, t_s)\},\end{aligned}$$

then $(v_i, t_i - 1) = (v_i, \lambda_{i_0} - 1) \in \max(I_1)$,

$$v_1 - 1 > v_2 - 1 > \cdots > v_{i-1} - 1 > v_i > v_{i+1} > \cdots > v_{s-1} > v_s \geq 0,$$
$$t_1 - v_1 - 1 > t_2 - v_2 - 1 > \cdots > t_{i-1} - v_{i-1} - 1 > t_i - v_i - 1 > t_{i+1} - v_{i+1}$$
$$> \cdots > t_{s-1} - v_{s-1} > t_s - v_s \geq 1.$$

Furthermore,

(3.10)
$$\begin{aligned}\boldsymbol{\nu}' = (t_1 - 2 > t_2 - 2 > \cdots > t_{i-1} - 2\\ > t_i - 1 = \lambda_{i_0} - 1 > t_{i+1} > \cdots > t_{s-1} > t_s),\end{aligned}$$
$$\begin{aligned}\boldsymbol{\nu}'/I_1 = \big((v_1 + t_2 - v_2 - 2) > (v_2 + t_3 - v_3 - 2) > \cdots > (v_{i-2} + t_{i-1} - v_{i-1} - 2)\\ > (v_{i-1} + t_i - v_i - 2 = v_{i-1} + \lambda_{i_0} - v_i - 2) > (v_i + t_{i+1} - v_{i+1})\\ > (v_{i+1} + t_{i+2} - v_{i+2}) \cdots > (v_{s-2} + t_{s-1} - v_{s-1})\\ > (v_{s-1} + t_s - v_s) > v_s\big).\end{aligned}$$

Now we notice that $\lambda_{i_0} - 1$ is a part of $\boldsymbol{\nu}'$. Hence it follows that $\lambda_{i_0} - 1$ is not a part of $\boldsymbol{\nu}'/I_1$ as it can be seen that $v_{i-1} + t_i - v_i - 2 = v_{i-1} + \lambda_{i_0} - v_i - 2 > \lambda_{i_0} - 1 > v_i + t_{i+1} - v_{i+1}$. The parts of $\boldsymbol{\nu}'$ interleave the parts of $\boldsymbol{\nu}'/I_1$.

Similarly we have $v_{i-1} - 1 > v_i$, or, equivalently, $v_{i-1} - v_i \geq 2$. Hence, $v_{i-1} + t_i - v_i = v_{i-1} + \lambda_{i_0} - v_i \geq \lambda_{i_0} + 2$. Moreover, $t_i - v_i - 1 > t_{i+1} - v_{i+1}$, which implies $\lambda_{i_0} - 1 > v_i + t_{i+1} - v_{i+1}$. So both $\lambda_{i_0} \pm 1$ are not parts of $\boldsymbol{\lambda}'/I$.

The number of parts of $\boldsymbol{\lambda}'$ is s. The number of parts of $\boldsymbol{\nu}'$ is also s. The number of parts of $\boldsymbol{\lambda}'/I$ is s if $v_s > 0$ and it is $s-1$ if $v_s = 0$. Note that the part $v_{i-1} + t_i - v_i - 2 = v_{i-1} + \lambda_{i_0} - v_i - 2$ of $\boldsymbol{\nu}'/I_1$ is greater than zero, since $v_{i-1} - v_i \geq 2$ and $\lambda_{i_0} \geq 1$. The number of parts of $\boldsymbol{\nu}'/I_1$ is s if $v_s > 0$, it is $s-1$ if $v_s = 0$. It cannot be $s-2$ in this case because the case $i = s$, $v_s = 0$, $v_{s-1} + t_s = 2$, that is, $i_0 = k$, $i = s$, $v_s = 0$, $v_{s-1} = 1$, $t_s = \lambda_k = 1$, is not allowed here since $i_0 = k$, $i = s$, $t_s = \lambda_k$, $(v_{s-1} - 1, t_{s-1} - 2), (v_s, t_s - 1) \in \max(I_1)$ imply $v_{s-1} - 1 > v_s$.

In the second case, if

$$\max(I_1) = \{(v_1 - 1, t_1 - 2), (v_2 - 1, t_2 - 2), \cdots, (v_{i-1} - 1, t_{i-1} - 2), (v_{i+1}, t_{i+1}), \cdots, (v_{s-1}, t_{s-1}), (v_s, t_s)\}$$

then $(v_i, t_i - 1) = (v_i, \lambda_{i_0} - 1) \notin \max(I_1)$,

$$v_1 - 1 > v_2 - 1 > \cdots > v_{i-1} - 1 > v_{i+1} > \cdots > v_{s-1} > v_s \geq 0,$$
$$t_1 - v_1 - 1 > t_2 - v_2 - 1 > \cdots > t_{i-1} - v_{i-1} - 1 > t_{i+1} - v_{i+1}$$
$$> \cdots > t_{s-1} - v_{s-1} > t_s - v_s \geq 1.$$

Furthermore,

(3.11)
$$\begin{aligned}\boldsymbol{\nu}' &= (t_1 - 2 > t_2 - 2 > \cdots > t_{i-1} - 2 > t_{i+1} > \cdots > t_{s-1} > t_s),\\ \boldsymbol{\nu}'/I_1 &= \big((v_1 + t_2 - v_2 - 2) > (v_2 + t_3 - v_3 - 2) > \cdots > (v_{i-2} + t_{i-1} - v_{i-1} - 2)\\ &\quad > (v_{i-1} + t_{i+1} - v_{i+1} - 1) > (v_{i+1} + t_{i+2} - v_{i+2}) > \cdots\\ &\quad > (v_{s-2} + t_{s-1} - v_{s-1}) > (v_{s-1} + t_s - v_s) > v_s\big).\end{aligned}$$

The number of parts of $\boldsymbol{\lambda}'$ is s. The number of parts of $\boldsymbol{\nu}'$ is $s-1$. The number of parts of $\boldsymbol{\lambda}'/I$ is s if $v_s > 0$ and it is $s-1$ if $v_s = 0$. If $i = s$ (for example if $i_0 = k, t_s = \lambda_{i_0} = \lambda_k$) then the number of parts of $\boldsymbol{\nu}'/I_1$ is $s-1$ if $v_{s-1} > 1$ and it is $s-2$ if $v_{s-1} = 1$. If $i < s$ then the number of parts of $\boldsymbol{\nu}'/I_1$ is $s-1$ if $v_s > 0$ and it is $s-2$ if $v_s = 0$.

The case $(v_i, t_i - 1) \notin \max(I_1)$ occurs if

(a) either $t_i - v_i - 1 = t_{i+1} - v_{i+1}$,
(b) or $v_{i-1} - 1 = v_i$.

In case (a) we have

(3.12)
$$\begin{aligned}\boldsymbol{\lambda}'/I &= \big((v_1 + t_2 - v_2) > (v_2 + t_3 - v_3) > \cdots > (v_{i-2} + t_{i-1} - v_{i-1})\\ &\quad > (v_{i-1} + t_i - v_i = v_{i-1} + \lambda_{i_0} - v_i)\\ &\quad > (v_i + t_{i+1} - v_{i+1} = t_i - 1 = \lambda_{i_0} - 1) > (v_{i+1} + t_{i+2} - v_{i+2}) > \cdots\\ &\quad > (v_{s-2} + t_{s-1} - v_{s-1}) > (v_{s-1} + t_s - v_s) > v_s\big),\\ \boldsymbol{\nu}'/I_1 &= \big((v_1 + t_2 - v_2 - 2) > (v_2 + t_3 - v_3 - 2) > \cdots > (v_{i-2} + t_{i-1} - v_{i-1} - 2)\\ &\quad > (v_{i-1} + t_{i+1} - v_{i+1} - 1 = v_{i-1} + t_i - v_i - 2 = v_{i-1} + \lambda_{i_0} - v_i - 2)\\ &\quad > (v_{i+1} + t_{i+2} - v_{i+2}) > \cdots > (v_{s-2} + t_{s-1} - v_{s-1})\\ &\quad > (v_{s-1} + t_s - v_s) > v_s\big).\end{aligned}$$

In case (b) we have

(3.13)
$$\begin{aligned}\boldsymbol{\lambda}'/I &= \big((v_1 + t_2 - v_2) > (v_2 + t_3 - v_3) > \cdots > (v_{i-2} + t_{i-1} - v_{i-1})\\ &\quad > (v_{i-1} + t_i - v_i = t_i + 1 = \lambda_{i_0} + 1) > (v_i + t_{i+1} - v_{i+1})\\ &\quad > (v_{i+1} + t_{i+2} - v_{i+2}) > \cdots\\ &\quad > (v_{s-2} + t_{s-1} - v_{s-1}) > (v_{s-1} + t_s - v_s) > v_s\big),\\ \boldsymbol{\nu}'/I_1 &= \big((v_1 + t_2 - v_2 - 2) > (v_2 + t_3 - v_3 - 2) > \cdots > (v_{i-2} + t_{i-1} - v_{i-1} - 2)\\ &\quad > (v_{i-1} + t_{i+1} - v_{i+1} - 1 = v_i + t_{i+1} - v_{i+1}) > (v_{i+1} + t_{i+2} - v_{i+2}) > \cdots\\ &\quad > (v_{s-2} + t_{s-1} - v_{s-1}) > (v_{s-1} + t_s - v_s) > v_s\big).\end{aligned}$$

From Equations (3.9)–(3.13), we observe that Proposition 3.10 follows. ■

Proposition 3.11. *Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ and λ_{i_0} be a part of $\boldsymbol{\lambda}$ for some $1 \leq i_0 \leq k$. Let*

$$\boldsymbol{\nu} = ((\lambda_1 - 2)^{\rho_1} > (\lambda_2 - 2)^{\rho_2} > \cdots > (\lambda_{i_0 - 1} - 2)^{\rho_{i_0 - 1}} \geq (\lambda_{i_0} - 1)^{\rho_{i_0}} \geq \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}).$$

Let $\mathcal{M} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. For $I \in \mathcal{M} \subseteq \mathcal{J}(P_{\boldsymbol{\lambda}})$ let $I_1 = \psi(I)$, where ψ is as defined in Theorem 3.4(2). Let $\boldsymbol{\lambda}'''$ be the partition obtained by inserting a new part λ_{i_0} into the partition $\boldsymbol{\lambda}'/I$. Let $\mathcal{N} = \{\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''}) \mid \max(\tilde{J}) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. Then there exists a lattice homomorphism $\phi : \mathcal{N} = \{\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''}) \mid \max(\tilde{J}) \cap P_{(\lambda_{i_0})} \neq \emptyset\} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ such that $\phi(\tilde{J}) = J_1$, where

$$\max(J_1) = \{(v-1, \lambda-2) \mid (v, \lambda) \in \max(\tilde{J}), \lambda > \lambda_{i_0}\} \cup \{(v, \lambda) \mid (v, \lambda) \in \max(\tilde{J}), \lambda < \lambda_{i_0}\}.$$

Proof. If $\max(\tilde{J}) \cap P_{(\lambda_{i_0})} \neq \emptyset$ then $(v, \lambda) \in \max(\tilde{J}), \lambda > \lambda_{i_0}$ imply $\lambda > \lambda_{i_0} + 1$, which in turn implies that $\lambda - 2$ is a part of $\boldsymbol{\nu}'/I_1$ using Proposition 3.10. We observe that if $\max(\tilde{J}) \cap P_{(\lambda_{i_0})} = \{(u, \lambda_{i_0})\}$ and $(v, \lambda) \in \max(\tilde{J})$, $\lambda > \lambda_{i_0}$ then $v > u$ implies $v \geq 1$ and $\lambda - v > \lambda_{i_0} - u \geq 1$ so that $\lambda - v \geq 2$ or, equivalently, $v - 1 < \lambda - 2$. So $(v-1, \lambda-2) \in P_{\boldsymbol{\nu}'/I_1}$.

If $\max(\tilde{J}) \cap P_{(\lambda_{i_0})} \neq \emptyset$ then $(v, \lambda) \in \max(\tilde{J})$, $\lambda < \lambda_{i_0}$ imply $\lambda < \lambda_{i_0} - 1$, which in turn implies that λ is a part of $\boldsymbol{\nu}'/I_1$ using Proposition 3.10.

Hence

$$\{(v-1, \lambda-2) \mid (v, \lambda) \in \max(\tilde{J}), \lambda > \lambda_{i_0}\} \cup \{(v, \lambda) \mid (v, \lambda) \in \max(\tilde{J}), \lambda < \lambda_{i_0}\}$$

is a subset of $P_{\boldsymbol{\nu}'/I_1}$. Now we prove that this set is an antichain. Clearly the set $\{(v-1, \lambda-2) \mid (v, \lambda) \in \max(\tilde{J}), \lambda > \lambda_{i_0}\}$ is an antichain and the set $\{(v, \lambda) \mid (v, \lambda) \in \max(\tilde{J}), \lambda < \lambda_{i_0}\}$ is also an antichain. Now, if $(v, \lambda) \in \max(\tilde{J}), \lambda > \lambda_{i_0}$ and $(w, \mu) \in \max(\tilde{J}), \mu < \lambda_{i_0}$, then we have $v > u > w, \lambda - v > \lambda_{i_0} - u > \mu - w$. This implies that $v - 1 > w, \lambda - v - 1 > \mu - w$ which further implies that $(v-1, \lambda-2), (w, \mu)$ are not comparable. Hence $\max(J_1)$ is an antichain and defines an ideal $J_1 \in \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$.

Now it is easy to see that the map $\phi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ is a lattice homomorphism. This proves the proposition. ■

In the following proposition we define another lattice homomorphism $\xi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$.

Proposition 3.12. *Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ and λ_{i_0} be a part of $\boldsymbol{\lambda}$ for some $1 \leq i_0 \leq k$. Let*

$$\boldsymbol{\nu} = \big((\lambda_1 - 2)^{\rho_1} > (\lambda_2 - 2)^{\rho_2} > \cdots > (\lambda_{i_0-1} - 2)^{\rho_{i_0-1}} \geq (\lambda_{i_0} - 1)^{\rho_{i_0}} \geq \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\big).$$

Let $\mathcal{M} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. For $I \in \mathcal{M} \subseteq \mathcal{J}(P_{\boldsymbol{\lambda}})$ let $I_1 = \psi(I)$, where ψ is as defined in Theorem 3.4(2). Let $\boldsymbol{\lambda}'''$ be the partition obtained by inserting a new part λ_{i_0} into the partition $\boldsymbol{\lambda}'/I$. Let $\mathcal{N} = \{\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''}) \mid \max(\tilde{J}) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. Then there exists a lattice homomorphism

$$\xi : \mathcal{N} = \{\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''}) \mid \max(\tilde{J}) \cap P_{(\lambda_{i_0})} \neq \emptyset\} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$$

such that $\xi(\tilde{J}) = J_3$, where $\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''}), J_3 \in \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ are given as follows. Let $\max([\tilde{J}]_{(\mu_i)}) = \{(a_i, \mu_i)\}$ if it is nonempty, for μ_i a part of $\boldsymbol{\lambda}'''$. Let $\max([\tilde{J}]_{(\mu_i = \lambda_{i_0})}) = \{(a_i = u, \mu_i = \lambda_{i_0})\} \in P_{(\lambda_{i_0})}$. Then

1. *$\max([\tilde{J}]_{(\lambda_{i_0}+1)}) = \{(u+1, \lambda_{i_0}+1)\}, \max([\tilde{J}]_{(\lambda_{i_0}-1)}) = \{(u, \lambda_{i_0}-1)\}$ if $u < \lambda_{i_0} - 1$ and $\max([\tilde{J}]_{(\lambda_{i_0}-1)})$ is empty if $u = \lambda_{i_0} - 1$.*
2. *$\max([J_3])_{(\mu_i-2)} = \{(a_i - 1, \mu_i - 2)\}$ for $\mu_i > \lambda_{i_0} + 1$ provided $a_i - 1 < \mu_i - 2$. Otherwise if $a_i = \mu_i - 1$ then $\max([J_3])_{(\mu_i-2)}$ is empty.*
3. *$\max([J_3])_{(\mu_i)} = \max([\tilde{J}]_{(\mu_i)})$ for $\mu_i < \lambda_{i_0} - 1$, that is, $\max([J_3])_{(\mu_i)} = \{(a_i, \mu_i)\}$ for $\mu_i < \lambda_{i_0} - 1$ and if $\max([\tilde{J}]_{(\mu_i)})$ is nonempty, $\max([J_3])_{(\mu_i)}$ is empty for $\mu_i < \lambda_{i_0} - 1$ and if $\max([\tilde{J}]_{(\mu_i)})$ is empty.*
4. *If $\lambda_{i_0} - 1$ is a part of $\boldsymbol{\nu}'/I_1$ then $\max([J_3])_{(\lambda_{i_0}-1)} = \max([\tilde{J}]_{(\lambda_{i_0}-1)})$.*

Let $\boldsymbol{\lambda}''''$ be a partition obtained by inserting both $\lambda_{i_0} \pm 1$ into $\boldsymbol{\lambda}'''$ (if either of them is not a part of $\boldsymbol{\lambda}'''$). Let $\boldsymbol{\nu}'''$ be a partition obtained by inserting $\lambda_{i_0} - 1$ into $\boldsymbol{\nu}'/I_1$ (if it is not a part of $\boldsymbol{\nu}'/I_1$). Let $\mathcal{R} = \{J' \in \mathcal{J}(P_{\boldsymbol{\lambda}''''}) \mid \max(J') \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. Then we have that ξ is the composition of the following maps:

$$\begin{aligned} &\mathcal{N} \underset{\cong}{\longrightarrow} \mathcal{R} \underset{\cong}{\xrightarrow{\psi}} \mathcal{J}(P_{\boldsymbol{\nu}'''}) \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1}), \\ &\tilde{J} \longrightarrow [\tilde{J}]_{\boldsymbol{\lambda}''''} \longrightarrow \psi([\tilde{J}]_{\boldsymbol{\lambda}''''}) \longrightarrow [\psi([\tilde{J}]_{\boldsymbol{\lambda}''''})]_{\boldsymbol{\nu}'/I_1}, \end{aligned} \tag{3.14}$$

where ψ is the map defined in Theorem 3.4(2) corresponding to $\boldsymbol{\lambda}''''$ and $\boldsymbol{\nu}'''$.

Proof. We prove (1). Since $(u, \lambda_{i_0}) \in \max(\tilde{J})$, (1) follows.

We prove (2), (3), (4). If μ_i is a part of $\boldsymbol{\lambda}'/I$ then using Proposition 3.10, we observe that $\mu_i - 2$ is a part of $\boldsymbol{\nu}'/I_1$ if $\mu_i > \lambda_{i_0} + 1$ and μ_i is a part of $\boldsymbol{\nu}'/I_1$ if $\mu_i < \lambda_{i_0} - 1$. If $\mu_i > \lambda_{i_0} + 1$ then $a_i > u$ implies $a_i \geq 1$ and $\mu_i - a_i > \lambda_{i_0} - u$, which in turn implies $\mu_i - a_i \geq 2$ or, equivalently, $0 \leq a_i - 1 \leq \mu_i - 2$. So $(a_i - 1, \mu_i - 2) \in P_{(\mu_i-2)}$ if $a_i < \mu_i - 1$. Now define J_3 such that $\max([J_3]_{(\mu)})$ is as given in (2), (3), (4) for μ a part of $\boldsymbol{\nu}'/I_1$. It is easy to check that this actually defines a well defined ideal in $\mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$. This proves (2), (3), (4).

Now it is easy to see that the map $\xi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ is a lattice homomorphism and is the composition of the maps given in 3.14. Hence the proposition follows. ■

Proposition 3.13. *Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ and λ_{i_0} be a part of $\boldsymbol{\lambda}$ for some $1 \leq i_0 \leq k$. Let*

$$\boldsymbol{\nu} = \big((\lambda_1 - 2)^{\rho_1} > (\lambda_2 - 2)^{\rho_2} > \cdots > (\lambda_{i_0-1} - 2)^{\rho_{i_0-1}} \geq (\lambda_{i_0} - 1)^{\rho_{i_0}} \geq \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\big).$$

Let $\mathcal{M} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. For $I \in \mathcal{M} \subseteq \mathcal{J}(P_{\boldsymbol{\lambda}})$ let $I_1 = \psi(I)$, where ψ is as defined in Theorem 3.4(2). Let $\boldsymbol{\lambda}'''$ be the partition obtained by inserting a new part λ_{i_0} into the partition $\boldsymbol{\lambda}'/I$. Let $\mathcal{N} = \{\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''}) \mid \max(\tilde{J}) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. Let $\phi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1}), \xi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ be the lattice homomorphisms as defined in Proposition 3.11 and Proposition 3.12, respectively. Then we have for any $\tilde{J} \in \mathcal{N}$ with $J_1 = \phi(\tilde{J}), J_3 = \xi(\tilde{J})$,

$$\max(J_1) = \max(\phi(\tilde{J})) \subseteq \max(\xi(\tilde{J})) = \max(J_3).$$

Proof. Let $\max([\tilde{J}]_{(\mu_i)}) = \{(a_i, \mu_i)\}$ if it is nonempty, for μ_i a part of $\boldsymbol{\lambda}'''$. Let $\max([\tilde{J}]_{(\mu_i = \lambda_{i_0})}) = \{(a_i = u, \mu_i = \lambda_{i_0})\} \in P_{(\lambda_{i_0})}$. Then both the sets $\max(J_1)$ and $\max(J_3)$ are contained in the set

$$U_1 = \{(a_i - 1, \mu_i - 2) \mid \mu_i > \lambda_{i_0} + 1, a_i < \mu_i - 1\} \cup \{(a_i, \mu_i) \mid \mu_i < \lambda_{i_0} - 1\} \cup \{(u, \lambda_{i_0} - 1)\}$$

if $u < \lambda_{i_0} - 1$ and both the sets $\max(J_1)$ and $\max(J_3)$ are contained in the set

$$U_2 = \{(a_i - 1, \mu_i - 2) \mid \mu_i > \lambda_{i_0} + 1, a_i < \mu_i - 1\}$$

if $u_0 = \lambda_{i_0} - 1$. Moreover any element in the set U_1 or U_2 is less than or equal to some element in the set $\max(J_3)$. Now let $J \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I})$ be such that $\max(J) = \max(\tilde{J}) \backslash \{(u, \lambda_{i_0})\}$. Now $(a, \mu) \in \max(J), \mu > \lambda_{i_0} + 1$ if and only if $(a - 1, \mu - 2) \in \max(J_1)$. Furthermore, $(a, \mu) \in \max(J), \mu < \lambda_{i_0} - 1$ if and only if $(a, \mu) \in \max(J_1)$. So in the initial case if $(a - 1, \mu - 2) \in \max(J_1)$ implies $(a - 1, \mu - 2) \in U_1$ or U_2 and $(a, \mu) \in \max(\tilde{J})$ and in the latter case if $(a, \mu) \in \max(J_1)$ implies $(a, \mu) \in U_1$ and $(a, \mu) \in \max(\tilde{J})$.

In the initial case for $(a - 1, \mu - 2) \in \max(J_1), \mu > \lambda_{i_0} + 1$ if $(a - 1, \mu - 2) \leq (b - 1, \lambda - 2) \in U_2$ then $(a, \mu) \leq (b, \lambda) \in \tilde{J}$ implies $(a, \mu) = (b, \lambda)$ since $(a, \mu) \in \max(\tilde{J})$ implies $(a - 1, \mu - 2) = (b - 1, \lambda - 2)$. If $\mu > \lambda_{i_0} + 1$, $\lambda \leq \lambda_{i_0} - 1$,

$(a-1,\mu-2) \in \max(J_1)$, $(b,\lambda) \in U_1$, then the case $(a-1,\mu-2) \leq (b,\lambda)$ does not arise. This is because:

- Case (a) (a,μ) and (u,λ_{i_0}) are not comparable as they are both in $\max(\tilde{J})$, (u,λ_{i_0}) and (b,λ) are not comparable in which case we have $a > u > b$ and $\mu - a > \lambda_{i_0} - u > \lambda - b$, which implies that the elements $(a-1,\mu-2)$ and (b,λ) are not comparable.
- Case (b) (a,μ) and (u,λ_{i_0}) are not comparable as they are both in $\max(\tilde{J})$, $(u,\lambda_{i_0}) \geq (b,\lambda)$ and $u = b$ since $(b,\lambda) \in U_1$, $\lambda \leq \lambda_{i_0} - 1$ imply $\max([\tilde{J}]_{(\lambda)}) = \{(b,\lambda)\}$. In this case we have $\mu - a > \lambda_{i_0} - u = \lambda_{i_0} - b > \lambda - b, a > u = b$. So we have $a - 1 \geq b$ and $\mu - a - 1 > \lambda - b$, which implies that either $(a-1,\mu-2),(b,\lambda)$ are not comparable or $(a-1,\mu-2)$ and (b,λ) are comparable and $(b,\lambda) < (a-1,\mu-2)$.

So we have $(a-1,\mu-2) \in \max(J_1)$, $\mu > \lambda_{i_0}+1$, hence $(a-1,\mu-2)$ is a maximal element in U_1 which implies that $(a-1,\mu-2) \in \max(J_3)$.

In the latter case for $(a,\mu) \in \max(J_1), \mu < \lambda_{i_0} - 1$ we have $(a,\mu) \in \max(\tilde{J})$. So $(a,\mu) \leq (b,\lambda) \in U_1$ for some $\lambda \leq \lambda_{i_0} - 1$, which implies $(b,\lambda) \in [\tilde{J}]_{(\lambda)}$ and $(a,\mu) = (b,\lambda)$. If $\lambda > \lambda_{i_0} + 1$ and $(b-1,\lambda-2) \in U_1$ then the case $(a,\mu) \leq (b-1,\lambda-2)$ does not arise. This is because:

- Case (A) $(u,\lambda_{i_0}),(a,\mu)$ are not comparable as they are both in $\max(\tilde{J})$ and (u,λ_{i_0}), (b,λ) are not comparable in which case we have $b > u > a, \lambda - b > \lambda_{i_0} - u > \mu - a$. This implies that the elements $(b-1,\lambda-2),(a,\mu)$ are not comparable.
- Case (B) $(u,\lambda_{i_0}),(a,\mu)$ are not comparable as they are both in $\max(\tilde{J}),(u,\lambda_{i_0}) \geq (b,\lambda)$ and $\lambda - b = \lambda_{i_0} - u$ since $(b-1,\lambda-2) \in U_1, \lambda > \lambda_{i_0}+1$, implying $\max([\tilde{J}]_{(\lambda)}) = \{(b,\lambda)\}$. In this case we have $\lambda - b = \lambda_{i_0} - u > \mu - a$, and therefore $\lambda - b - 1 \geq \mu - a$. Now $\lambda - b = \lambda_{i_0} - u$ and $\lambda > \lambda_{i_0}$, hence $b > u$. So we have $b > u > a$, hence $b - 1 > a$. So $\lambda - b - 1 \geq \mu - a, b - 1 > a$ implies that either $(b-1,\lambda-2),(a,\mu)$ are not comparable or $(b-1,\lambda-2),(a,\mu)$ are comparable and $(b-1,\lambda-2) < (a,\mu)$.

So we have $(a,\mu) \in \max(J_1)$, $\mu < \lambda_{i_0} - 1$, therefore (a,μ) is a maximal element in U_1 which implies that $(a,\mu) \in \max(J_3)$.

This proves the proposition that

$$\max(J_1) = \max(\phi(\tilde{J})) \subseteq \max(\xi(\tilde{J})) = \max(J_3). \qquad \blacksquare$$

Remark 3.14. The map $\xi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\nu'/I_1})$ is called the upper lattice homomorphism and $\phi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\nu'/I_1})$ is called the lower lattice homomorphism since for every $\tilde{J} \in \mathcal{N}$ we have $\max(\phi(\tilde{J})) \subseteq \max(\xi(\tilde{J}))$ implying $\phi(\tilde{J}) \subseteq \xi(\tilde{J})$.

Remark 3.15. For $(u, \lambda_{i_0}) \in \max(\tilde{J}), \phi(\tilde{J}) = J_1, \xi(\tilde{J}) = J_3 = [\psi([\tilde{J}]_{\boldsymbol{\lambda}''''})]_{\boldsymbol{\nu}'/I_1}$, we have $[J_1 \cup \langle\{(u, \lambda_{i_0} - 1)\}\rangle]_{\boldsymbol{\nu}'''} = \psi([\tilde{J}]_{\boldsymbol{\lambda}''''})$ if $u < \lambda_{i_0} - 1$ and if $u = \lambda_{i_0} - 1$ then $J_1 = J_3$. This is because of the following facts.

In case $u < \lambda_{i_0} - 1$, if $(a, \mu) \in \max(J_3) \backslash \max(J_1)$ and $\mu < \lambda_{i_0} - 1$ then $(a, \mu) \in \tilde{J}$ and $(a, \mu) \le (u, \lambda_{i_0})$, which implies $u \le a, \mu - a < \lambda_{i_0} - a \le \lambda_{i_0} - u$, hence $\mu - a \le \lambda_{i_0} - u - 1$. So $(a, \mu) \le (u, \lambda_{i_0} - 1)$. Moreover, if $(a - 1, \mu - 2) \in \max(J_3) \backslash \max(J_1)$ and $\mu > \lambda_{i_0} + 1$ then $(a, \mu) \in \tilde{J}, (a, \mu) \le (u, \lambda_{i_0})$. So $u \le a, \mu - a \le \lambda_{i_0} - u$. Now we can get better inequalities using the fact that $\{(u + 1, \lambda_{i_0} + 1)\} = \max([\tilde{J}]_{(\lambda_{i_0}+1)}), \{(a, \mu)\} = \max([\tilde{J}]_{(\mu)})$. Here we have $a \ge u + 1, \mu - a \ge \lambda_{i_0} - u$. So we get $\mu - a = \lambda_{i_0} - u, u \le a - 1$, which implies $(a - 1, \mu - 2) \le (u, \lambda_{i_0} - 1)$. This proves that $\max(J_3) \subset [J_1 \cup \langle\{(u, \lambda_{i_0} - 1)\}\rangle]_{\boldsymbol{\nu}'''}$, and therefore $[J_1 \cup \langle\{(u, \lambda_{i_0} - 1)\}\rangle]_{\boldsymbol{\nu}'''} = \psi([\tilde{J}]_{\boldsymbol{\lambda}''''})$.

In case $u = \lambda_{i_0} - 1$ then if $(a, \mu) \le (u = \lambda_{i_0} - 1, \lambda_{i_0})$ and $\mu \ge \lambda_{i_0} + 1$ then $a = \mu - 1$. Moreover, if $\mu < \lambda_{i_0}$ then $\max([\tilde{J}]_{(\mu)}) = \emptyset$. So $\max(J_1) = \max(J_3)$, implying $J_1 = J_3$.

Now we prove that there exists a lattice homomorphism $\zeta : \mathcal{J}(P_{\boldsymbol{\nu}'/I_1}) \longrightarrow \mathcal{N}$ in the reverse direction satisfying certain nice properties.

Proposition 3.16. *Let* $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ *and* λ_{i_0} *be a part of* $\boldsymbol{\lambda}$ *for some* $1 \le i_0 \le k$. *Let*

$$\boldsymbol{\nu} = \big((\lambda_1 - 2)^{\rho_1} > (\lambda_2 - 2)^{\rho_2} > \cdots > (\lambda_{i_0-1} - 2)^{\rho_{i_0-1}} \ge (\lambda_{i_0} - 1)^{\rho_{i_0}} \\ \ge \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\big).$$

Let $\mathcal{M} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. *For* $I \in \mathcal{M} \subseteq \mathcal{J}(P_{\boldsymbol{\lambda}})$ *let* $I_1 = \psi(I)$, *where* ψ *is as defined in Theorem 3.4(2). Let* $\boldsymbol{\lambda}'''$ *be the partition obtained by inserting a new part* λ_{i_0} *into the partition* $\boldsymbol{\lambda}'/I$. *Let* $\mathcal{N} = \{\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''}) \mid \max(\tilde{J}) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. *Then there exists a lattice homomorphism* $\zeta : \mathcal{J}(P_{\boldsymbol{\nu}'/I_1}) \longrightarrow \mathcal{N}$ *such that* $\zeta(J_3) = J_5$, *where* $J_3 \in \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ *with (say)* $\max([J_3]_{(\delta_i)}) = \{(b_i, \delta_i)\}$ *if nonempty, for* δ_i *a part of* $\boldsymbol{\nu}'/I_1$ *and (say)* $\max([J_3]_{(\lambda_{i_0}-1)}) = \{(b, \lambda_{i_0} - 1)\}$ *if nonempty, irrespective of whether* $\lambda_{i_0} - 1$ *is a part of* $\boldsymbol{\nu}'/I_1$. *Then* J_5 *is given as follows:*

(1) *If* $\delta_i + 2 > \lambda_{i_0} + 1$ *and* $\max([J_3]_{(\delta_i)}) = \{(b_i, \delta_i)\}$ *then* $\max([J_5]_{(\delta_i+2)}) = \{(b_i + 1, \delta_i + 2)\}$.
(2) *If* $\delta_i + 2 > \lambda_{i_0} + 1$ *and* $\max([J_3]_{(\delta_i)})$ *is empty then* $\max([J_5]_{(\delta_i+2)}) = \{(\delta_i + 1, \delta_i + 2)\}$.
(3) *If* $\delta_i < \lambda_{i_0} - 1$ *and* $\max([J_5]_{(\delta_i)}) = \max([J_3]_{(\delta_i)})$, *that is,* $\max([J_3]_{(\delta_i)}) = \{(b_i, \delta_i)\}$ *then* $\max([J_5]_{(\delta_i)}) = \{(b_i, \delta_i)\}$ *and if* $\max([J_3]_{(\delta_i)})$ *is empty then* $\max([J_5]_{(\delta_i)})$ *is empty.*

(4) *If* $\lambda_{i_0}+1$ *is a part of* $\boldsymbol{\lambda}'/I$ *then* $\max([J5]_{(\lambda_{i_0}+1)}) = \{(b+1,\lambda_{i_0}+1)\}$ *if* $\max([J3]_{(\lambda_{i_0}-1)}) = \{(b,\lambda_{i_0}-1)\}$. *If* $\max([J3]_{(\lambda_{i_0}-1)})$ *is empty then* $\max([J5]_{(\lambda_{i_0}+1)}) = \{(\lambda_{i_0},\lambda_{i_0}+1)\}$

(5) *If* $\lambda_{i_0}-1$ *is a part of* $\boldsymbol{\lambda}'/I$ *then* $\max([J5]_{(\lambda_{i_0}-1)}) = \max([J3]_{(\lambda_{i_0}-1)})$.

(6) $\max([J5]_{(\lambda_{i_0})}) = \{(b,\lambda_{i_0})\}$ *if* $\max([J3]_{(\lambda_{i_0}-1)}) = \{(b,\lambda_{i_0}-1)\}$. *If* $\max([J3]_{(\lambda_{i_0}-1)})$ *is empty then* $\max([J5]_{(\lambda_{i_0})}) = \{(\lambda_{i_0}-1,\lambda_{i_0})\}$.

Let $\boldsymbol{\lambda}''''$ *be a partition obtained by inserting both* $\lambda_{i_0}\pm 1$ *into* $\boldsymbol{\lambda}'''$ *(if either of them is not a part of* $\boldsymbol{\lambda}'''$*). Let* $\boldsymbol{\nu}'''$ *be a partition obtained by inserting* $\lambda_{i_0}-1$ *into* $\boldsymbol{\nu}'/I_1$ *(if it is not a part of* $\boldsymbol{\nu}'/I_1$*). Let* $\mathcal{R} = \{J' \in \mathcal{J}(P_{\boldsymbol{\lambda}''''}) \mid \max(J') \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. *Then we have that* ζ *is a composition of the following maps:*

$$\begin{array}{c} \mathcal{J}(P_{\boldsymbol{\nu}'/I_1}) \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'''}) \xrightarrow[\cong]{\psi^{-1}} \mathcal{R} \xrightarrow[\cong]{} \mathcal{N} \\ J3 \longrightarrow [J3]_{\boldsymbol{\nu}'''} \longrightarrow \psi^{-1}([J3]_{\boldsymbol{\nu}'''}) \longrightarrow [\psi^{-1}([J3]_{\boldsymbol{\nu}'''})]_{\boldsymbol{\lambda}'''}, \end{array} \tag{3.15}$$

where ψ *is the map defined in Theorem 3.4(2) corresponding to* $\boldsymbol{\lambda}''''$ *and* $\boldsymbol{\nu}'''$.

Proof. We observe that $J5$ is an ideal in $\mathcal{J}(P_{\boldsymbol{\lambda}'''})$. If $\max([J3]_{(\lambda_{i_0}-1)}) = \{(b,\lambda_{i_0}-1)\}$ then $(b,\lambda_{i_0}) \in \max(J5)$. If $\max([J3]_{(\lambda_{i_0}-1)})$ is empty then $(\lambda_{i_0}-1,\lambda_{i_0}) \in \max(J5)$. So we have $\max(J5) \cap P_{(\lambda_{i_0})} \neq \emptyset$, implying $J5 \in \mathcal{N}$.

It is clear that the map $\zeta : \mathcal{J}(P_{\boldsymbol{\nu}'/I_1}) \longrightarrow \mathcal{N}$ is a lattice homomorphism and is a composition of maps given in 3.15. This proves the proposition. ■

Remark 3.17. If $\lambda_{i_0}-1$ is a part of $\boldsymbol{\nu}'/I_1$, that is, $\boldsymbol{\nu}''' = \boldsymbol{\nu}'/I_1$ then the maps $\xi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ and $\zeta : \mathcal{J}(P_{\boldsymbol{\nu}'/I_1}) \longrightarrow \mathcal{N}$ are inverses of each other.

Proposition 3.18. *Let* $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ *and* λ_{i_0} *be a part of* $\boldsymbol{\lambda}$ *for some* $1 \leq i_0 \leq k$ *with* $\rho_{i_0} = 2$. *Let*

$$\begin{aligned} \boldsymbol{\nu} = \big((\lambda_1-2)^{\rho_1} > (\lambda_2-2)^{\rho_2} > \cdots > (\lambda_{i_0-1}-2)^{\rho_{i_0-1}} \geq (\lambda_{i_0}-1)^{\rho_{i_0}} \\ \geq \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\big). \end{aligned}$$

Let $\mathcal{M} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. *For* $I \in \mathcal{M} \subseteq \mathcal{J}(P_{\boldsymbol{\lambda}})$ *let* $I_1 = \psi(I)$, *where* ψ *is as defined in Theorem 3.4(2). Correspondingly we have partitions* $\boldsymbol{\lambda}', \boldsymbol{\lambda}'', \boldsymbol{\lambda}'/I, \boldsymbol{\nu}', \boldsymbol{\nu}'', \boldsymbol{\nu}'/I_1$. *Let* $\boldsymbol{\lambda}'''$ *be the partition obtained by inserting a new part* λ_{i_0} *into the partition* $\boldsymbol{\lambda}'/I$. *Let* $\mathcal{N} = \{\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''}) \mid \max(\tilde{J}) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. *Let* $L \in \mathcal{M}, K \in \mathcal{R} = \{T \in \mathcal{J}(P_{\boldsymbol{\lambda}''}) \mid \max(T) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$ *and* $\tilde{J} \in \mathcal{N}$ *be such that* $J \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I})$ *with* $\max(J) = \max(\tilde{J}) \backslash P_{(\lambda_{i_0})}$, $[J \cup K]_{\boldsymbol{\lambda}} = L$ *and* $[\tilde{J}]_{(\lambda_{i_0})} = [K]_{(\lambda_{i_0})}$. *Let* $L_1 = \psi(L) \in \mathcal{J}(P_{\boldsymbol{\nu}})$. *Furthermore, let* $K_1 = \psi(K) \in \mathcal{J}(P_{\boldsymbol{\nu}''})$ *for the lattice homomorphism* $\psi : \mathcal{R} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}''})$ *which is as defined in Theorem 3.4(2) for the partitions* $\boldsymbol{\lambda}'', \boldsymbol{\nu}''$. *Let* $J_1 = \phi(\tilde{J}) \in \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$, *where* $\phi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ *is the* *lower lattice homomorphism as defined in Proposition 3.11. Let* $S = \max(K) \cap [J]_{\lambda''}$ *and let* $S_1 = \{(a-1, \lambda-2) \mid (a,\lambda) \in S, \lambda > \lambda_{i_0}+1\} \cup \{(a,\lambda) \mid (a,\lambda) \in S, \lambda < \lambda_{i_0}+1\}$*. Let* $J_3 = \xi(\tilde{J}) \in \mathcal{J}(P_{\nu'/I_1})$*, where* $\xi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\nu'/I_1})$ *is the upper lattice homomorphism as defined in Proposition 3.12. Then we have*

1. $[K_1]_\nu \cup [J_1]_\nu = [K_1]_\nu \cup [J_3]_\nu = L_1$.
2. $\max(K_1) \cap [J_3]_{\nu''} \supseteq \max(K_1) \cap [J_1]_{\nu''} \supseteq S_1$.
3. $S_1 \cup \{(u, \lambda_{i_0}-1)\} \supseteq \max(K_1) \cap [J_1]_{\nu''} \supseteq S_1$ *when* $u < \lambda_{i_0}-1$ *and* $\{(u,\lambda_{i_0})\} = [\tilde{J}]_{(\lambda_{i_0})} = [K]_{(\lambda_{i_0})}$.
4. *When* $u = \lambda_{i_0}-1$ *and* $\{(u,\lambda_{i_0})\} = [\tilde{J}]_{(\lambda_{i_0})} = [K]_{(\lambda_{i_0})}$ *then* $\max(K_1) \cap [J_1]_{\nu''} = S_1$

Proof. We prove (1). We observe that if $\{(u,\lambda_{i_0})\} = [\tilde{J}]_{(\lambda_{i_0})} = [K]_{(\lambda_{i_0})}$ and $u < \lambda_{i_0}-1$ then $(u, \lambda_{i_0}-1) \in J_1 \cup K_1$. Hence using Remark 3.15 we get $J_1 \cup K_1 = J_3 \cup K_1$, that is, they generate the same ideal in the big fundamental poset P defined in 2.1. If $u = \lambda_{i_0}-1$ then we have $J_1 = J_3$, and therefore $J_1 \cup K_1 = J_3 \cup K_1$. Now we have $[\tilde{J}]_\lambda \cup [K]_\lambda = L$. We observe that $\psi([\tilde{J}]_\lambda) = [J_3 \cup \langle\{(u, \lambda_{i_0}-1)\}\rangle]_\nu, \psi([K]_\lambda) = [K_1]_\nu$. So we obtain $L_1 = \psi(L) = \psi([\tilde{J}]_\lambda \cup [K]_\lambda) = \psi([\tilde{J}]_\lambda) \cup \psi([K]_\lambda) = [J_3 \cup K_1]_\nu = [J_1 \cup K_1]_\nu$. Hence (1) follows.

We prove (2). Let $(a,\lambda) \in S, \lambda > \lambda_{i_0}+1$. Then $(a,\lambda) \in \max(K)$. If $(a,\lambda) \leq (v,\delta) \in \max(J)$ then either $\delta > \lambda_{i_0}+1$ or $\delta < \lambda_{i_0}-1$. We show that δ cannot be smaller than λ_{i_0}. Since $(a,\lambda), (u,\lambda_{i_0}) \in \max(K)$ we have $a > u, \lambda - a > \lambda_{i_0} - u$. Moreover, $(v,\delta), (u,\lambda_{i_0}) \in \max(\tilde{J})$, hence $u > v, \lambda_{i_0} - u > \delta - v$. So we have $a > u > v, \lambda - a > \lambda_{i_0} - u > \delta - v$ which implies that (a,λ) and (v,δ) are not comparable with each other. So $\delta > \lambda_{i_0}$. If $\delta > \lambda_{i_0}$ then $(a-1,\lambda-2) \leq (v-1,\delta-2)$. Here $(a-1,\lambda-2) \in \max(K_1)$ and $(v-1,\delta-2) \in \max(J_1)$. So $(a-1,\lambda-2) \in \max(K_1) \cap [J_1]_{\nu''}$. Now consider the case $(a,\lambda) \in S, \lambda < \lambda_{i_0}-1$. Here also $(a,\lambda) \in \max(K)$. If $(a,\lambda) \leq (v,\delta) \in \max(J)$ then either $\delta > \lambda_{i_0}+1$ or $\delta < \lambda_{i_0}-1$. We show that δ cannot be bigger than λ_{i_0}. Since $(a,\lambda), (u,\lambda_{i_0}) \in \max(K)$ they are not comparable with each other. Since $(v,\delta), (u,\lambda_{i_0}) \in \max(\tilde{J})$ they are not also comparable with each other. So if $\delta > \lambda_{i_0}$ then (a,λ) and (v,δ) are not comparable with each other. So $\delta < \lambda_{i_0}$. If $\delta < \lambda_{i_0}$ and $(a,\lambda) \leq (v,\delta)$, then we have $(a,\lambda) \in \max(K_1), (v,\delta) \in \max(J_1)$ and therefore $(a,\lambda) \in \max(K_1) \cap [J_1]_{\nu''} \subseteq \max(K_1) \cap [J_3]_{\nu''}$. Hence $S_1 \subseteq \max(K_1) \cap [J_1]_{\nu''}$. This proves (2).

We prove (3) and (4). Now suppose $(a,\lambda) \in \max(K_1) \cap [J_1]_{\nu''}, \lambda > \lambda_{i_0}-1$ and $(a,\lambda) \leq (b,\delta) \in \max(J_1)$. Now either $\delta < \lambda_{i_0}-1$ or $\delta > \lambda_{i_0}-1$. If $\delta > \lambda_{i_0}-1$ then $(a+1,\lambda+2) \leq (b+1,\delta+2) \in \max(J)$ and $(a+1,\lambda+2) \in \max(K)$. So $(a,\lambda) \in S_1$. If $\delta < \lambda_{i_0}-1$ then $(b,\delta) \in \max(J) \subseteq \max(\tilde{J})$. So

$u > b, \lambda_{i_0} - u > \delta - b$. Moreover, $(a+1, \lambda+2) \in \max(K)$, hence $a+1 > u, \lambda - a + 1 > \lambda_{i_0} - u$. So $a+1 > u > b, \lambda - a + 1 > \lambda_{i_0} - u > \delta - b$ and therefore $a > b, \lambda - a > \delta - b$, which implies that (a, λ) and (b, δ) are not comparable. So the case $\delta < \lambda_{i_0} - 1$ does not occur. Now assume that $\lambda < \lambda_{i_0} - 1$ and $(a, \lambda) \leq (b, \delta) \in \max(J_1)$. Here again either $\delta < \lambda_{i_0} - 1$ or $\delta > \lambda_{i_0} - 1$. If $\delta < \lambda_{i_0} - 1$ then $(a, \lambda) \in \max(K), (b, \delta) \in \max(J)$ and $(a, \lambda) \leq (b, \delta)$ implying $(a, \lambda) \in S$, and thus $(a, \lambda) \in S_1$. If $\delta > \lambda_{i_0} - 1$ then $(b+1, \delta+2) \in \max(J) \subseteq \max(\tilde{J})$. So $b+1 > u, \delta - b + 1 > \lambda_{i_0} - u$. Moreover, $(a, \lambda) \in \max(K)$, from which it follows that $u > a$ and $\lambda_{i_0} - u > \lambda - a$. Therefore we have $b+1 > u > a$, $\delta - b + 1 > \lambda_{i_0} - u > \lambda - a$ and therefore $b > a$, $\delta - b > \lambda - a$, which implies that (a, λ) and (b, δ) are not comparable. So the case $\delta > \lambda_{i_0} - 1$ does not occur. Now we assume that $\lambda = \lambda_{i_0} - 1$. Then $(a, \lambda) = (u, \lambda_{i_0} - 1) \in \max(K_1)$, implying $u < \lambda_{i_0} - 1$. So we get

$$S_1 \cup \{(u, \lambda_{i_0} - 1)\} \supseteq \max(K_1) \cap [J_1]_{\nu''} \supseteq S_1 \text{ if } u < \lambda_{i_0} - 1.$$

Otherwise if $u = \lambda_{i_0} - 1$ then $S_1 = \max(K_1) \cap [J_1]_{\nu''}$. This proves (3), (4).

Hence the proposition follows. ∎

Proposition 3.19. *Let* $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ *and* λ_{i_0} *be a part of* $\boldsymbol{\lambda}$ *for some* $1 \leq i_0 \leq k$ *with* $\rho_{i_0} = 2$. *Let*

$$\boldsymbol{\nu} = ((\lambda_1 - 2)^{\rho_1} > (\lambda_2 - 2)^{\rho_2} > \cdots > (\lambda_{i_0-1} - 2)^{\rho_{i_0-1}} \geq (\lambda_{i_0} - 1)^{\rho_{i_0}} \\ \geq \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}).$$

Let $\mathcal{M} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. *For* $I \in \mathcal{M} \subseteq \mathcal{J}(P_{\boldsymbol{\lambda}})$ *let* $I_1 = \psi(I)$, *where* ψ *is as defined in Theorem 3.4(2). Correspondingly we have partitions* $\boldsymbol{\lambda}', \boldsymbol{\lambda}'', \boldsymbol{\lambda}'/I, \boldsymbol{\nu}', \boldsymbol{\nu}'', \boldsymbol{\nu}'/I_1$. *Let* $\boldsymbol{\lambda}'''$ *be the partition obtained by inserting a new part* λ_{i_0} *into the partition* $\boldsymbol{\lambda}'/I$. *Let* $\mathcal{N} = \{\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''}) \mid \max(\tilde{J}) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. *Let* $\mathcal{R} = \{T \in \mathcal{J}(P_{\boldsymbol{\lambda}''}) \mid \max(T) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. *Let* $J_4 \in \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ *and* $K_2 \in \mathcal{J}(P_{\boldsymbol{\nu}''})$. *Let* $L_1 = [J_4]_{\boldsymbol{\nu}} \cup [K_2]_{\boldsymbol{\nu}} \in \mathcal{J}(P_{\boldsymbol{\nu}})$. *Let* $L \in \mathcal{M}$ *be such that* $\psi(L) = L_1$, *where the lattice homomorphism* $\psi : \mathcal{M} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}})$ *is as defined in Theorem 3.4(2). Let* $[L]_{(\lambda_{i_0})} = \{(u, \lambda_{i_0})\} \in \max(L)$. *Let* $K \in \mathcal{R} \subseteq \mathcal{J}(P_{\boldsymbol{\lambda}''})$ *be such that* $K = \psi^{-1}(K_2) \cup [\langle\{(u, \lambda_{i_0})\}\rangle]_{\boldsymbol{\lambda}''}$. *Let* $K_1 = \psi(K) \in \mathcal{J}(P_{\boldsymbol{\nu}''})$, *where the lattice homomorphism* $\psi : \mathcal{R} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}''})$ *is as defined in Theorem 3.4(2) for the partitions* $\boldsymbol{\lambda}'', \boldsymbol{\nu}''$. *Let* $J_5 = \zeta(J_4) \in \mathcal{N}$, *where the lattice homomorphism* $\zeta : \mathcal{J}(P_{\boldsymbol{\nu}'/I_1}) \longrightarrow \mathcal{N}$ *is as defined in Proposition 3.16. Let* $\tilde{J} = J_5 \cup [\langle\{(u, \lambda_{i_0})\}\rangle]_{\boldsymbol{\lambda}'''}$. *Then* $\tilde{J} \in \mathcal{N}$. *Let* $J \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I})$ *be such that* $\max(J) = \max(\tilde{J}) \backslash \{(u, \lambda_{i_0})\}$. *Let* $J_1 \in \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ *be such that* $\phi(\tilde{J}) = J_1$. *Let* $J_3 = \xi(\tilde{J}) \in \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$, *where the upper lattice homomorphism* $\xi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ *is as defined in Proposition 3.12. Let* $S = \max(K) \cap [J]_{\boldsymbol{\lambda}''}$ *and* *let* $S_1 = \{(a-1,\lambda-2) \mid (a,\lambda) \in S, \lambda > \lambda_{i_0}\} \cup \{(a,\lambda) \mid (a,\lambda) \in S, \lambda < \lambda_{i_0}\}$. *Then we have*

(1) $(u,\lambda_{i_0}) \in \max(K)$.
(2) $(u,\lambda_{i_0}) \in \max(\tilde{J})$.
(3) $J_1 \subseteq J_4 \subseteq J_3$.
(4) $K_2 \subseteq K_1$.
(5) $\max(K_2) \cap [J_4]_{\nu''} \supseteq S_1$.
(6) $[K_1]_\nu \cup [J_1]_\nu = L_1$.
(7) *Additional consequences are*
   - $[K_1]_\nu \cup [J_4]_\nu = [K_1]_\nu \cup [J_3]_\nu = L_1$.
   - $\max(K_1) \cap [J_3]_{\nu''} \supseteq \max(K_1) \cap [J_4]_{\nu''} \supseteq \max(K_1) \cap [J_1]_{\nu''} \supseteq S_1$.
   - $\max(K_2) \cap [J_3]_{\nu''} \supseteq S_1$.

Proof. We prove (1). $(u,\lambda_{i_0}) \in \max(L), L = \psi^{-1}([J_4]_\nu) \cup \psi^{-1}([K_2]_\nu)$. Hence $(u,\lambda_{i_0}) \in \max(\psi^{-1}([K_2]_\nu) \cup [\langle\{(u,\lambda_{i_0})\}\rangle]_\lambda) = \max([K]_\lambda)$. Now λ_{i_0} is a part of λ'' as well. Hence $(u,\lambda_{i_0}) \in \max(K)$. This proves (1).

We prove (2). First observe that $\lambda_{i_0}-1$ is a part of ν. Hence $[[J_4]_\nu]_{(\lambda_{i_0}-1)} = [J_4]_{(\lambda_{i_0}-1)}$. Suppose $u < \lambda_{i_0}-1$ and $\max([L_1]_{(\lambda_{i_0}-1)}) = \{(u,\lambda_{i_0}-1)\}$. Then either $\max([J_4]_{(\lambda_{i_0}-1)})$ is empty or $\{(b,\lambda_{i_0}-1)\} = \max([J_4]_{(\lambda_{i_0}-1)}) \leq \{(u,\lambda_{i_0}-1)\}$, that is, $b \geq u$. If $u = \lambda_{i_0}-1$ then $\max([L_1]_{(\lambda_{i_0}-1)})$ is empty and also $\max([J_4]_{(\lambda_{i_0}-1)})$ is empty. So if $\max([J_4]_{(\lambda_{i_0}-1)})$ is empty then $\max([J_5])_{(\lambda_{i_0})} = \{(\lambda_{i_0}-1,\lambda_{i_0})\}$ and $u \leq \lambda_{i_0}-1$. If $\max([J_4]_{(\lambda_{i_0}-1)}) = \{(b,\lambda_{i_0}-1)\}$ then $\max([J_5])_{(\lambda_{i_0})} = \{(b,\lambda_{i_0})\}$ and $u \leq b$. Now $J_5 \in \mathcal{N}$. Hence $\tilde{J} \in \mathcal{N}$ and $(u,\lambda_{i_0}) \in \max(\tilde{J})$. This proves (2).

We prove (3). Since $J_5 \subset \tilde{J}$, we have from Equations (3.14) and (3.15)

$$\begin{aligned}
\zeta(J_4) \subseteq \tilde{J} \text{ if and only if } & [\psi^{-1}([J_4]_{\nu'''})]_{\lambda'''} \subseteq \tilde{J} \text{ by definition,}\\
& \text{if and only if } \psi^{-1}([J_4]_{\nu'''}) \subseteq [\tilde{J}]_{\lambda''''} \text{ under isomorphism,}\\
& \text{if and only if } [J_4]_{\nu'''} \subseteq \psi([\tilde{J}]_{\lambda''''}) \text{ under isomorphism,}\\
& \text{implying } [[J_4]_{\nu'''}]_{\nu'/I_1} \subseteq [\psi([\tilde{J}]_{\lambda''''})]_{\nu'/I_1},\\
& \text{implying } J_4 \subseteq J_3.
\end{aligned}$$

The last implication follows because $J_4 \in \mathcal{J}(P_{\nu'/I_1}), \nu'/I_1 \subseteq \nu'''$ and hence $J_4 = [[J_4]_{\nu'''}]_{\nu'/I_1}$ and we have $J_3 = [\psi([\tilde{J}]_{\lambda''''})]_{\nu'/I_1}$ by definition.

Now for any part μ_i of ν'/I_1, let $\max([J_4]_{(\mu_i)}) = \{(b_i,\mu_i)\}$ if nonempty and choose $b_i = \mu_i$ if $\max([J_4]_{(\mu_i)})$ is empty. Then we have for the ideal $J_5 \in \mathcal{J}(P_{\lambda'''})$, if $\mu_i \geq \lambda_{i_0}-1$ then $\max([J_5]_{(\mu_i+2)}) = \{(b_i+1,\mu_i+2)\}$ and if $\mu_i \leq \lambda_{i_0}-1$ then

$\max([J_5]_{(\mu_i)}) = \max([J_4]_{(\mu_i)})$. We also have $\max([J_5]_{(\lambda_{i_0})}) = \{(b, \lambda_{i_0})\}$, where $b = b_i$ such that $\mu_i = \lambda_{i_0} - 1$. Therefore we have

$$\max(J_5) \subseteq \{(b_i + 1, \mu_i + 2) \mid \mu_i \geq \lambda_{i_0} - 1\} \cup \{(b, \lambda_{i_0})\} \\ \cup \{(b_i, \mu_i) \mid \mu_i \leq \lambda_{i_0} - 1, b_i < \mu_i\} \subseteq J_5.$$

So we have

$$(u, \lambda_{i_0}) \in \max(\tilde{J}) \subseteq \{(b_i + 1, \mu_i + 2) \mid \mu_i > \lambda_{i_0} - 1\} \cup \{(u, \lambda_{i_0})\} \\ \cup \{(b_i, \mu_i) \mid \mu_i < \lambda_{i_0} - 1, b_i < \mu_i\} \subseteq \tilde{J}.$$

So

$$\max(J) = \max(\tilde{J}) \backslash \{(u, \lambda_{i_0})\} \subseteq \{(b_i + 1, \mu_i + 2) \mid \mu_i > \lambda_{i_0} - 1\} \\ \cup \{(b_i, \mu_i) \mid \mu_i < \lambda_{i_0} - 1, b_i < \mu_i\}$$

which generates an ideal contained in J_5. Therefore

$$\max(J_1) \subseteq \{(b_i, \mu_i) \mid \mu_i > \lambda_{i_0} - 1\} \cup \{(b_i, \mu_i) \mid \mu_i < \lambda_{i_0} - 1, b_i < \mu_i\}$$

which generates an ideal contained in J_4. So $J_1 \subseteq J_4$. This proves (3).

We prove (4). Since $K = \psi^{-1}(K_2) \cup [\langle\{(u, \lambda_{i_0})\}\rangle]_{\lambda''}$ and $K_1 = \psi(K)$ we have $K_2 \subseteq K_1$. This proves (4).

We prove (5). Let $(a, \lambda) \in S, \lambda > \lambda_{i_0} + 1$. Then $(a, \lambda) \in [J]_{\lambda''}$, which implies that λ is a part of $\boldsymbol{\lambda}''$ and $(a, \lambda) \leq (b, \mu)$ for some $(b, \mu) \in \max(J)$. Now $(a, \lambda), (u, \lambda_{i_0}) \in \max(K)$ and hence are not comparable. Moreover, (u, λ_{i_0}), $(b, \mu) \in \max(\tilde{J})$ are not comparable. Hence if $\mu < \lambda_{i_0} - 1$ then we have $a > u > b, \lambda - a > \lambda_{i_0} - u > \mu - b$. So (a, λ) and (b, μ) are not comparable. Therefore $\mu > \lambda_{i_0} + 1$ and $(b - 1, \mu - 2) \in \max(J_1)$. Then $(a - 1, \lambda - 2) \leq (b - 1, \mu - 2)$. So $(a - 1, \lambda - 2) \in [J_1]_{\nu''}$. If $(a, \lambda) \in \max(K)$ and $\lambda > \lambda_{i_0} + 1$ then (a, λ) is not comparable with (u, λ_{i_0}). We have $K = \psi^{-1}(K_2) \cup [\langle\{(u, \lambda_{i_0})\}\rangle]_{\lambda''}$. So $(a, \lambda) \in \psi^{-1}(K_2)$, which implies $(a - 1, \lambda - 2) \in \max(K_1) \cap K_2 \subseteq \max(K_2)$. Hence $(a - 1, \lambda - 2) \in \max(K_2) \cap [J_4]_{\nu''}$. Now let $(a, \lambda) \in S, \lambda < \lambda_{i_0} - 1$. Then $(a, \lambda) \in [J]_{\lambda''}$, which implies that λ is a part of $\boldsymbol{\lambda}''$ and $(a, \lambda) \leq (b, \mu)$ for some $(b, \mu) \in \max(J)$. Now $(a, \lambda), (u, \lambda_{i_0}) \in \max(K)$ and hence are not comparable. Moreover, $(u, \lambda_{i_0}), (b, \mu) \in \max(\tilde{J})$ are not comparable. Hence if $\mu > \lambda_{i_0} - 1$ then we have $b > u > a, \mu - b > \lambda_{i_0} - u > \lambda - a$. So (a, λ) and (b, μ) are not comparable. Therefore $\mu < \lambda_{i_0} - 1$ and $(b, \mu) \in \max(J_1)$. So $(a, \lambda) \leq (b, \mu)$ implies $(a, \lambda) \in [J_1]_{\nu''}$. If $(a, \lambda) \in \max(K)$ and $\lambda < \lambda_{i_0} + 1$ then (a, λ) is not comparable with (u, λ_{i_0}). We have $K = \psi^{-1}(K_2) \cup [\langle\{(u, \lambda_{i_0})\}\rangle]_{\lambda''}$. So $(a, \lambda) \in \psi^{-1}(K_2)$, and therefore $(a,\lambda) \in \max(K_1) \cap K_2 \subseteq \max(K_2)$. Hence $(a,\lambda) \in \max(K_2) \cap [J_4]_{\nu''}$. This implies that $S_1 \subseteq \max(K_2) \cap [J_4]_{\nu''}$. This proves (5).

We prove (6). We observe that $[K]_\lambda \cup [J]_\lambda = [K]_\lambda \cup [\tilde{J}]_\lambda = L$. So using Proposition 3.18(1), we obtain $[K_1]_\nu \cup [J_1]_\nu = L_1$.

Now the additional consequences in (7) also follow. This completes the proof of the proposition. ∎

Proposition 3.20. *Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ and λ_{i_0} be a part of $\boldsymbol{\lambda}$ for some $1 \leq i_0 \leq k$ with $\rho_{i_0} = 2$. Let*

$$\boldsymbol{\nu} = \big((\lambda_1 - 2)^{\rho_1} > (\lambda_2 - 2)^{\rho_2} > \cdots > (\lambda_{i_0-1} - 2)^{\rho_{i_0-1}} \geq (\lambda_{i_0} - 1)^{\rho_{i_0}} \geq \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\big).$$

Let $\mathcal{M} = \{I \in \mathcal{J}(P_{\boldsymbol{\lambda}}) \mid \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\}$. For $I \in \mathcal{M} \subseteq \mathcal{J}(P_{\boldsymbol{\lambda}})$ let $I_1 = \psi(I)$, where ψ is the lattice homomorphism defined in Theorem 3.4(2). Correspondingly we have partitions $\boldsymbol{\lambda}', \boldsymbol{\lambda}'', \boldsymbol{\lambda}'/I, \boldsymbol{\nu}', \boldsymbol{\nu}'', \boldsymbol{\nu}'/I_1$.

(1)
$$\frac{|\mathcal{A}_{\boldsymbol{\lambda}'}|}{|\mathcal{A}_{\boldsymbol{\lambda}'/I}|} = q\frac{|\mathcal{A}_{\boldsymbol{\nu}'}|}{|\mathcal{A}_{\boldsymbol{\nu}'/I_1}|}.$$

(2) *For $L \in \mathcal{M}$ and $L_1 = \psi(L) \in \mathcal{J}(P_{\boldsymbol{\nu}})$, we have*
$$|(\mathcal{A}_{\boldsymbol{\lambda}})_L| = q^{\#(\textit{parts of } \boldsymbol{\lambda} \textit{ greater than or equal to } \lambda_{i_0})}\left|(\mathcal{A}_{\boldsymbol{\nu}})_{L_1}\right|.$$

Proof. We prove (1). In Equations (3.9)–(3.13), we observe that
$$|\boldsymbol{\lambda}'| - \left|\boldsymbol{\lambda}'/I\right| = t_1 - v_1 = 1 + |\boldsymbol{\nu}'| - |\boldsymbol{\nu}'/I_1|.$$

We prove (2). We observe that if L corresponds to
$$(\mathcal{A}_{\boldsymbol{\lambda}})_L = \bigoplus_{i=1}^{i_0-1}\left(\pi^{r_i}(\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i} \oplus \pi^{r_{i_0}}(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})^{\rho_{i_0}}\right) \oplus \bigoplus_{i=i_0+1}^{k} \pi^{r_i}(\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i}$$
then L_1 corresponds to
$$(\mathcal{A}_{\boldsymbol{\nu}})_{L_1} = \bigoplus_{i=1}^{i_0-1}\pi^{r_i-1}\left((\mathcal{R}/\pi^{\lambda_i-2}\mathcal{R})^{\rho_i} \oplus \pi^{r_{i_0}}(\mathcal{R}/\pi^{\lambda_{i_0}-1}\mathcal{R})^{\rho_{i_0}}\right) \oplus \bigoplus_{i=i_0+1}^{k} \pi^{r_i}(\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i}.$$

So
$$|(\mathcal{A}_{\boldsymbol{\lambda}})_L| = q^{\sum_{i=1}^{i_0}\rho_i}\left|(\mathcal{A}_{\boldsymbol{\nu}})_{L_1}\right| = q^{\#(\text{parts of } \boldsymbol{\lambda} \text{ greater than or equal to } \lambda_{i_0})}\left|(\mathcal{A}_{\boldsymbol{\nu}})_{L_1}\right|.$$

This proves the proposition. ∎

3.2.3. **Proof of Theorem 3.8.**

Proof. We consider the case when $\max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset, I \in \mathcal{J}(P_{\boldsymbol{\lambda}})$. These ideals are in bijection with the ideals in $\mathcal{J}(P_{\boldsymbol{\nu}})$ using Theorem 3.4(2). We have to prove the identity in Equation (3.3).

We consider the scenario

$$\{(v, \lambda_{i_0})\} = \max([J]_{(\lambda_{i_0})}) < \max([K]_{(\lambda_{i_0})}) = \{(u, \lambda_{i_0})\}, \text{ that is, } v > u$$

or

$$\emptyset = \max([J]_{(\lambda_{i_0})}) < \max([K]_{(\lambda_{i_0})}) = \{(u, \lambda_{i_0})\},$$

where $J \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I}), K \in \mathcal{J}(P_{\boldsymbol{\lambda}''})$ is such that $\max(K) \cap P_{(\lambda_{i_0})} \neq \emptyset$. Let $\boldsymbol{\lambda}'''$ be the partition obtained by inserting the part λ_{i_0} into $\boldsymbol{\mu}'/I$. Let $\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''})$ be an ideal such that $(u, \lambda_{i_0}) \in \max(\tilde{J})$. Let $J \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I})$ be such that $\max(J) = \max(\tilde{J}) \backslash \{(u, \lambda_{i_0})\}$. Then clearly $\max([J]_{(\lambda_{i_0})}) < \max([K]_{(\lambda_{i_0})}) = \{(u, \lambda_{i_0})\}$. Now consider ideals $J_2 \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I})$ such that $J \subseteq J_2 \subseteq \tilde{J} \backslash P_{(\lambda_{i_0})} \subsetneq \tilde{J}$. Note that for every ideal $J_2 \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I})$ such that $\max([J_2]_{(\lambda_{i_0})}) < \max([K]_{(\lambda_{i_0})}) = \{(u, \lambda_{i_0})\}$ there is a unique ideal $\tilde{J}$ with $(u, \lambda_{i_0}) \in \max(\tilde{J})$ such that $J \subseteq J_2 \subseteq \tilde{J} \backslash P_{(\lambda_{i_0})} \subsetneq \tilde{J}$, where $\max(J) = \max(\tilde{J}) \backslash \{(u, \lambda_{i_0})\}$. The unique ideal $\tilde{J} \in \mathcal{J}(P_{\boldsymbol{\lambda}'''})$ is given by $\tilde{J} = [J_2]_{\boldsymbol{\lambda}'''} \cup [\langle \{(u, \lambda_{i_0})\} \rangle]_{\boldsymbol{\lambda}'''}$.

For such an ideal J_2 we have $[J_2 \cup K]_{\boldsymbol{\lambda}'} = [J \cup K]_{\boldsymbol{\lambda}'}$ since $J \cup K, J_2 \cup K$ define the same ideal in the big fundamental poset P defined in Equation (2.1). We also have $\max(K) \backslash [J_2]_{\boldsymbol{\lambda}''} = \max(K) \backslash [J]_{\boldsymbol{\lambda}''}$. So for all $J \subseteq J_2 \subseteq \tilde{J} \backslash P_{(\lambda_{i_0})}$

$$\begin{aligned} \alpha^{\boldsymbol{\lambda}}_{I,J_2,K}(q) &= \alpha^{\boldsymbol{\lambda}}_{I,J,K}(q) \\ &= \left|(\mathcal{A}_{\boldsymbol{\lambda}})_{J \cup K}\right| \prod_{\lambda \in \boldsymbol{\lambda}'' \text{ such that there exists } (u,\lambda) \in \max(K) \backslash [J]_{\boldsymbol{\lambda}''}} \left(1 - \frac{1}{q^{m(\lambda)}}\right), \end{aligned}$$

where $m(\lambda)$ is the multiplicity of the part λ in $\boldsymbol{\lambda}''$.

Now we observe that

$$\begin{aligned} \bigsqcup_{J \subseteq J_2 \subseteq \tilde{J} \backslash P_{(\lambda_{i_0})}} \left|X^{\boldsymbol{\lambda}}_{I,J_2,K}\right| &= \frac{|\mathcal{A}_{\boldsymbol{\lambda}'}|}{|\mathcal{A}_{\boldsymbol{\lambda}'/I}|} \left|(\mathcal{A}_{\boldsymbol{\lambda}''})^*_K\right| \left(\sum_{J \subseteq J_2 \subseteq \tilde{J} \backslash P_{(\lambda_{i_0})}} \left|(\mathcal{A}_{\boldsymbol{\lambda}'/I})^*_{J_2}\right|\right) \\ &= \frac{|\mathcal{A}_{\boldsymbol{\lambda}'}|}{|\mathcal{A}_{\boldsymbol{\lambda}'/I}|} \left|(\mathcal{A}_{\boldsymbol{\lambda}''})^*_K\right| \frac{\left|(\mathcal{A}_{\boldsymbol{\lambda}'''})^*_{\tilde{J}}\right|}{\left(q^{\lambda_{i_0} - u} - q^{\lambda_{i_0} - u - 1}\right)}. \end{aligned}$$

Therefore we get

$$\sum_{J\subseteq J_2\subseteq \tilde{J}\backslash P_{(\lambda_{i_0})}} \frac{\left|X^{\lambda}_{I,J_2,K}\right|}{\alpha^{\lambda}_{I,J_2,K}(q)} = \frac{\frac{|\mathcal{A}_{\lambda'}|}{|\mathcal{A}_{\lambda'/I}|}\frac{1}{\left|(\mathcal{A}_{\lambda})_{J\cup K}\right|}\frac{\left|(\mathcal{A}_{\lambda'''})^*_{\tilde{J}}\right|}{\left(q^{\lambda_{i_0}-u}-q^{\lambda_{i_0}-u-1}\right)}\left|(\mathcal{A}_{\lambda''})^*_K\right|}{\prod\limits_{\lambda\in\lambda''\text{ such that there exists }(u,\lambda)\in\max(K)\backslash[J]_{\lambda''}}\left(1-\frac{1}{q^{m(\lambda)}}\right)}$$

$$= \frac{|\mathcal{A}_{\lambda'}|}{|\mathcal{A}_{\lambda'/I}|}\frac{1}{\left|(\mathcal{A}_{\lambda})_{J\cup K}\right|}\frac{\left|(\mathcal{A}_{\lambda'''})^*_{\tilde{J}}\right|}{\left(q^{\lambda_{i_0}-u}-q^{\lambda_{i_0}-u-1}\right)}\left|(\mathcal{A}_{\lambda''})_K\right| \times \prod_{\lambda\in\lambda''\text{ such that there exists }(u,\lambda)\in\max(K)\cap[J]_{\lambda''}}\left(1-\frac{1}{q^{m(\lambda)}}\right).$$

Let $I_1 = \psi(I) \in \mathcal{J}(P_\nu)$, where the lattice homomorphism $\psi : \mathcal{M} = \{T \in \mathcal{J}(P_\lambda) \mid \max(T)\cap P_{(\lambda_{i_0})} \neq \emptyset\} \longrightarrow \mathcal{J}(P_\nu)$ is as defined in Theorem 3.4(2). Let $J_4 \in \mathcal{J}(P_{\nu'/I_1}), K_2 \in \mathcal{J}(P_{\nu''})$ be any two ideals. Then using Proposition 3.19, there are unique ideals $\tilde{J} \in \mathcal{N} = \{T \in \mathcal{J}(P_{\lambda'''}) \mid \max(T)\cap P_{(\lambda_{i_0})} \neq \emptyset\} \subsetneq \mathcal{J}(P_{\lambda'''})$ and $K \in \mathcal{R} = \{T \in \mathcal{J}(P_{\lambda''}) \mid \max(T)\cap P_{(\lambda_{i_0})} \neq \emptyset\} \subsetneq \mathcal{J}(P_{\lambda''})$ such that $\max([\tilde{J}]_{\lambda_{i_0}}) = \max([K]_{(\lambda_{i_0})})$.

Now we begin with an ideal $\tilde{J} \in \mathcal{N}$ and an ideal $K \in \mathcal{R}$ such that $\max([\tilde{J}]_{\lambda_{i_0}}) = \max([K]_{(\lambda_{i_0})}) = \{(u,\lambda_{i_0})\}$. Let $J \in \mathcal{J}(P_{\lambda'/I})$ be such that $\max(J) = \max(\tilde{J})\backslash P_{(\lambda_{i_0})}$, $[J\cup K]_\lambda = L$. Let $L_1 = \psi(L) \in \mathcal{J}(P_\nu)$ and $K_1 = \psi(K) \in \mathcal{J}(P_{\nu''})$. Let $J_1 = \phi(\tilde{J}) \in \mathcal{J}(P_{\nu'/I_1})$. Let $S = \max(K)\cap[J]_{\lambda''}$ and let

$$S_1 = \{(a-1,\lambda-2) \mid (a,\lambda)\in S, \lambda > \lambda_{i_0}\} \cup \{(a,\lambda) \mid (a,\lambda)\in S, \lambda < \lambda_{i_0}\}.$$

Let $J_3 = \xi(\tilde{J})$, where the upper lattice homomorphism $\xi : \mathcal{N} \longrightarrow \mathcal{J}(P_{\nu'/I_1})$ is as defined in Proposition 3.12. Let $J_4 \in \mathcal{J}(P_{\nu'/I_1})$ denote any ideal such that $J_1 \subseteq J_4 \subseteq J_3$. Let $K_2 \in \mathcal{J}(P_{\nu''})$ denote any ideal such that $K_2 \subseteq K_1$. Then we

prove the following:
(3.16)

$$\frac{|\mathcal{A}_{\lambda'}|}{|\mathcal{A}_{\lambda'/I}|}\frac{1}{|(\mathcal{A}_{\lambda})_L|}\frac{\left|(\mathcal{A}_{\lambda'''})^*_{\tilde{J}}\right|}{\left(q^{\lambda_{i_0}-u}-q^{\lambda_{i_0}-u-1}\right)}\left|(\mathcal{A}_{\lambda''})_K\right| \times \prod_{\lambda\in\lambda'' \text{ such that there exists } (u,\lambda)\in\max(K)\cap[J]_{\lambda''}}\left(1-\frac{1}{q^{m(\lambda)}}\right)$$

$$=\frac{|\mathcal{A}_{\nu'}|}{|\mathcal{A}_{\nu'/I_1}|}\frac{1}{|(\mathcal{A}_{\nu})_{L_1}|} \times \sum_{\substack{J_1\subseteq J_4\subseteq J_3, K_2\subseteq K_1\\ [J_4\cup K_2]_\nu=L_1\\ \max(K_2)\cap[J_4]_{\nu''}\supseteq S_1}}\left(\left|\left(\mathcal{A}_{\nu'/I_1}\right)^*_{J_4}\right|\left|(\mathcal{A}_{\nu''})_{K_2}\right|\prod_{\substack{\lambda\in\nu''\\ \text{such that there exists}\\ (u,\lambda)\in\max(K_2)\cap[J_4]_{\nu''}}}\left(1-\frac{1}{q^{m(\lambda)}}\right)\right).$$

We have $[J_4\cup K_2]_\nu = L_1$, implying $[J_4]_{\nu''}\cup K_2 = [L_1]_{\nu''}$. We also have $[J_4]_{\nu''}\cup K_1 = [L_1]_{\nu''}$. Hence $[L_1]_{\nu''}\backslash[J_4]_{\nu''} = K_1\backslash[J_4]_{\nu''} = K_2\backslash[J_4]_{\nu''}$. Now we show that $\max(K_1)\backslash[J_4]_{\nu''} = \max(K_2)\backslash[J_4]_{\nu''}$. Since $K_2\subseteq K_1$ we have $\max(K_1)\cap K_2\subseteq\max(K_2)$. Now $(c,\mu)\in\max(K_1)\backslash[J_4]_{\nu''}$, hence $(c,\mu)\in K_1\backslash[J_4]_{\nu''} = K_2\backslash[J_4]_{\nu''}$. So $(c,\mu)\in\max(K_1)\cap K_2\subseteq\max(K_2)$, from which it follows that $(c,\mu)\in\max(K_2)\backslash[J_4]_{\nu''}$. Hence $\max(K_1)\backslash[J_4]_{\nu''}\subseteq\max(K_2)\backslash[J_4]_{\nu''}$. Now $(c,\mu)\in\max(K_2)\backslash[J_4]_{\nu''}$ and $[J_4]_{\nu''}\cup K_2 = [L_1]_{\nu''}$, therefore $(c,\mu)\in\max([L_1]_{\nu''})\backslash[J_4]_{\nu''}$. Now $[J_4]_{\nu''}\cup K_1 = [L_1]_{\nu''}$, implying $(c,\mu)\in\max(K_1)\backslash[J_4]_{\nu''}$. So $\max(K_1)\backslash[J_4]_{\nu''} = \max(K_2)\backslash[J_4]_{\nu''}$.

Now let $K_3\in\mathcal{J}(P_{\nu''})$ be such that $\max(K_3) = \max(K_1)\backslash[J_4]_{\nu''}\cup S_1$. Note that $S_1\subseteq\max(K_1)\cap[J_4]_{\nu''}$ using Proposition 3.19(7). Hence $\max(K_3)\subseteq\max(K_1)$ is indeed an antichain. Since L_1 is generated by $[J_4]_\nu$ and $\max(K_1)\backslash[J_4]_\nu = \max(K_1)\backslash[J_4]_{\nu''} = \max(K_3)\backslash[J_4]_{\nu''} = \max(K_3)\backslash[J_4]_\nu$ we have $L_1 = [K_3]_\nu\cup[J_4]_\nu$. We also observe that $\max(K_3)\cap[J_4]_{\nu''}\supseteq S_1$, in fact equal to S_1. Now let $K_2\in\mathcal{J}(P_{\nu''})$ be any ideal such that $K_3\subseteq K_2\subseteq K_1$. Then we have $L_1 = [K_2]_\nu\cup[J_4]_\nu$ and $(\mathcal{A}_{\nu''})^*_{K_3}\subseteq(\mathcal{A}_{\nu''})^*_{K_2}\subseteq(\mathcal{A}_{\nu''})^*_{K_1}$. So $\max(K_3)\subseteq\max(K_1)$, and therefore $\max(K_3)\subseteq\max(K_2)$. So $\max(K_2)\cap[J_4]_{\nu''}\supseteq\max(K_3)\cap[J_4]_{\nu''}\supseteq S_1$.

Conversely if $K_2\subseteq K_1$ such that $L_1 = [K_2]_\nu\cup[J_4]_\nu$ and $\max(K_2)\cap[J_4]_{\nu''}\supseteq S_1$ then $\max(K_2)\supseteq S_1\cup\max(K_1)\backslash[J_4]_{\nu''} = \max(K_3)$. So we have for a fixed J_4 such that $J_1 \subseteq J_4 \subseteq J_3$,

$$\sum_{\substack{K_2 \subseteq K_1 \\ [J_4 \cup K_2]_{\boldsymbol{\nu}} = L_1 \\ \max(K_2) \cap [J_4]_{\boldsymbol{\nu}''} \supseteq S_1}} \left(\left|(\mathcal{A}_{\boldsymbol{\nu}'/I_1})^*_{J_4}\right| \left|(\mathcal{A}_{\boldsymbol{\nu}''})_{K_2}\right| \prod_{\substack{\lambda \in \boldsymbol{\nu}'' \\ \text{such that there exists} \\ (u,\lambda) \in \max(K_2) \cap [J_4]_{\boldsymbol{\nu}''}}} \left(1 - \frac{1}{q^{m(\lambda)}}\right) \right)$$
$$= \left|(\mathcal{A}_{\boldsymbol{\nu}'/I_1})^*_{J_4}\right| \sum_{K_3 \subseteq K_2 \subseteq K_1} \left(\left|(\mathcal{A}_{\boldsymbol{\nu}''})_{K_2}\right| \prod_{\substack{\lambda \in \boldsymbol{\nu}'' \\ \text{such that there exists} \\ (u,\lambda) \in \max(K_2) \cap [J_4]_{\boldsymbol{\nu}''}}} \left(1 - \frac{1}{q^{m(\lambda)}}\right) \right)$$
$$= \left|(\mathcal{A}_{\boldsymbol{\nu}'/I_1})^*_{J_4}\right| \sum_{K_3 \subseteq K_2 \subseteq K_1} \left(\frac{\left|(\mathcal{A}_{\boldsymbol{\nu}''})^*_{K_2}\right|}{\prod_{\substack{\lambda \in \boldsymbol{\nu}'' \\ \text{such that there exists} \\ (u,\lambda) \in \max(K_2) \setminus [J_4]_{\boldsymbol{\nu}''}}} \left(1 - \frac{1}{q^{m(\lambda)}}\right)} \right)$$
$$= \left|(\mathcal{A}_{\boldsymbol{\nu}'/I_1})^*_{J_4}\right| \sum_{K_3 \subseteq K_2 \subseteq K_1} \left(\frac{\left|(\mathcal{A}_{\boldsymbol{\nu}''})^*_{K_2}\right|}{\prod_{\substack{\lambda \in \boldsymbol{\nu}'' \\ \text{such that there exists} \\ (u,\lambda) \in \max(K_1) \setminus [J_4]_{\boldsymbol{\nu}''}}} \left(1 - \frac{1}{q^{m(\lambda)}}\right)} \right)$$
$$= \frac{\left|(\mathcal{A}_{\boldsymbol{\nu}'/I_1})^*_{J_4}\right|}{\left(\prod_{\substack{\lambda \in \boldsymbol{\nu}'' \\ \text{such that there exists} \\ (u,\lambda) \in \max(K_1) \setminus [J_4]_{\boldsymbol{\nu}''}}} \left(1 - \frac{1}{q^{m(\lambda)}}\right) \right)} \sum_{K_3 \subseteq K_2 \subseteq K_1} \left(\left|(\mathcal{A}_{\boldsymbol{\nu}''})^*_{K_2}\right| \right).$$

Now $\max(K_3) \subseteq \max(K_2) \cap \max(K_1)$ implies that

$$\sum_{K_3 \subseteq K_2 \subseteq K_1} \left|(\mathcal{A}_{\boldsymbol{\nu}''})^*_{K_2}\right| = \left|(\mathcal{A}_{\boldsymbol{\nu}''})_{K_1}\right| \left(\prod_{\substack{\lambda \in \boldsymbol{\nu}'' \\ \text{such that there exists} \\ (u,\lambda) \in \max(K_3)}} \left(1 - \frac{1}{q^{m(\lambda)}}\right) \right).$$

Hence

$$\frac{\left|(\mathcal{A}_{\nu'/I_1})^*_{J_4}\right|}{\left(\prod\limits_{\substack{\lambda\in\nu''\\ \text{such that there exists}\\ (u,\lambda)\in\max(K_1)\backslash[J_4]_{\nu''}}}\left(1-\frac{1}{q^{m(\lambda)}}\right)\right)}\sum_{K_3\subseteq K_2\subseteq K_1}\left(\left|(\mathcal{A}_{\nu''})^*_{K_2}\right|\right)$$

$$=\frac{\left|(\mathcal{A}_{\nu'/I_1})^*_{J_4}\right|\left|(\mathcal{A}_{\nu''})_{K_1}\right|}{\left(\prod\limits_{\substack{\lambda\in\nu''\\ \text{such that there exists}\\ (u,\lambda)\in\max(K_1)\backslash[J_4]_{\nu''}}}\left(1-\frac{1}{q^{m(\lambda)}}\right)\right)}\left(\prod_{\substack{\lambda\in\nu''\\ \text{such that there exists}\\ (u,\lambda)\in\max(K_3)}}\left(1-\frac{1}{q^{m(\lambda)}}\right)\right)$$

$$=\left|(\mathcal{A}_{\nu'/I_1})^*_{J_4}\right|\left|(\mathcal{A}_{\nu''})_{K_1}\right|\left(\prod_{\substack{\lambda\in\nu''\\ \text{such that there exists}\\ (u,\lambda)\in S_1}}\left(1-\frac{1}{q^{m(\lambda)}}\right)\right).$$

Now $J_1\subseteq J_4\subseteq J_3$ and $\max(J_1)\subseteq\max(J_3)$, which imply $\max(J_1)\subseteq\max(J_4)$. Hence

$$\sum_{J_1\subseteq J_4\subseteq J_3}\left|(\mathcal{A}_{\nu'/I_1})^*_{J_4}\right|=\left|(\mathcal{A}_{\nu'/I_1})_{J_3}\right|\left(\prod_{\substack{\lambda\in\nu'/I_1\\ \text{such that there exists}\\ (u,\lambda)\in\max(J_1)}}\left(1-\frac{1}{q}\right)\right).$$

So

$$\frac{|\mathcal{A}_{\nu'}|}{|\mathcal{A}_{\nu'/I_1}|}\frac{1}{\left|(\mathcal{A}_{\nu})_{L_1}\right|}$$
$$\times\sum_{\substack{J_1\subseteq J_4\subseteq J_3, K_2\subseteq K_1\\ [J_4\cup K_2]_{\nu}=L_1\\ \max(K_2)\cap[J_4]_{\nu''}\supseteq S_1}}\left(\left|(\mathcal{A}_{\nu'/I_1})^*_{J_4}\right|\left|(\mathcal{A}_{\nu''})_{K_2}\right|\prod_{\substack{\lambda\in\nu''\\ \text{such that there exists}\\ (u,\lambda)\in\max(K_2)\cap[J_4]_{\nu''}}}\left(1-\frac{1}{q^{m(\lambda)}}\right)\right)$$
$$=\frac{|\mathcal{A}_{\nu'}|}{|\mathcal{A}_{\nu'/I_1}|}\frac{1}{\left|(\mathcal{A}_{\nu})_{L_1}\right|}\left|(\mathcal{A}_{\nu'/I_1})_{J_3}\right|\left(\prod_{\substack{\lambda\in\nu'/I_1\\ \text{such that there exists}\\ (u,\lambda)\in\max(J_1)}}\left(1-\frac{1}{q}\right)\right)$$
$$\times\left|(\mathcal{A}_{\nu''})_{K_1}\right|\left(\prod_{\substack{\lambda\in\nu''\\ \text{such that there exists}\\ (u,\lambda)\in S_1}}\left(1-\frac{1}{q^{m(\lambda)}}\right)\right)$$

$$= \frac{|\mathcal{A}_{\boldsymbol{\lambda}'}|}{|\mathcal{A}_{\boldsymbol{\lambda}'/I}|} \frac{q^{\substack{-1+\#(\text{parts of } \boldsymbol{\lambda} \\ \text{greater than or equal to } \lambda_{i_0})}}}{|(\mathcal{A}_{\boldsymbol{\lambda}})_L|} \left|(\mathcal{A}_{\boldsymbol{\nu}'/I_1})_{J_3}\right| \left(\prod_{\substack{\lambda \in \boldsymbol{\nu}'/I_1 \\ \text{such that there exists} \\ (u,\lambda)\in \max(J_1)}} \left(1-\frac{1}{q}\right)\right)$$

$$\times \left|(\mathcal{A}_{\boldsymbol{\nu}''})_{K_1}\right| \left(\prod_{\substack{\lambda \in \boldsymbol{\nu}'' \\ \text{such that there exists} \\ (u,\lambda)\in S_1}} \left(1-\frac{1}{q^{m(\lambda)}}\right)\right).$$

The last equality follows using Proposition 3.20.

Now $(u,\lambda) \in S$ if and only if $(u-1,\lambda-2) \in S_1$ if $\lambda > \lambda_{i_0}$ and $(u,\lambda) \in S_1$ if $\lambda < \lambda_{i_0}$. Here we observe that if $\lambda > \lambda_{i_0}$ then $m_{\boldsymbol{\lambda}''}(\lambda)$, the multiplicity of $\lambda \in \boldsymbol{\lambda}''$ equals $m_{\boldsymbol{\nu}''}(\lambda-2)$, the multiplicity of $(\lambda-2) \in \boldsymbol{\nu}''$ and if $\lambda < \lambda_{i_0}$ then $m_{\boldsymbol{\lambda}''}(\lambda) = m_{\boldsymbol{\nu}''}(\lambda)$.

Hence we have

$$\left(\prod_{\substack{\lambda \in \boldsymbol{\nu}'' \\ \text{such that there exists} \\ (u,\lambda)\in S_1}} \left(1-\frac{1}{q^{m(\lambda)}}\right)\right) = \left(\prod_{\substack{\lambda \in \boldsymbol{\lambda}'' \\ \text{such that there exists} \\ (u,\lambda)\in S}} \left(1-\frac{1}{q^{m(\lambda)}}\right)\right).$$

We also see that $(u,\lambda) \in \max(\tilde{J}), \lambda \neq \lambda_{i_0}, \lambda > \lambda_{i_0}+1$ if and only if $(u-1,\lambda-2) \in \max(J_1)$ and $(u,\lambda) \in \max(\tilde{J}), \lambda \neq \lambda_{i_0}, \lambda < \lambda_{i_0}-1$ if and only if $(u,\lambda) \in \max(J_1)$. Hence we have

$$\left(\prod_{\substack{\lambda \in \boldsymbol{\nu}'/I_1 \\ \text{such that there exists} \\ (u,\lambda)\in \max(J_1)}} \left(1-\frac{1}{q}\right)\right)$$

$$= \left(\prod_{\substack{\lambda \in \boldsymbol{\lambda}'/I \\ \text{such that there exists} \\ (u,\lambda)\in \max(J)}} \left(1-\frac{1}{q}\right)\right) = \frac{\left|(\mathcal{A}_{\boldsymbol{\lambda}'''})^*_{\tilde{J}}\right|}{\left|(\mathcal{A}_{\boldsymbol{\lambda}'''})_{\tilde{J}}\right| \left(1-\frac{1}{q}\right)}.$$

So to prove Equation (3.16), it is enough to prove the following:

$$q^{\substack{-1+\#(\text{parts of } \boldsymbol{\lambda} \\ \text{greater than or equal to } \lambda_{i_0})}} \left|(\mathcal{A}_{\boldsymbol{\nu}'/I_1})_{J_3}\right| \left|(\mathcal{A}_{\boldsymbol{\nu}''})_{K_1}\right| = \frac{\left|(\mathcal{A}_{\boldsymbol{\lambda}'''})_{\tilde{J}}\right|}{\left(q^{\lambda_{i_0}-u}\right)} \left|(\mathcal{A}_{\boldsymbol{\lambda}''})_K\right|. \tag{3.17}$$

From Equations (3.9) and (3.10) we get that the first $i-1$ parts of $\boldsymbol{\lambda}'/I$ are greater than $t_i = \lambda_{i_0}$ and the remaining parts are less than λ_{i_0}. In $\boldsymbol{\nu}'/I$ the first $i-1$ parts are also greater than $t_i - 1 = \lambda_{i_0} - 1$ and the remaining parts are less than $\lambda_{i_0} - 1$. If $\max([J_3]_{(\mu)}) = \{b, \mu\}$ with $\mu > \lambda_{i_0} - 1$ then $\max([\tilde{J}]_{(\mu+2)}) =$

$\{b+1,\mu+2\}$. If $\max([J_3]_{(\mu)})$ is empty for $\mu > \lambda_{i_0}-1$ then $\max([\tilde{J}]_{(\mu+2)}) = \{(\mu+1,\mu+2)\}$. If $\mu \leq \lambda_{i_0}-1$ then $\max([J_3]_{(\mu)}) = \max([\tilde{J}]_{(\mu)})$. So there is a decrement by one that occurs in exactly $i-1$ parts of J_3 with respect to $\tilde{J}$ because $\mu - b + 1 = (\mu+2)-(b+1)$. Now we observe that the first i parts of $\boldsymbol{\lambda}'$ are greater than or equal to λ_{i_0} and the remaining parts of $\boldsymbol{\lambda}'$ are less than λ_{i_0}. For the ideals K_1 and K, there is decrement by one that occurs in those parts of $\boldsymbol{\lambda}''$ which are greater than or equal to λ_{i_0}. So the total decrement is

$$\begin{aligned}
(i-1) &+ \#(\text{parts of } \boldsymbol{\lambda}'' \text{ greater than or equal to } \lambda_{i_0}) \\
&= \#(\text{parts of } \boldsymbol{\lambda}' \text{ greater than or equal to } \lambda_{i_0}) - 1 \\
&\qquad\qquad + \#(\text{parts of } \boldsymbol{\lambda}'' \text{ greater than or equal to } \lambda_{i_0}) \\
&= -1 + \#(\text{parts of } \boldsymbol{\lambda} \text{ greater than or equal to } \lambda_{i_0}).
\end{aligned}$$

In $(\mathcal{A}_{\boldsymbol{\lambda}'''})_{\tilde{J}}$ there is an extra component $\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})$ corresponding to (u,λ_{i_0}). Hence in this case from Equations (3.9) and (3.10), Equation (3.17) follows.

In Equation (3.11), a similar computation yields that

$$\frac{|(\mathcal{A}_{\boldsymbol{\nu}''})_{K_1}|}{q^{\lambda_{i_0}-u-1}} = \frac{|(\mathcal{A}_{\boldsymbol{\lambda}''})_K|}{q^{\#(\text{parts of } \boldsymbol{\lambda}'' \text{ greater than or equal to } \lambda_{i_0})}} \tag{3.18}$$

as there is an extra component $\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}-u-1}\mathcal{R})$ in K_1 since $\lambda_{i_0}-1 = t_i - 1$ is not a part of $\boldsymbol{\nu}'$.

In Equation (3.12) $\lambda_{i_0}-1$ is a part of $\boldsymbol{\lambda}'/I$. So a similar computation yields that

$$\left|(\mathcal{A}_{\boldsymbol{\nu}'/I_1})_{J_3}\right| = \frac{\left|(\mathcal{A}_{\boldsymbol{\lambda}'''})_{\tilde{J}}\right|}{q^{i-1}q^{\lambda_{i_0}-u}q^{\lambda_{i_0}-u-1}}, \tag{3.19}$$

where i is the number of parts of $\boldsymbol{\lambda}'$ that are greater than or equal to $\lambda_{i_0} = t_i$. Here in this case there are extra components $\pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R}) \oplus \pi^u(\mathcal{R}/\pi^{\lambda_{i_0}-1}\mathcal{R})$ in $(\mathcal{A}_{\boldsymbol{\lambda}'''})_{\tilde{J}}$.

In Equation (3.13), $\lambda_{i_0}+1$ is a part of $\boldsymbol{\lambda}'/I$. So a similar computation yields that

$$\left|(\mathcal{A}_{\boldsymbol{\nu}'/I_1})_{J_3}\right| = \frac{\left|(\mathcal{A}_{\boldsymbol{\lambda}'''})_{\tilde{J}}\right|}{q^{i-2}q^{\lambda_{i_0}-u}q^{\lambda_{i_0}-u}}, \tag{3.20}$$

where i is the number of parts of $\boldsymbol{\lambda}'$ that are greater than or equal to $\lambda_{i_0} = t_i$. Here in this case there are extra components $\pi^{u+1}(\mathcal{R}/\pi^{\lambda_{i_0}+1}\mathcal{R}) \oplus \pi^u(\mathcal{R}/\pi^{\lambda_{i_0}}\mathcal{R})$ in $(\mathcal{A}_{\boldsymbol{\lambda}'''})_{\tilde{J}}$.

Hence by combining Equations (3.19) and (3.20) individually with Equation (3.18) we get that Equation (3.17) follows in the case where $t_i - 1 = \lambda_{i_0} - 1$ is not a part of ν'. This completes the proof of Equation (3.16).

We observe that for $J_4 \in \mathcal{J}(P_{\nu'/I_1}), K_2 \in \mathcal{J}(P_{\nu''})$, we have

$$\frac{\left|X^{\nu}_{I_1,J_4,K_2}\right|}{\alpha^{\nu}_{I_1,J_4,K_2}} = \frac{|\mathcal{A}_{\nu'}|}{|\mathcal{A}_{\nu'/I_1}|}\frac{1}{|(\mathcal{A}_{\nu})_{L_1}|}\left(\left|(\mathcal{A}_{\nu'/I_1})^*_{J_4}\right| |(\mathcal{A}_{\nu''})_{K_2}| \prod_{\substack{\lambda\in\nu'' \\ \text{such that there exists} \\ (u,\lambda)\in\max(K_2)\cap[J_4]_{\nu''}}} \left(1-\frac{1}{q^{m(\lambda)}}\right)\right).$$

Therefore we have

$$\sum_{\substack{J_1\subseteq J_4\subseteq J_3, K_2\subseteq K_1 \\ [J_4\cup K_2]_\nu = L_1 \\ \max(K_2)\cap[J_4]_{\nu''}\supseteq S_1}} \frac{\left|X^{\nu}_{I_1,J_4,K_2}\right|}{\alpha^{\nu}_{I_1,J_4,K_2}} = \frac{|\mathcal{A}_{\nu'}|}{|\mathcal{A}_{\nu'/I_1}|}\frac{1}{|(\mathcal{A}_{\nu})_{L_1}|}$$
$$\times \sum_{\substack{J_1\subseteq J_4\subseteq J_3, K_2\subseteq K_1 \\ [J_4\cup K_2]_\nu = L_1 \\ \max(K_2)\cap[J_4]_{\nu''}\supseteq S_1}} \left(\left|(\mathcal{A}_{\nu'/I_1})^*_{J_4}\right| |(\mathcal{A}_{\nu''})_{K_2}| \prod_{\substack{\lambda\in\nu'' \\ \text{such that there exists} \\ (u,\lambda)\in\max(K_2)\cap[J_4]_{\nu''}}} \left(1-\frac{1}{q^{m(\lambda)}}\right)\right).$$

Using Proposition 3.19, we see that every $J_4 \in \mathcal{J}(P_{\nu'/I_1}), K_2 \in \mathcal{J}(P_{\nu''})$ appears in a unique such sum exactly once. Now we observe that summing all such equations over all possibilities for a fixed I_1 we get

$$\left(\sum_{J\in\mathcal{J}(P_{\nu'/I_1}), K\in\mathcal{J}(P_{\nu''})} \frac{\left|X^{\nu}_{I_1,J,K}\right|}{\alpha^{\nu}_{I_1,J,K}}\right).$$

Furthermore correspondingly summing the below expressions

$$\frac{|\mathcal{A}_{\lambda'}|}{|\mathcal{A}_{\lambda'/I}|}\frac{1}{|(\mathcal{A}_{\lambda})_L|}\frac{\left|(\mathcal{A}_{\lambda'''})^*_{\tilde{J}}\right|}{\left(q^{\lambda_{i_0}-u}-q^{\lambda_{i_0}-u-1}\right)}|(\mathcal{A}_{\lambda''})_K| \prod_{\substack{\lambda\in\lambda'' \\ \text{such that there exists} \\ (u,\lambda)\in\max(K)\cap[J]_{\lambda''}}} \left(1-\frac{1}{q^{m(\lambda)}}\right)$$

over all possibilities for a fixed $I \in \mathcal{J}(P_{\lambda})$ such that $\max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset$, we get

$$\left(\sum_{\substack{J\in\mathcal{J}(P_{\lambda'/I}), K\in\mathcal{J}(P_{\lambda''}) \\ \max([J]_{(\lambda_{i_0})})<\max([K]_{(\lambda_{i_0})}), \max(K)\cap P_{(\lambda_{i_0})}\neq\emptyset}} \frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}\right).$$

This gives us, under the association $I \in \mathcal{J}(P_{\lambda})$ such that $\max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset$ with $\psi(I) = I_1 \in \mathcal{J}(P_{\nu})$,

$$\left(\sum_{\substack{J \in \mathcal{J}(P_{\lambda'/I}), K \in \mathcal{J}(P_{\lambda''}) \\ \max([J]_{(\lambda_{i_0})}) < \max([K]_{(\lambda_{i_0})}), \max(K) \cap P_{(\lambda_{i_0})} \neq \emptyset}} \frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}\right) = \left(\sum_{J \in \mathcal{J}(P_{\nu'/I_1}), K \in \mathcal{J}(P_{\nu''})} \frac{\left|X^{\nu}_{I_1,J,K}\right|}{\alpha^{\nu}_{I_1,J,K}}\right). \tag{3.21}$$

So upon summation over $I \in \mathcal{J}(P_{\lambda})$ such that $\max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset$, we get the identity in Equation (3.3). This proves Theorem 3.8. ■

4. The polynomial $n^0_{\lambda}(q) = \left|\mathcal{G}_{\lambda} \backslash (\{\textit{Height Zero Elements}\} \times \mathcal{A}_{\lambda})\right|$ for a partition λ

We say an element $a \in \mathcal{A}_{\lambda}$ is of height zero if the equation $a = px$ has no solution for $x \in \mathcal{A}_{\lambda}$. We say an ideal $I \in \mathcal{J}(P_{\lambda})$ is a height zero ideal if one of the co-ordinates of e_I is 1, that is, $v_i = 0$ for some $1 \leq i \leq s$ in 2.3. Note in this case all elements of $(\mathcal{A}_{\lambda})^*_I$ are of height zero. Let $\mathcal{J}^0(P_{\lambda}) \subseteq \mathcal{J}(P_{\lambda})$ be the set of height zero ideals in $\mathcal{J}(P_{\lambda})$. Let

$$\mathcal{H}^0_{\lambda} = \bigsqcup_{I \in \mathcal{J}^0(P_{\lambda})} (\mathcal{A}_{\lambda})^*_I.$$

If $\lambda = \emptyset$ then $\mathcal{H}^0_{\lambda} = \emptyset$. Define for $\lambda \neq \emptyset$,

$$n^0_{\lambda}(q) = \left|\mathcal{G}_{\lambda} \backslash (\mathcal{H}^0_{\lambda} \times \mathcal{A}_{\lambda})\right|.$$

If $\lambda = \emptyset$ then define $n^0_{\lambda}(q) = 1$ and not zero for the sake of some consistency which we will see in this section. The following facts about $n^0_{\lambda}(q)$ can be deduced immediately.

(1) The number $n^0_{\lambda}(q)$ is a polynomial in q with integer coefficients (cf. [1, Thm. 6.4]).
(2) The degree of $n^0_{\lambda}(q)$ is λ_1, the highest part of λ. This can be deduced by adapting the proof of Theorem 5.11 in [1], using the maximal ideal generated by P_{λ} which is a height zero principal ideal. The polynomial $n^0_{\lambda}(q)$ is monic as well.
(3) The polynomial $n^0_{\lambda}(q) = n^0_{\lambda^2}(q)$, where

$$\lambda^2 = (\lambda_1^{\min(\rho_1,2)} > \lambda_2^{\min(\rho_2,2)} > \cdots > \lambda_k^{\min(\rho_k,2)})$$

(cf. [1, Cor. 4.5 and Rem. 4.7]). So, for the computation of the polynomial $n^0_{\boldsymbol{\lambda}}(q)$, any part of the partition which repeats more than twice can be reduced to two.

(4) An alternative expression for $n^0_{\boldsymbol{\lambda}}(q)$ for $\boldsymbol{\lambda} \neq \emptyset$ is given as

$$n^0_{\boldsymbol{\lambda}}(q) = \sum_{I \in \mathcal{J}^0(P_{\boldsymbol{\lambda}})} |(\mathcal{G}_{\boldsymbol{\lambda}})_I \backslash \mathcal{A}_{\boldsymbol{\lambda}}| = \sum_{I \in \mathcal{J}^0(P_{\boldsymbol{\lambda}})} \left(\sum_{J \in \mathcal{J}(P_{\boldsymbol{\lambda}'}/_I), K \in \mathcal{J}(P_{\boldsymbol{\lambda}''})} \frac{\left|X^{\boldsymbol{\lambda}}_{I,J,K}\right|}{\alpha^{\boldsymbol{\lambda}}_{I,J,K}} \right), \tag{4.1}$$

where $\alpha^{\boldsymbol{\lambda}}_{I,J,K}$ is cardinality of the valuative set

$$(\mathcal{A}_{\boldsymbol{\lambda}'})_{J \cup K} \oplus \left((\mathcal{A}_{\boldsymbol{\lambda}''})^*_K + (\mathcal{A}_{\boldsymbol{\lambda}''})_J\right) \subseteq \mathcal{A}_{\boldsymbol{\lambda}'} \oplus \mathcal{A}_{\boldsymbol{\lambda}''} = \mathcal{A}_{\boldsymbol{\lambda}}$$

and $X^{\boldsymbol{\lambda}}_{I,J,K} = \{(x', x'') \in \mathcal{A}_{\boldsymbol{\lambda}} \mid I(\overline{x}') = J, I(x'') = K\}$.

(5) We have $|\mathcal{G}_{\boldsymbol{\lambda}} \backslash (\pi\mathcal{A}_{\boldsymbol{\lambda}} \times \pi\mathcal{A}_{\boldsymbol{\lambda}})| = \left|\mathcal{G}_{\boldsymbol{\mu}} \backslash (\mathcal{A}_{\boldsymbol{\mu}} \times \mathcal{A}_{\boldsymbol{\mu}})\right|$, where $\boldsymbol{\mu}$ is the partition corresponding to the finite $\mathcal{R}$-module $\pi\mathcal{A}_{\boldsymbol{\lambda}}$. Therefore

$$|\mathcal{G}_{\boldsymbol{\lambda}} \backslash (\mathcal{A}_{\boldsymbol{\lambda}} \times \mathcal{A}_{\boldsymbol{\lambda}})| = \left|\mathcal{G}_{\boldsymbol{\mu}} \backslash (\mathcal{A}_{\boldsymbol{\mu}} \times \mathcal{A}_{\boldsymbol{\mu}})\right| + \left|\mathcal{G}_{\boldsymbol{\lambda}} \backslash (\mathcal{H}^0_{\boldsymbol{\lambda}} \times \mathcal{A}_{\boldsymbol{\lambda}})\right| + \left|\mathcal{G}_{\boldsymbol{\lambda}} \backslash (\pi\mathcal{A}_{\boldsymbol{\lambda}} \times \mathcal{H}^0_{\boldsymbol{\lambda}})\right|.$$

We also have

$$\begin{aligned}\left|\mathcal{G}_{\boldsymbol{\lambda}} \backslash (\pi\mathcal{A}_{\boldsymbol{\lambda}} \times \mathcal{H}^0_{\boldsymbol{\lambda}})\right| &= \sum_{I \in \left(\mathcal{J}(P_{\boldsymbol{\lambda}}) \backslash \mathcal{J}^0(P_{\boldsymbol{\lambda}})\right)} \left|(\mathcal{G}_{\boldsymbol{\lambda}})_I \backslash \mathcal{H}^0_{\boldsymbol{\lambda}}\right| \\ &= \sum_{I \in \left(\mathcal{J}(P_{\boldsymbol{\lambda}}) \backslash \mathcal{J}^0(P_{\boldsymbol{\lambda}})\right)} \left(\sum_{\substack{\text{either } J \in \mathcal{J}(P_{\boldsymbol{\lambda}'}/_I) \text{ is of height zero or} \\ K \in \mathcal{J}(P_{\boldsymbol{\lambda}''}) \text{ is of height zero}}} \frac{\left|X^{\boldsymbol{\lambda}}_{I,J,K}\right|}{\alpha^{\boldsymbol{\lambda}}_{I,J,K}} \right).\end{aligned}$$

So we have

$$\begin{aligned}n_{\boldsymbol{\lambda}}(q) &= n_{\boldsymbol{\mu}}(q) + n^0_{\boldsymbol{\lambda}}(q) + \left|\mathcal{G}_{\boldsymbol{\lambda}} \backslash (\pi\mathcal{A}_{\boldsymbol{\lambda}} \times \mathcal{H}^0_{\boldsymbol{\lambda}})\right| \\ &= n_{\boldsymbol{\mu}}(q) + n^0_{\boldsymbol{\lambda}}(q) \\ &\quad + \sum_{I \in \left(\mathcal{J}(P_{\boldsymbol{\lambda}}) \backslash \mathcal{J}^0(P_{\boldsymbol{\lambda}})\right)} \left(\sum_{(J,K) \in \mathcal{J}^0(P_{\boldsymbol{\lambda}'}/_I) \times \mathcal{J}(P_{\boldsymbol{\lambda}''}) \cup \mathcal{J}(P_{\boldsymbol{\lambda}'}/_I) \times \mathcal{J}^0(P_{\boldsymbol{\lambda}''})} \frac{\left|X^{\boldsymbol{\lambda}}_{I,J,K}\right|}{\alpha^{\boldsymbol{\lambda}}_{I,J,K}} \right).\end{aligned} \tag{4.2}$$

Remark 4.1. Here below is the table of values of $n^0_{\boldsymbol{\lambda}}(q)$ for partitions whose parts sum up to 6. From the table, we observe that $n^0_{(\lambda)} = q^{\lambda}$. So it is always true that the polynomial $n_{\boldsymbol{\lambda}}(q)$ can be expressed as an integer linear combination of $n^0_{\boldsymbol{\mu}}(q)$ for some partitions $\boldsymbol{\mu} = (\mu), \mu \geq 0$. We are not interested in the trivial manner of expression where we do not know, a priori, the coefficients of the summands whether they are positive or negative or zero. Most of the computations in the table are hand computations except for very few of them which are done using *Mathematica*.

The Partition $\boldsymbol{\lambda}$	The Height Zero Polynomial $n^0_{\boldsymbol{\lambda}}(q)$	The Orbit Polynomial $n_{\boldsymbol{\lambda}}(q)$
$\emptyset$	1(using definition)	1
(1)	q	$q+2$
(1^2)	$q+1$	$q+3$
(2)	q^2	q^2+2q+2
$(2>1)$	q^2+3q+1	q^2+5q+5
$(2>1^2)$	q^2+3q+2	q^2+5q+6
(2^2)	q^2+q+1	q^2+3q+5
$(2^2>1)$	q^2+4q+2	q^2+6q+8
$(2^2>1^2)$	q^2+4q+3	q^2+6q+9
(3)	q^3	q^3+2q^2+2q+2
$(3>1)$	q^3+3q^2+3q	q^3+5q^2+7q+4
$(3>1^2)$	q^3+3q^2+4q+2	q^3+5q^2+8q+6
$(3>2)$	q^3+3q^2+2q+1	$q^3+5q^2+10q+7$
$(3>2>1)$	q^3+6q^2+9q+3	$q^3+8q^2+19q+13$
(3^2)	q^3+q^2+q+1	q^3+3q^2+5q+7
(4)	q^4	$q^4+2q^3+2q^2+2q+2$
$(4>1)$	$q^4+3q^3+3q^2+2q$	$q^4+5q^3+7q^2+6q+4$
$(4>1^2)$	$q^4+3q^3+4q^2+4q+2$	$q^4+5q^3+8q^2+8q+6$
$(4>2)$	$q^4+3q^3+4q^2+q$	$q^4+5q^3+12q^2+11q+4$
(5)	q^5	$q^5+2q^4+2q^3+2q^2+2q+2$
$(5>1)$	$q^5+3q^4+3q^3+2q^2+2q$	$q^5+5q^4+7q^3+6q^2+6q+4$
(6)	q^6	$q^6+2q^5+2q^4+2q^3+2q^2+2q+2$

Now we state a theorem which may be used to express the polynomial $n_{\boldsymbol{\lambda}}(q)$ as a sum, in terms of $n^0_{\boldsymbol{\mu}}(q)$ for some suitable partitions $\boldsymbol{\mu}$, where we know that the coefficients of the summands are positive. If $\boldsymbol{\lambda}$ has distinct parts then parts of the suitable partitions $\boldsymbol{\mu}$ are also distinct.

Theorem 4.2. *Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ be a nonempty partition.*

(1) *Suppose $\lambda_k > 1$. Let $\boldsymbol{\mu} = \big((\lambda_1 - 1)^{\rho_1} > \cdots > (\lambda_k - 1)^{\rho_k}\big)$. Then we have*

$$n_{\boldsymbol{\lambda}}(q) = n^0_{\boldsymbol{\lambda}}(q) + n_{\boldsymbol{\mu}}(q) + n^0_{\boldsymbol{\mu}}(q).$$

(2) *Suppose $\lambda_k = 1$. Let $\boldsymbol{\mu} = \big((\lambda_1 - 1)^{\rho_1} > \cdots > (\lambda_{k-1} - 1)^{\rho_{k-1}}\big)$. Then we have*

$$n_{\boldsymbol{\lambda}}(q) = n^0_{\boldsymbol{\lambda}}(q) + 2n_{\boldsymbol{\mu}}(q).$$

In particular, if $\boldsymbol{\lambda}$ has distinct parts then $\boldsymbol{\mu}$ also has distinct parts in both cases.

We prove Theorem 4.2 after illustrating it in the following example.

Example 4.3.

$$\begin{aligned} n_{(3^2>2^2)}(q) &= n^0_{(3^2>2^2)}(q) + n_{(2^2>1^2)}(q) + n^0_{(2^2>1^2)}(q) \\ &= n^0_{(3^2>2^2)}(q) + n^0_{(2^2>1^2)}(q) + 2n_{(1^2)}(q) + n^0_{(2^2>1^2)}(q) \\ &= n^0_{(3^2>2^2)}(q) + 2n^0_{(2^2>1^2)}(q) + 2n_{(1^2)}(q) \\ &= n^0_{(3^2>2^2)}(q) + 2n^0_{(2^2>1^2)}(q) + 2(n^0_{(1^2)}(q) + 2n_{\emptyset}(q)) \\ &= n^0_{(3^2>2^2)}(q) + 2n^0_{(2^2>1^2)}(q) + 2n^0_{(1^2)}(q) + 4n^0_{\emptyset}(q). \end{aligned}$$

This identity can be verified by the actual polynomials

$$\begin{aligned} n_{(3^2>2^2)}(q) &= q^3 + 6q^2 + 14q + 15, \\ n^0_{(3^2>2^2)}(q) &= q^3 + 4q^2 + 4q + 3, \\ n^0_{(2^2>1^2)}(q) &= q^2 + 4q + 3, \\ n^0_{(1^2)}(q) &= q + 1, \\ n_{\emptyset}(q) &= 1 = n^0_{\emptyset}(q). \end{aligned}$$

Proof of Theorem 4.2. To prove this theorem we use Equation (4.2). To prove (1), it is enough to show that

$$n^0_\mu(q) = \left|\mathcal{G}_\mu \backslash (\mathcal{A}_\mu \times \mathcal{H}^0_\mu)\right| = \left|\mathcal{G}_\lambda \backslash (\pi\mathcal{A}_\lambda \times \mathcal{H}^0_\lambda)\right|.$$

We first show that $\left|\mathcal{G}_\lambda \backslash (\pi\mathcal{A}_\lambda \times \mathcal{H}^0_\lambda)\right| = \left|\mathcal{G}_\lambda \backslash (\pi\mathcal{A}_\lambda \times \pi\mathcal{H}^0_\lambda)\right|$. Let $x_1, y_1 \in \mathcal{A}_\lambda, x_2, y_2 \in \mathcal{H}^0_\lambda$. Suppose $g \in \mathcal{G}_\lambda$ is such that $g\pi x_1 = \pi y_1, gx_2 = y_2$. Then we have $g\pi x_1 = \pi y_1, g\pi x_2 = \pi y_2$. Conversely suppose $g\pi x_1 = \pi y_1, g\pi x_2 = \pi y_2$. Let $z = gx_2 - y_2$. Then we have $\pi z = 0$. So $z \in (\mathcal{A}_\lambda)_J$, where $J \in \mathcal{J}(P_\lambda)$ is an ideal such that $(\mathcal{A}_\lambda)_J = \bigoplus_{i=1}^k \pi^{\lambda_i - 1}(\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i} \subset \mathcal{A}_\lambda$. Note that x_2 is an element of height zero in $\mathcal{A}_\lambda$. Any height zero ideal, say K, (here K is the ideal of x_2) in the big fundamental poset P defined in Equation (2.1) contains the ideal generated by J in the big fundamental poset P. So there is an endomorphism $h \in \mathrm{Hom}(\mathcal{A}_\lambda, \bigoplus_{i=1}^k \pi^{\lambda_i - 1}(\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i})$ such that $hx_2 = z$. Here we have $\pi h = 0, (g-h)(x_2) = gx_2 - hx_2 = gx_2 - z = y_2$. Now note that, since $\lambda_i \geq \lambda_k > 1$ for $1 \leq i \leq k$ we also have $h \mod \pi = 0$. Hence we have $g \mod \pi = (g-h) \mod \pi$. Now any endomorphism $k \in \mathrm{End}(\mathcal{A}_\lambda)$ is invertible if and only $k \mod \pi$ is invertible. So the endomorphism $g - h \in \mathrm{End}(\mathcal{A}_\lambda)$ is invertible. Moreover $(g-h)\pi x_1 = gx_1 - \pi h x_1 = gx_1 = y_1$. This proves that $(\pi x_1, x_2)$ and $(\pi y_1, y_2)$ are in the same $\mathcal{G}_\lambda$ orbit if and only if $(\pi x_1, \pi x_2), (\pi y_1, \pi y_2)$ are in the same $\mathcal{G}_\lambda$ orbit. Hence we have $\left|\mathcal{G}_\lambda \backslash (\pi\mathcal{A}_\lambda \times \mathcal{H}^0_\lambda)\right| = \left|\mathcal{G}_\lambda \backslash (\pi\mathcal{A}_\lambda \times \pi\mathcal{H}^0_\lambda)\right|$. Since $\lambda_k > 1$, the set $\pi\mathcal{H}^0_\lambda$ is precisely the set of height zero elements in $\pi\mathcal{A}_\lambda \cong \mathcal{A}_\mu$, that is, $\pi\mathcal{H}^0_\lambda \cong \mathcal{H}^0_\mu$. Now we have $\left|\mathcal{G}_\lambda\backslash(\pi\mathcal{A}_\lambda \times \pi\mathcal{H}^0_\lambda)\right| = \left|\mathcal{G}_\mu\backslash(\mathcal{A}_\mu \times \mathcal{H}^0_\mu)\right|$ because the map $\mathcal{G}_\lambda \longrightarrow \mathcal{G}_\mu$ which takes g to $g_{|_{\pi\mathcal{A}_\lambda}|}$ is surjective. This proves (1).

Now we prove (2). Here $\lambda_k = 1$. In this case we have $\pi\mathcal{H}^0_\lambda = \pi\mathcal{A}_\lambda \cong \mathcal{A}_\mu$. Using Equation (4.2), it is enough to show that

$$n_\mu(q) = \left|\mathcal{G}_\mu\backslash(\mathcal{A}_\mu \times \mathcal{A}_\mu)\right| = \left|\mathcal{G}_\lambda\backslash(\pi\mathcal{A}_\lambda \times \mathcal{H}^0_\lambda)\right|.$$

The proof is similar to that of the previous case except a bit longer. Let $x_1, y_1 \in \mathcal{A}_\lambda, x_2, y_2 \in \mathcal{H}^0_\lambda$. Suppose $g \in \mathcal{G}_\lambda$ is such that $g\pi x_1 = \pi y_1, gx_2 = y_2$. Then we have $g\pi x_1 = \pi y_1, g\pi x_2 = \pi y_2$. Conversely suppose $g\pi x_1 = \pi y_1, g\pi x_2 = \pi y_2$. So $\pi x_2, \pi y_2$ are of the same height in $\pi\mathcal{A}_\lambda \cong \mathcal{A}_\mu$. Let t stand for transpose. Let the column vectors $x_2 = (x^t_{21}, x^t_{22}, \cdots, x^t_{2k})^t, y_2 = (y^t_{21}, y^t_{22}, \cdots, y^t_{2k})^t \in \bigoplus_{i=1}^k (\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i} = \mathcal{A}_\lambda$ with the column vectors $x_{2i} = (x^1_{2i}, x^2_{2i} \cdots, x^{\rho_i}_{2i})^t, y_{2i} = (y^1_{2i}, y^2_{2i} \cdots, y^{\rho_i}_{2i})^t \in (\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i}, 1 \leq i \leq k$. Let the column vector $gx_2 = ((gx)^t_{21}, (gx)^t_{22}, \cdots, (gx)^t_{2k})^t \in \bigoplus_{i=1}^k (\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i} = \mathcal{A}_\lambda$ with the column vectors $(gx)_{2i} = ((gx)^1_{2i}, (gx)^2_{2i} \cdots, (gx)^{\rho_i}_{2i})^t \in (\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i}$, $1 \leq i \leq k$. Let the column vector $z = gx_2 - y_2 = (z^t_1, z^t_2, \cdots, z^t_k)^t \in \bigoplus_{i=1}^k \pi^{\lambda_i - 1}(\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i}$.

Claim 4.4. *There exists $g_1 \in \mathcal{G}_\lambda$ such that $g_1\pi x_1 = \pi y_1, g_1\pi x_2 = \pi y_2$ and if the column vector $w = g_1x_2 - y_2 = (w^t_1, w^t_2, \cdots, w^t_k)^t \in \bigoplus_{i=1}^k \pi^{\lambda_i-1}(\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i}$ then $w_k = 0$.*

Proof of Claim. If z_k is already zero then we take $g = g_1$. If $z_k \neq 0$ then there are two cases to consider.

We consider the first case. Let x_{2i_0} contain a unit in one of its coordinates for some $1 \leq i_0 \leq k-1$. We choose a homomorphism $e = [e^{ij}]_{k\times k} : \mathcal{A}_\lambda \longrightarrow \mathcal{A}_\lambda$, where $e^{ij} = [e^{ij}_{mn}]_{\rho_i \times \rho_j} : (\mathcal{R}/\pi^{\lambda_j}\mathcal{R})^{\rho_j} \longrightarrow (\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i}$ such that e^{ij} is nonzero only when $i = k, j = i_0$ and $e^{ki_0}(x_{2i_0}) = z_k$. We observe that $\pi e = 0$ because $\pi e^{ki_0} = 0$. We have $i_0 \neq k$ and the diagonal blocks of $g - e$ are all invertible modulo π. Hence $g - e$ is invertible. Moreover we have $(g-e)\pi x_1 = g\pi x_1 = \pi y_1, (g-e)\pi x_2 = g\pi x_2 = \pi y_2, (g-e)x_2 - y_2 = (z^t_1, z^t_2, \cdots, z^t_{k-1}, \mathbf{0}^t)^t$. Hence choose $g_1 = g - e$.

We consider the second case. Let x_{2i} does not contain a unit in any of its coordinates for all $1 \leq i_0 \leq k-1$. So x_{2k} must contain a unit in one if its coordinates. Hence $(gx)_{2k}$ must also contain a unit in one of its coordinates. As $g\pi x_2 = \pi y_2, \pi x_2, \pi y_2$ are of the same height in $\pi\mathcal{A}_\lambda \cong \mathcal{A}_\mu$ and therefore y_{2i} also does not contain a unit in any of its coordinates for all $1 \leq i_0 \leq k-1$. So y_{2k} must contain a unit in one if its coordinates. Hence we choose an automorphism $e = [e^{ij}]_{k\times k} : \mathcal{A}_\lambda \longrightarrow \mathcal{A}_\lambda$, where $e^{ij} = [e^{ij}_{mn}]_{\rho_i\times\rho_j} : (\mathcal{R}/\pi^{\lambda_j}\mathcal{R})^{\rho_j} \longrightarrow (\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i}$ such that e^{ij} is the identity block for $1 \leq i = j \leq k-1$, e^{ij} is the zero block for $1 \leq$

$i \neq j \leq k$ and $e^{kk} : (\mathcal{R}/\pi^{\lambda_k}\mathcal{R})^{\rho_k} \longrightarrow (\mathcal{R}/\pi^{\lambda_k}\mathcal{R})^{\rho_k}$ is an automorphism such that $e^{kk}((gx)_{2k}) = y_{2k}$. Now we have $eg(x_2) - y_2 = (z_1^t, z_2^t, \cdots, z_{k-1}^t, \mathbf{0}^t)^t, eg(\pi x_1) = \pi y_1, eg(\pi x_2) = \pi y_2$. Hence choose $g_1 = eg$ which is an automorphism.

This proves the claim. ∎

Continuing with the proof of (2), we assume without loss of generality that $g\pi x_1 = \pi y_1, g\pi x_2 = \pi y_2$ and $gx_2 - y_2 = z = (z_1^t, z_2^t, \cdots, z_{k-1}^t, z_k^t)^t$ with $z_k = 0$ and $z \in (\mathcal{A}_\lambda)_J = \bigoplus_{i=1}^k \pi^{\lambda_i - 1}(\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i} \subset \mathcal{A}_\lambda$, where J is defined by its association to $(\mathcal{A}_\lambda)_J$. Note that x_2 is an element of height zero in $\mathcal{A}_\lambda$. Any height zero ideal, say K, (here K is the ideal of x_2) in the big fundamental poset P defined in Equation (2.1) contains the ideal generated by J in the big fundamental poset P. So there is an endomorphism $h \in \mathrm{Hom}(\mathcal{A}_\lambda, \bigoplus_{i=1}^k \pi^{\lambda_i-1}(\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i})$ such that $hx_2 = z$ and for this h we have $\pi h = 0$. Now we left multiply h by a homomorphism $f = [f^{ij}]_{k\times k} : \mathcal{A}_\lambda \longrightarrow \mathcal{A}_\lambda$, where $f^{ij} = [f^{ij}_{mn}]_{\rho_i \times \rho_j} : (\mathcal{R}/\pi^{\lambda_j}\mathcal{R})^{\rho_j} \longrightarrow (\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i}$ such that f^{ij} is the identity block for $1 \leq i = j \leq k-1$, f^{ij} is the zero block for $1 \leq i \neq j \leq k$ and $f^{kk} : (\mathcal{R}/\pi^{\lambda_k}\mathcal{R})^{\rho_k} \longrightarrow (\mathcal{R}/\pi^{\lambda_k}\mathcal{R})^{\rho_k}$ is also zero. Then we have $fh(x_2) = z$, the last row of blocks in fh are all zero blocks, the remaining blocks of fh are the same as that of h. So $\pi fh = 0$ and also $fh \mod \pi = 0$. Hence we have $g \mod \pi = (g - fh) \mod \pi$. So $g - fh$ is invertible. Furthermore, $(g - fh)(\pi x_1) = g\pi x_1 = \pi y_1, (g - fh)(x_2) - y_2 = z - fh(x_2) = z - z = 0$.

This proves that $(\pi x_1, x_2)$ and $(\pi y_1, y_2)$ are in the same $\mathcal{G}_\lambda$ orbit if and only if $(\pi x_1, \pi x_2), (\pi y_1, \pi y_2)$ are in the same $\mathcal{G}_\lambda$ orbit. Hence we have

$$\left|\mathcal{G}_\lambda \backslash (\pi\mathcal{A}_\lambda \times \mathcal{H}^0_\lambda)\right| = \left|\mathcal{G}_\lambda \backslash (\pi\mathcal{A}_\lambda \times \pi\mathcal{H}^0_\lambda)\right|$$
$$= |\mathcal{G}_\lambda \backslash (\pi\mathcal{A}_\lambda \times \pi\mathcal{A}_\lambda)| = \left|\mathcal{G}_\mu \backslash (\mathcal{A}_\mu \times \mathcal{A}_\mu)\right| = n_\mu(q).$$

This proves (2).

We have completed the proof of Theorem 4.2. ∎

Now we state a theorem which along with Theorem 4.2, can be used to express the polynomial $n^0_\lambda(q)$ for a partition λ with a repeated part as a sum, in terms of the polynomials $n^0_\mu(q)$ for some suitable partitions μ which has all their parts distinct, where we know that the coefficients of the summands are positive.

Theorem 4.5. *Let $\lambda = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ be a partition such that $\rho_{i_0} = 2$ for some $1 \leq i_0 \leq k$. Let*

$$\mu = \left(\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \cdots > \lambda_{i_0-1}^{\rho_{i_0-1}} > \lambda_{i_0}^{\rho_{i_0}-1} = \lambda_{i_0}^1 > \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\right).$$

Let

$$\nu = \big((\lambda_1-2)^{\rho_1} > (\lambda_2-2)^{\rho_2} > \cdots > (\lambda_{i_0-1}-2)^{\rho_{i_0-1}} \geq (\lambda_{i_0}-1)^{\rho_{i_0}} \\ \geq \lambda_{i_0+1}^{\rho_{i_0+1}} > \cdots > \lambda_{k-1}^{\rho_{k-1}} > \lambda_k^{\rho_k}\big).$$

(1) *Suppose* $\lambda_{i_0} > 1$. *Then we have*

$$n^0_\lambda(q) = n^0_\mu(q) + n^0_\nu(q).$$

(2) *Suppose* $\lambda_{i_0} = 1$ *which implies in particular* $i_0 = k$. *Then we have*

$$n^0_\lambda(q) = n^0_\mu(q) + n_\nu(q).$$

We prove Theorem 4.5 after illustrating it in the following example.

Example 4.6.

$$\begin{aligned} n^0_{(3^2>2^2)}(q) &= n^0_{(3>2^2)}(q) + n^0_{(2^2)}(q) \\ &= n^0_{(3>2)}(q) + n^0_{(1^2)(q)} + n^0_{(2)}(q) + n^0_{(1^2)(q)} \\ &= n^0_{(3>2)}(q) + n^0_{(2)}(q) + 2n^0_{(1^2)(q)} \\ &= n^0_{(3>2)}(q) + n^0_{(2)}(q) + 2(n^0_{(1)(q)} + n_\emptyset(q)) \\ &= n^0_{(3>2)}(q) + n^0_{(2)}(q) + 2n^0_{(1)}(q) + 2n^0_\emptyset(q). \end{aligned}$$

This identity can be verified by the actual polynomials

$$\begin{aligned} n^0_{(3^2>2^2)}(q) &= q^3 + 4q^2 + 4q + 3, \\ n^0_{(3>2)}(q) &= q^3 + 3q^2 + 2q + 1, \\ n^0_{(2)}(q) &= q^2, \\ n^0_{(1)}(q) &= q, \\ n_\emptyset(q) &= 1 = n^0_\emptyset(q). \end{aligned}$$

Proof of Theorem 4.5. To prove this theorem, we use the proof of Theorem Δ. We prove (1) and (2) simultaneously. We use the alternative expressions for $n^0_\lambda(q), n^0_\mu(q), n^0_\nu(q)$ as defined in Equation (4.1).

In Equation (3.1), if we sum over only height zero ideals $I \in \mathcal{J}^0(P_\lambda) = \mathcal{J}^0(P_\mu)$ such that $\max(I) \cap P_{(\lambda_{i_0})} = \emptyset$, then we get

(4.3)

$$\sum_{I\in\mathcal{J}^0(P_\lambda),\max(I)\cap P_{(\lambda_{i_0})}=\emptyset}\left(\sum_{J\in\mathcal{J}(P_{\lambda'/I}),K\in\mathcal{J}(P_{\lambda''})}\frac{\left|X^\lambda_{I,J,K}\right|}{\alpha^\lambda_{I,J,K}}\right) \\ = \sum_{I\in\mathcal{J}^0(P_\mu),\max(I)\cap P_{(\lambda_{i_0})}=\emptyset}\left(\sum_{J\in\mathcal{J}(P_{\mu'/I}),K\in\mathcal{J}(P_{\mu''})}\frac{\left|X^\mu_{I,J,K}\right|}{\alpha^\mu_{I,J,K}}\right).$$

In Equation (3.2), if we sum over only height zero ideals $I \in \mathcal{J}^0(P_\lambda) = \mathcal{J}^0(P_\mu)$ such that $\max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset$, then we get

$$\sum_{I \in \mathcal{J}^0(P_\lambda), \max(I)\cap P_{(\lambda_{i_0})} \neq \emptyset} \left(\sum_{\substack{J \in \mathcal{J}(P_{\lambda'/I}), K \in \mathcal{J}(P_{\lambda''}) \\ \max([J]_{(\lambda_{i_0})}) \geq \max([K]_{(\lambda_{i_0})}) \text{ or} \\ \max([J]_{(\lambda_{i_0})}) < \max([K]_{(\lambda_{i_0})}), \max(K)\cap P_{(\lambda_{i_0})} = \emptyset}} \frac{\left| X^\lambda_{I,J,K} \right|}{\alpha^\lambda_{I,J,K}} \right)$$
$$= \sum_{\substack{I \in \mathcal{J}^0(P_\mu) \\ \max(I)\cap P_{(\lambda_{i_0})} \neq \emptyset}} \left(\sum_{J \in \mathcal{J}(P_{\mu'/I}), K \in \mathcal{J}(P_{\mu''})} \frac{\left| X^\mu_{I,J,K} \right|}{\alpha^\mu_{I,J,K}} \right). \tag{4.4}$$

In Theorem 3.4(2), we have defined a lattice isomorphism $\psi : \mathcal{M} = \{I \in \mathcal{J}(P_\lambda) \mid \max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset\} \longrightarrow \mathcal{J}(P_\nu)$. If $\lambda_{i_0} = 1$, then $\mathcal{M} \subseteq \mathcal{J}^0(P_\lambda)$, since in this case, any ideal in $\mathcal{M}$ is of height zero. If $\lambda_{i_0} > 1$ then the we have a restricted lattice isomorphism ψ given as

$$\psi\big|_{\mathcal{M}\cap\mathcal{J}^0(P_\lambda)} : \mathcal{M} \cap \mathcal{J}^0(P_\lambda) \longrightarrow \mathcal{J}^0(P_\nu)$$

onto the set $\mathcal{J}^0(P_\nu)$ of height zero ideals.

Hence in Equations (3.21) and (3.3), if we sum over only height zero ideals $I \in \mathcal{J}^0(P_\lambda)$ such that $\max(I) \cap P_{(\lambda_{i_0})} \neq \emptyset$, then we get for $\lambda_{i_0} > 1$,

$$\sum_{I \in \mathcal{J}^0(P_\lambda), \max(I)\cap P_{(\lambda_{i_0})} \neq \emptyset} \left(\sum_{\substack{J \in \mathcal{J}(P_{\lambda'/I}), K \in \mathcal{J}(P_{\lambda''}) \\ \max([J]_{(\lambda_{i_0})}) < \max([K]_{(\lambda_{i_0})}), \max(K)\cap P_{(\lambda_{i_0})} \neq \emptyset}} \frac{\left| X^\lambda_{I,J,K} \right|}{\alpha^\lambda_{I,J,K}} \right)$$
$$= \sum_{I_1 \in \mathcal{J}^0(P_\nu)} \left(\sum_{J \in \mathcal{J}(P_{\nu'/I_1}), K \in \mathcal{J}(P_{\nu''})} \frac{\left| X^\nu_{I_1,J,K} \right|}{\alpha^\nu_{I_1,J,K}} \right) \tag{4.5}$$

and for $\lambda_{i_0} = 1$,

$$\sum_{I \in \mathcal{J}^0(P_\lambda), \max(I)\cap P_{(1)} \neq \emptyset} \left(\sum_{\substack{J \in \mathcal{J}(P_{\lambda'/I}), K \in \mathcal{J}(P_{\lambda''}) \\ \max([J]_{(1)}) < \max([K]_{(1)}), \max(K)\cap P_{(1)} \neq \emptyset}} \frac{\left| X^\lambda_{I,J,K} \right|}{\alpha^\lambda_{I,J,K}} \right)$$
$$= \sum_{I_1 \in \mathcal{J}(P_\nu)} \left(\sum_{J \in \mathcal{J}(P_{\nu'/I_1}), K \in \mathcal{J}(P_{\nu''})} \frac{\left| X^\nu_{I_1,J,K} \right|}{\alpha^\nu_{I_1,J,K}} \right). \tag{4.6}$$

So, upon summing Equations (4.3), (4.4), (4.5), we obtain (1), and, upon summing Equations (4.3), (4.4), (4.6), we obtain (2). This proves Theorem 4.5. ∎

Corollary 4.7. *Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ be a partition. Then the polynomial $n_{\boldsymbol{\lambda}}(q)$ is expressible as the sum of $n^0_{\boldsymbol{\mu}}(q)$ for partitions $\boldsymbol{\mu}$ which has all their parts distinct and $\boldsymbol{\mu} \subseteq \boldsymbol{\lambda}$, where we know that the coefficients of the summands are positive.*

Proof. By repeatedly using Theorem 4.2 we can express $n_{\boldsymbol{\lambda}}(q)$ as a sum of $n^0_{\boldsymbol{\mu}}(q)$ and $n_{\boldsymbol{\mu}}(q)$ for $\boldsymbol{\mu} \subseteq \boldsymbol{\lambda}$. Here while using Theorem 4.2 repetitions in the parts of partitions may not reduce. Now the repetitions in the parts can be reduced to unrepeated parts or distinct parts by repeatedly using Theorem 4.5. We also note that $n_{\emptyset}(q) = 1 = n^0_{\emptyset}(q)$. ∎

We illustrate the above corollary with an example.

Example 4.8.

$$\begin{aligned}
n_{(2^2>1^2)}(q) &= n^0_{(2^2>1^2)}(q) + 2n_{(1^2)}(q)\\
&= n^0_{(2^2>1^2)}(q) + 2(n^0_{(1^2)}(q) + 2n_{\emptyset}(q))\\
&= (n^0_{(2>1^2)}(q) + n^0_{(1^2)}(q)) + 2n^0_{(1^2)}(q) + 4n^0_{\emptyset}(q)\\
&= n^0_{(2>1^2)}(q) + 3n^0_{(1^2)}(q) + 4n^0_{\emptyset}(q)\\
&= (n^0_{(2>1)}(q) + n_{\emptyset}(q)) + 3(n^0_{(1)}(q) + n_{\emptyset}(q)) + 4n^0_{\emptyset}(q)\\
&= n^0_{(2>1)}(q) + 3n^0_{(1)}(q) + 8n^0_{\emptyset}(q).
\end{aligned}$$

This identity can be verified by the actual polynomials

$$\begin{aligned}
n_{(2^2>1^2)}(q) &= q^2 + 6q + 9,\\
n^0_{(2>1)}(q) &= q^2 + 3q + 1,\\
n^0_{(1)}(q) &= q,\\
n_{\emptyset}(q) &= 1 = n^0_{\emptyset}(q).
\end{aligned}$$

Now we state a theorem which may be used to express $n^0_{\boldsymbol{\lambda}}(q)$ for a partition $\boldsymbol{\lambda}$ which has all its parts distinct as a sum, in terms of $n^0_{\boldsymbol{\mu}}(q)$ for partitions $\boldsymbol{\mu}$ which has all its parts distinct such that $\boldsymbol{\mu} \subsetneq \boldsymbol{\lambda}$ with coefficients of the summands being polynomials in q with nonnegative integer coefficients.

Theorem 4.9. *Let $\boldsymbol{\lambda} = (\lambda_1 > \lambda_2 > \cdots > \lambda_k) \in \Lambda_0$ be a nonempty partition with distinct parts. Let*

$$\begin{aligned}
\boldsymbol{\mu} &= (\lambda_1 - 1 > \lambda_2 - 1 > \cdots > \lambda_k - 1),\\
\boldsymbol{\nu} &= (\lambda_1 - \lambda_k > \lambda_2 - \lambda_k > \cdots > \lambda_{k-1} - \lambda_k),\\
\boldsymbol{\delta} &= (\lambda_1 - \lambda_k - 1 > \lambda_2 - \lambda_k - 1 > \cdots > \lambda_{k-1} - \lambda_k - 1).
\end{aligned}$$

(1) *Suppose $k \geq 2$ and $\lambda_{k-1} - \lambda_k > 1$. Then we have*

$$n^0_{\boldsymbol{\lambda}}(q) = qn^0_{\boldsymbol{\mu}}(q) + n^0_{\boldsymbol{\nu}}(q) + n^0_{\boldsymbol{\delta}}(q) \text{ if } \lambda_k > 1$$
$$n^0_{\boldsymbol{\lambda}}(q) = qn_{\boldsymbol{\mu}}(q) + n^0_{\boldsymbol{\nu}}(q) + n^0_{\boldsymbol{\delta}}(q) \text{ if } \lambda_k = 1 \text{ (here } \boldsymbol{\mu} = \boldsymbol{\nu}).$$

(2) *Suppose $k \geq 2$ and $\lambda_{k-1} - \lambda_k = 1$. Then we have*

$$n^0_{\boldsymbol{\lambda}}(q) = qn^0_{\boldsymbol{\mu}}(q) + n^0_{\boldsymbol{\nu}}(q) + n_{\boldsymbol{\delta}}(q) \text{ if } \lambda_k > 1$$
$$n^0_{\boldsymbol{\lambda}}(q) = qn_{\boldsymbol{\mu}}(q) + n^0_{\boldsymbol{\nu}}(q) + n_{\boldsymbol{\delta}}(q) \text{ if } \lambda_k = 1 \text{ (here } \boldsymbol{\mu} = \boldsymbol{\nu}).$$

(3) *Suppose $k = 1, \lambda_1 = \lambda_k > 1$. Let $\boldsymbol{\mu} = (\lambda_1 - 1)$. Then we have*

$$n^0_{\boldsymbol{\lambda}}(q) = qn^0_{\boldsymbol{\mu}}(q).$$

We prove Theorem 4.9 after illustrating it in the following example and proving Theorems 4.11, 4.12, 4.13, 4.14, 4.15.

4.1. An Example Illustrating Theorem 4.9.

Example 4.10. (1) Examples for Theorem 4.9(1), $\lambda_k > 1, \lambda_{k-1} - \lambda_k > 1$:

- Let $\boldsymbol{\lambda} = (4 > 2)$. We have

  $$n^0_{(4>2)}(q) = qn^0_{(3>1)}(q) + n^0_{(2)}(q) + n^0_{(1)}(q).$$

  This identity holds as

  $$\begin{aligned} n^0_{(4>2)}(q) &= q^4 + 3q^3 + 4q^2 + q, \\ n^0_{(3>1)}(q) &= q^3 + 3q^2 + 3q, \\ n^0_{(2)}(q) &= q^2, \\ n^0_{(1)}(q) &= q. \end{aligned}$$

- Let $\boldsymbol{\lambda} = (5 > 4 > 2)$. We have

  $$n^0_{(5>4>2)}(q) = qn^0_{(4>3>1)}(q) + n^0_{(3>2)}(q) + n^0_{(2>1)}(q).$$

  This identity holds as

  $$\begin{aligned} n^0_{(5>4>2)}(q) &= q^5 + 6q^4 + 15q^3 + 16q^2 + 7q + 2, \\ n^0_{(4>3>1)}(q) &= q^4 + 6q^3 + 14q^2 + 12q + 2, \\ n^0_{(3>2)}(q) &= q^3 + 3q^2 + 2q + 1, \\ n^0_{(2>1)}(q) &= q^2 + 3q + 1. \end{aligned}$$

- Let $\boldsymbol{\lambda} = (6 > 4 > 2)$. We have

  $$n^0_{(6>4>2)}(q) = qn^0_{(5>3>1)}(q) + n^0_{(4>2)}(q) + n^0_{(3>1)}(q).$$

This identity holds as

$$\begin{aligned} n^0_{(6>4>2)}(q) &= q^6 + 6q^5 + 17q^4 + 22q^3 + 15q^2 + 4q, \\ n^0_{(5>3>1)}(q) &= q^5 + 6q^4 + 16q^3 + 18q^2 + 8q, \\ n^0_{(4>2)}(q) &= q^4 + 3q^3 + 4q^2 + q, \\ n^0_{(3>1)}(q) &= q^3 + 3q^2 + 3q. \end{aligned}$$

(2) Examples for Theorem 4.9(1), $\lambda_k = 1, \lambda_{k-1} - \lambda_k > 1$:

- Let $\boldsymbol{\lambda} = (3 > 1)$. We have

$$n^0_{(3>1)}(q) = qn_{(2)}(q) + n^0_{(2)}(q) + n^0_{(1)}(q).$$

This identity holds as

$$\begin{aligned} n^0_{(3>1)}(q) &= q^3 + 3q^2 + 3q, \\ n_{(2)}(q) &= q^2 + 2q + 2, \\ n^0_{(2)}(q) &= q^2, \\ n^0_{(1)}(q) &= q. \end{aligned}$$

- Let $\boldsymbol{\lambda} = (4 > 3 > 1)$. We have

$$n^0_{(4>3>1)}(q) = qn_{(3>2)}(q) + n^0_{(3>2)}(q) + n^0_{(2>1)}(q).$$

This identity holds as

$$\begin{aligned} n^0_{(4>3>1)}(q) &= q^4 + 6q^3 + 14q^2 + 12q + 2, \\ n_{(3>2)}(q) &= q^3 + 5q^2 + 10q + 7, \\ n^0_{(3>2)}(q) &= q^3 + 3q^2 + 2q + 1, \\ n^0_{(2>1)}(q) &= q^2 + 3q + 1. \end{aligned}$$

- Let $\boldsymbol{\lambda} = (5 > 3 > 1)$. We have

$$n^0_{(5>3>1)}(q) = qn_{(4>2)}(q) + n^0_{(4>2)}(q) + n^0_{(3>1)}(q).$$

This identity holds as

$$\begin{aligned} n^0_{(5>3>1)}(q) &= q^5 + 6q^4 + 16q^3 + 18q^2 + 8q, \\ n_{(4>2)}(q) &= q^4 + 5q^3 + 12q^2 + 11q + 4, \\ n^0_{(4>2)}(q) &= q^4 + 3q^3 + 4q^2 + q, \\ n^0_{(3>1)}(q) &= q^3 + 3q^2 + 3q. \end{aligned}$$

(3) Examples for Theorem 4.9(2), $\lambda_k > 1, \lambda_{k-1} - \lambda_k = 1$:

- Let $\boldsymbol{\lambda} = (3 > 2)$. We have
$$n^0_{(3>2)}(q) = qn^0_{(2>1)}(q) + n^0_{(1)}(q) + n_{\emptyset}(q).$$
This identity holds as
$$\begin{aligned} n^0_{(3>2)}(q) &= q^3 + 3q^2 + 2q + 1, \\ n^0_{(2>1)}(q) &= q^2 + 3q + 1, \\ n^0_{(1)}(q) &= q, \\ n_{\emptyset}(q) &= 1. \end{aligned}$$
- Let $\boldsymbol{\lambda} = (4 > 3 > 2)$. We have
$$n^0_{(4>3>2)}(q) = qn^0_{(3>2>1)}(q) + n^0_{(2>1)}(q) + n_{(1)}(q).$$
This identity holds as
$$\begin{aligned} n^0_{(4>3>2)}(q) &= q^4 + 6q^3 + 10q^2 + 7q + 3, \\ n^0_{(3>2>1)}(q) &= q^3 + 6q^2 + 9q + 3, \\ n^0_{(2>1)}(q) &= q^2 + 3q + 1, \\ n_{(1)}(q) &= q + 2. \end{aligned}$$
- Let $\boldsymbol{\lambda} = (5 > 3 > 2)$. We have
$$n^0_{(5>3>2)}(q) = qn^0_{(4>2>1)}(q) + n^0_{(3>1)}(q) + n_{(2)}(q).$$
This identity holds as
$$\begin{aligned} n^0_{(5>3>2)}(q) &= q^5 + 6q^4 + 12q^3 + 13q^2 + 7q + 2, \\ n^0_{(4>2>1)}(q) &= q^4 + 6q^3 + 11q^2 + 9q + 2, \\ n^0_{(3>1)}(q) &= q^3 + 3q^2 + 3q, \\ n_{(2)}(q) &= q^2 + 2q + 2. \end{aligned}$$

(4) Examples for Theorem 4.9(2), $\lambda_k = 1, \lambda_{k-1} - \lambda_k = 1$:
- Let $\boldsymbol{\lambda} = (2 > 1)$. We have
$$n^0_{(2>1)}(q) = qn_{(1)}(q) + n^0_{(1)}(q) + n_{\emptyset}(q).$$
This identity holds as
$$\begin{aligned} n^0_{(2>1)}(q) &= q^2 + 3q + 1, \\ n_{(1)}(q) &= q + 2, \\ n^0_{(1)}(q) &= q, \\ n_{\emptyset}(q) &= 1. \end{aligned}$$

- Let $\boldsymbol{\lambda} = (3 > 2 > 1)$. We have

$$n^0_{(3>2>1)}(q) = qn_{(2>1)}(q) + n^0_{(2>1)}(q) + n_{(1)}(q).$$

This identity holds as

$$\begin{aligned} n^0_{(3>2>1)}(q) &= q^3 + 6q^2 + 9q + 3, \\ n_{(2>1)}(q) &= q^2 + 5q + 5, \\ n^0_{(2>1)}(q) &= q^2 + 3q + 1, \\ n_{(1)}(q) &= q + 2. \end{aligned}$$

- Let $\boldsymbol{\lambda} = (4 > 2 > 1)$. We have

$$n^0_{(4>2>1)}(q) = qn_{(3>1)}(q) + n^0_{(3>1)}(q) + n_{(2)}(q).$$

This identity holds as

$$\begin{aligned} n^0_{(4>2>1)}(q) &= q^4 + 6q^3 + 11q^2 + 9q + 2, \\ n_{(3>1)}(q) &= q^3 + 5q^2 + 7q + 4, \\ n^0_{(3>1)}(q) &= q^3 + 3q^2 + 3q, \\ n_{(2)}(q) &= q^2 + 2q + 2. \end{aligned}$$

(5) Example for Theorem 4.9(3): Let $\boldsymbol{\lambda} = (2)$. We have

$$n^0_{(2)}(q) = qn^0_{(1)}(q).$$

This identity holds as $n^0_{(2)}(q) = q^2, n^0_{(1)}(q) = q$.

4.2. Theorems 4.11, 4.12, 4.13, 4.14, 4.15.

Theorem 4.11. *Let $\boldsymbol{\lambda} = (\lambda_1 > \lambda_2 > \cdots > \lambda_k) \in \Lambda_0$ be a nonempty partition with distinct parts. Let $\boldsymbol{\mu} = (\lambda_1 - 1 > \lambda_2 - 1 > \cdots > \lambda_k - 1)$. Let $I \in \mathcal{J}(P_{\boldsymbol{\lambda}}), J \in \mathcal{J}(P_{\boldsymbol{\lambda}'/I}), K \in \mathcal{J}(P_{\boldsymbol{\lambda}''})$. Define $X^{\boldsymbol{\lambda}}_{I,J,K} = \{(x', x'') \in \mathcal{A}_{\boldsymbol{\lambda}} \mid I(\overline{x}') = J, I(x'') = K\}$ and $\alpha^{\boldsymbol{\lambda}}_{I,J,K} = \left|(\mathcal{A}_{\boldsymbol{\lambda}'})_{J\cup K} \oplus \left((\mathcal{A}_{\boldsymbol{\lambda}''})^*_K + (\mathcal{A}_{\boldsymbol{\lambda}''})_J\right)\right|$. Let*

$$X^{\boldsymbol{\lambda}}_I = \bigsqcup_{J \in \left(\mathcal{J}(P_{\boldsymbol{\lambda}'/I}) \backslash \mathcal{J}^0(P_{\boldsymbol{\lambda}'/I})\right), K \in \left(\mathcal{J}(P_{\boldsymbol{\lambda}''}) \backslash \mathcal{J}^0(P_{\boldsymbol{\lambda}''})\right)} X^{\boldsymbol{\lambda}}_{I,J,K}$$

. Then we have

(a) $X^{\boldsymbol{\lambda}}_I = \mathcal{R}e_I + \pi\mathcal{A}_{\boldsymbol{\lambda}}$.

(b)

$$\sum_{I\in\mathcal{J}^0(P_\lambda)}\left|(\mathcal{G}_\lambda)_I\backslash X_I^\lambda\right| \tag{4.7}$$
$$= \sum_{I\in\mathcal{J}^0(P_\lambda)}\left(\sum_{J\in\left(\mathcal{J}(P_{\lambda'/I})\backslash\mathcal{J}^0(P_{\lambda'/I})\right),K\in\left(\mathcal{J}(P_{\lambda''})\backslash\mathcal{J}^0(P_{\lambda''})\right)}\frac{\left|X^\lambda_{I,J,K}\right|}{\alpha^\lambda_{I,J,K}}\right).$$

(c) $\left|(\mathcal{G}_\lambda)_I\backslash X_I^\lambda\right| = q\left|(\mathcal{G}_\lambda)_I\backslash\pi\mathcal{A}_\lambda\right|$.

(d) *If* $\lambda_k > 1$ *then*

$$\begin{aligned}\sum_{I\in\mathcal{J}^0(P_\lambda)}\left|(\mathcal{G}_\lambda)_I\backslash X_I^\lambda\right| &= q\sum_{I\in\mathcal{J}^0(P_\lambda)}\left|(\mathcal{G}_\lambda)_I\backslash\pi\mathcal{A}_\lambda\right| = q\left|\mathcal{G}_\lambda\backslash(\mathcal{H}^0_\lambda\times\pi\mathcal{A}_\lambda)\right|\\ &= q\left|\mathcal{G}_\lambda\backslash(\pi\mathcal{H}^0_\lambda\times\pi\mathcal{A}_\lambda)\right| = q\left|\mathcal{G}_\mu\backslash(\mathcal{H}^0_\mu\times\mathcal{A}_\mu)\right| = qn^0_\mu(q).\end{aligned}$$

(e) *If* $\lambda_k = 1$ *then*

$$\begin{aligned}\sum_{I\in\mathcal{J}^0(P_\lambda)}\left|(\mathcal{G}_\lambda)_I\backslash X_I^\lambda\right| &= q\sum_{I\in\mathcal{J}^0(P_\lambda)}\left|(\mathcal{G}_\lambda)_I\backslash\pi\mathcal{A}_\lambda\right| = q\left|\mathcal{G}_\lambda\backslash(\mathcal{H}^0_\lambda\times\pi\mathcal{A}_\lambda)\right|\\ &= q\left|\mathcal{G}_\lambda\backslash(\pi\mathcal{H}^0_\lambda\times\pi\mathcal{A}_\lambda)\right| = q\left|\mathcal{G}_\mu\backslash(\mathcal{A}_\mu\times\mathcal{A}_\mu)\right| = qn_\mu(q).\end{aligned}$$

Proof. We prove (a). We have

$$\begin{aligned}X_I^\lambda =&\{(x',x'')\in\mathcal{A}_{\lambda'}\oplus\mathcal{A}_{\lambda''}\mid \quad I(\overline{x}') = J\in\left(\mathcal{J}(P_{\lambda'/I})\backslash\mathcal{J}^0(P_{\lambda'/I})\right),\\ &\qquad I(x'') = K\in\left(\mathcal{J}(P_{\lambda''})\backslash\mathcal{J}^0(P_{\lambda''})\right)\}\\ &=\{(x',x'')\in\mathcal{A}_{\lambda'}\oplus\mathcal{A}_{\lambda''}\mid \quad \overline{x}'\in\pi\mathcal{A}_{\lambda'/I}, x''\in\pi\mathcal{A}_{\lambda''}\}\\ &=\{(x',x'')\in\mathcal{A}_{\lambda'}\oplus\mathcal{A}_{\lambda''}\mid \quad x'\in\mathcal{R}e'_I+\pi\mathcal{A}_{\lambda'}, x''\in\pi\mathcal{A}_{\lambda''}\}\\ &=\mathcal{R}e_I+\pi\mathcal{A}_\lambda.\end{aligned}$$

This proves (a).

We prove (b). Consider the sum in the right-hand side of Equation (4.7). This sum is exactly given by the action of $(\mathcal{G}_\lambda)_I$ on X_I^λ for all $I\in\mathcal{J}^0(P_\lambda)$. This proves (b).

We prove (c). To prove this we use the fact the residue field $\mathcal{R}/\pi\mathcal{R}$ is a field with exactly q elements. Let $s:\mathcal{R}/\pi\mathcal{R}\longrightarrow\mathcal{R}$ be a section for the projection map $\mathcal{R}\longrightarrow\mathcal{R}/\pi\mathcal{R}$. Assume $s(0)=0, s(1)=1$. Given a transitive $(\mathcal{G}_\lambda)_I$-orbit $O\subseteq\pi\mathcal{A}_\lambda$ we have q transitive $(\mathcal{G}_\lambda)_I$-orbits

$$s(a)e_I+O\subseteq\mathcal{R}e_I+\pi\mathcal{A}_\lambda \text{ for every } a\in\mathcal{R}/\pi\mathcal{R}.$$

This is because: For any $x,y\in\pi\mathcal{A}_\lambda, g\in(\mathcal{G}_\lambda)_I$ we have $gx=y$ if and only if $g(s(a)e_I+x)=s(a)e_I+y$. Moreeover, $s(a_1)e_I+O_1\cap s(a_2)e_I+O_2=\emptyset$ unless

$s(a_1) = s(a_2), O_1 = O_2$ which is if and only if $a_1 = a_2, O_1 = O_2$. We also observe that the orbit decomposition of X_I^{λ} is given by

$$\bigsqcup_{a \in \mathcal{R}/\pi\mathcal{R}, O \in (\mathcal{G}_{\lambda})_I \backslash \pi\mathcal{A}_{\lambda}} s(a)e_I + O = \mathcal{R}e_I + \pi\mathcal{A}_{\lambda} = X_I^{\lambda}.$$

This proves (c).

We prove (d) and (e). Note that if $\lambda_k > 1$, we have $\pi\mathcal{H}_{\lambda}^0 = \mathcal{H}_{\mu}^0$. If $\lambda_k = 1$, we have $\pi\mathcal{H}_{\lambda}^0 = \pi\mathcal{A}_{\lambda} = \mathcal{A}_{\mu}$. Now we use the proof of Theorem 4.2. This proves (d) and (e).

Hence Theorem 4.11 follows. ■

Now we introduce some notation.

(1) Let $\boldsymbol{\lambda} = (\lambda_1 > \lambda_2 > \cdots > \lambda_k) \in \Lambda_0$ be a nonempty partition with distinct parts.
(2) Let $\boldsymbol{\nu} = (\lambda_1 - \lambda_k > \lambda_2 - \lambda_k > \cdots > \lambda_{k-1} - \lambda_k)$.
(3) For $I \in \mathcal{J}(P_{\lambda}), J \in \mathcal{J}(P_{\lambda'/I}), K \in \mathcal{J}(P_{\lambda''})$, let

$$X_{I,J,K}^{\lambda} = \{(x', x'') \in \mathcal{A}_{\lambda} \mid I(\overline{x}') = J, I(x'') = K\}$$

and $\alpha_{I,J,K}^{\lambda} = \left|(\mathcal{A}_{\lambda'})_{J \cup K} \oplus \left((\mathcal{A}_{\lambda''})_K^* + (\mathcal{A}_{\lambda''})_J\right)\right|$.
(4) $I_1 \in \mathcal{J}(P_{\nu}), J_1 \in \mathcal{J}(P_{\nu'/I_1}), K_1 \in \mathcal{J}(P_{\nu''})$, let

$$X_{I_1,J_1,K_1}^{\nu} = \{(x', x'') \in \mathcal{A}_{\nu} \mid I(\overline{x}') = J_1, I(x'') = K_1\}$$

and $\alpha_{I_1,J_1,K_1}^{\nu} = \left|(\mathcal{A}_{\nu'})_{J_1 \cup K_1} \oplus \left((\mathcal{A}_{\nu''})_{K_1}^* + (\mathcal{A}_{\nu''})_{J_1}\right)\right|$
(5) For any partition δ, let $\mathcal{J}^0(P_{\delta})$ be the set of height zero ideals in $\mathcal{J}(P_{\delta})$.

Now we state four lattice identities in the following four theorems.

Theorem 4.12. *Using the notation as above, we have*

$$\sum_{\substack{I \in \mathcal{J}^0(P_{\lambda}) \\ (0,\lambda_k) \notin \max(I)}} \left(\sum_{(J,K) \in \mathcal{J}^0(P_{\lambda'/I}) \times \mathcal{J}(P_{\lambda''})} \frac{\left|X_{I,J,K}^{\lambda}\right|}{\alpha_{I,J,K}^{\lambda}} \right) = \sum_{I_1 \in \mathcal{J}^0(P_{\nu})} \left(\sum_{(J_1,K_1) \in \mathcal{J}^0(P_{\nu'/I_1}) \times \mathcal{J}(P_{\nu''})} \frac{\left|X_{I_1,J_1,K_1}^{\nu}\right|}{\alpha_{I_1,J_1,K_1}^{\nu}} \right). \tag{4.8}$$

Theorem 4.13. *Using the notation as above, we have*

$$\sum_{\substack{I\in\mathcal{J}^0(P_{\boldsymbol{\lambda}})\\(0,\lambda_k)\in\max(I)}}\left(\sum_{J\in\mathcal{J}^0(P_{\boldsymbol{\lambda}'/I}),K\in\mathcal{J}(P_{\boldsymbol{\lambda}''})}\frac{\left|X^{\boldsymbol{\lambda}}_{I,J,K}\right|}{\alpha^{\boldsymbol{\lambda}}_{I,J,K}}\right)$$
$$(4.9)\qquad = \sum_{I_1\in\mathcal{J}(P_{\boldsymbol{\nu}})\backslash\mathcal{J}^0(P_{\boldsymbol{\nu}})}\left(\sum_{J_1\in\mathcal{J}^0(P_{\boldsymbol{\nu}'/I_1}),K_1\in\mathcal{J}(P_{\boldsymbol{\nu}''})}\frac{\left|X^{\boldsymbol{\nu}}_{I_1,J_1,K_1}\right|}{\alpha^{\boldsymbol{\nu}}_{I_1,J_1,K_1}}\right).$$

Theorem 4.14. *Using the notation as above, we have*

$$\sum_{\substack{I\in\mathcal{J}^0(P_{\boldsymbol{\lambda}})\\(0,\lambda_k)\notin\max(I)}}\left(\sum_{J\in\mathcal{J}(P_{\boldsymbol{\lambda}'/I})\backslash\mathcal{J}^0(P_{\boldsymbol{\lambda}'/I}),K\in\mathcal{J}^0(P_{\boldsymbol{\lambda}''})}\frac{\left|X^{\boldsymbol{\lambda}}_{I,J,K}\right|}{\alpha^{\boldsymbol{\lambda}}_{I,J,K}}\right)$$
$$(4.10)\qquad = \sum_{I_1\in\mathcal{J}^0(P_{\boldsymbol{\nu}})}\left(\sum_{J_1\in\mathcal{J}(P_{\boldsymbol{\nu}'/I_1})\backslash\mathcal{J}^0(P_{\boldsymbol{\nu}'/I_1}),K_1\in\mathcal{J}(P_{\boldsymbol{\nu}''})}\frac{\left|X^{\boldsymbol{\nu}}_{I_1,J_1,K_1}\right|}{\alpha^{\boldsymbol{\nu}}_{I_1,J_1,K_1}}\right).$$

Theorem 4.15. *Using the notation as above, we have*

$$\sum_{\substack{I\in\mathcal{J}^0(P_{\boldsymbol{\lambda}})\\(0,\lambda_k)\in\max(I)}}\left(\sum_{J\in\mathcal{J}(P_{\boldsymbol{\lambda}'/I})\backslash\mathcal{J}^0(P_{\boldsymbol{\lambda}'/I}),K\in\mathcal{J}^0(P_{\boldsymbol{\lambda}''})}\frac{\left|X^{\boldsymbol{\lambda}}_{I,J,K}\right|}{\alpha^{\boldsymbol{\lambda}}_{I,J,K}}\right)$$
$$(4.11)\qquad = \sum_{I_1\in\mathcal{J}(P_{\boldsymbol{\nu}})\backslash\mathcal{J}^0(P_{\boldsymbol{\nu}})}\left(\sum_{J_1\in\mathcal{J}(P_{\boldsymbol{\nu}'/I_1})\backslash\mathcal{J}^0(P_{\boldsymbol{\nu}'/I_1}),K_1\in\mathcal{J}^0(P_{\boldsymbol{\nu}''})}\frac{\left|X^{\boldsymbol{\nu}}_{I_1,J_1,K_1}\right|}{\alpha^{\boldsymbol{\nu}}_{I_1,J_1,K_1}}\right).$$

4.2.1. **Proof of Theorem 4.12.**

Proof. The number of ideals $I\in\mathcal{J}(P_{\boldsymbol{\lambda}})$ of height zero, that is, $I\in\mathcal{J}^0(P_{\boldsymbol{\lambda}})$ is $(\lambda_1-\lambda_2+1)(\lambda_2-\lambda_3+1)\cdots(\lambda_{k-1}-\lambda_k+1)$ if $k>1$ and one if $k=1$. These ideals are in bijection with the ideals $I_1\in\mathcal{J}(P_{\boldsymbol{\nu}})$, where $\boldsymbol{\nu}=\big((\lambda_1-\lambda_k)>(\lambda_2-\lambda_k)>\cdots>(\lambda_{k-1}-\lambda_k)\big)$ with the bijection being

$$\max(I)\longrightarrow\max(I_1)=\{(v,\lambda-\lambda_k)\mid(v,\lambda)\in\max(I),(v,\lambda)\neq(0,\lambda_k)\}.$$

The inverse bijection is given as follows:

(4.12)
$$\max(I_1)\longrightarrow\max(I)=\{(v,\nu+\lambda_k)\mid(v,\nu)\in\max(I_1)\}\text{ if }I_1\text{ is of height zero,}$$
$$\max(I_1)\longrightarrow\max(I)=\{(v,\nu+\lambda_k)\mid(v,\nu)\in\max(I_1)\}\cup\{(0,\lambda_k)\}$$
$$\text{if }I_1\text{ is of positive height.}$$

Let $I \in \mathcal{J}^0(P_{\boldsymbol{\lambda}})$ with

$$\max(I) = \{(v_1, t_1), (v_2, t_2), \cdots, (v_s = 0, t_s) \mid t_1 > t_2 > \cdots > t_s\}.$$

We have $v_1 > v_2 > \cdots > v_{s-1} > v_s = 0$ and

$$t_1 - v_1 > t_2 - v_2 > \cdots > t_{s-1} - v_{s-1} > t_s - v_s = t_s \geq \lambda_k.$$

Therefore,

$$\begin{aligned} \boldsymbol{\lambda}' &= (t_1 > t_2 > \cdots > t_{s-1} > t_s), \\ \boldsymbol{\lambda}'/I &= \big((v_1 + t_2 - v_2) > (v_2 + t_3 - v_3) > \cdots \\ &\qquad > (v_{s-2} + t_{s-1} - v_{s-1}) > (v_{s-1} + t_s)\big). \end{aligned} \tag{4.13}$$

Consider the case $t_s > \lambda_k$. Under the above bijection the ideal $I \longrightarrow I_1 \in \mathcal{J}(P_{\boldsymbol{\nu}})$, where

$$\max(I_1) = \{(v_1, t_1 - \lambda_k), (v_2, t_2 - \lambda_k), \cdots, (v_{s-1}, t_{s-1} - \lambda_k), (v_s = 0, t_s - \lambda_k)\}.$$

The ideal I_1 is of height zero and we have

$$\begin{aligned} \boldsymbol{\nu}' &= (t_1 - \lambda_k > t_2 - \lambda_k > \cdots > t_{s-1} - \lambda_k > t_s - \lambda_k), \\ \boldsymbol{\nu}'/I_1 &= \big((v_1 + t_2 - v_2 - \lambda_k) > (v_2 + t_3 - v_3 - \lambda_k) > \cdots \\ &\qquad > (v_{s-2} + t_{s-1} - v_{s-1} - \lambda_k) > (v_{s-1} + t_s - \lambda_k)\big). \end{aligned} \tag{4.14}$$

Consider the case $t_s = \lambda_k$. Here

$$\max(I_1) = \{(v_1, t_1 - \lambda_k), (v_2, t_2 - \lambda_k), \cdots, (v_{s-1}, t_{s-1} - \lambda_k)\}$$

is a positive height ideal and we have

$$\begin{aligned} \boldsymbol{\nu}' &= (t_1 - \lambda_k > t_2 - \lambda_k > \cdots > t_{s-1} - \lambda_k), \\ \boldsymbol{\nu}'/I_1 &= \big((v_1 + t_2 - v_2 - \lambda_k) > (v_2 + t_3 - v_3 - \lambda_k) > \cdots \\ &\qquad > (v_{s-2} + t_{s-1} - v_{s-1} - \lambda_k) > v_{s-1} = v_{s-1} + t_s - \lambda_k\big). \end{aligned} \tag{4.15}$$

Note that the last part of $\boldsymbol{\lambda}'/I$ which is $v_{s-1} + t_s$ is always greater than λ_k.

Suppose $I \in \mathcal{J}^0(P_{\boldsymbol{\lambda}})$ is a height zero ideal, such that $t_s > \lambda_k$, which occurs if and only if $I_1 \in \mathcal{J}^0(P_{\boldsymbol{\nu}})$ is a height zero ideal, then λ_k is a part of $\boldsymbol{\lambda}''$ and it is the last part of $\boldsymbol{\lambda}''$. So the set of ideals of height zero, that is, the set $\mathcal{J}^0(P_{\boldsymbol{\lambda}''})$ is in bijection with the set $\mathcal{J}(P_{\boldsymbol{\nu}''})$ of all ideals.

Let $J \in \mathcal{J}^0(P_{\boldsymbol{\lambda}'/I}), K \in \mathcal{J}(P_{\boldsymbol{\lambda}''})$ and

$$\max(J) = \{(w_1, r_1), (w_2, r_2), \cdots, (w_l = 0, r_l)\},.$$

Then $r_l - w_l = r_l \geq v_{s-1} + t_s > \lambda_k$ and therefore for $1 \leq i \leq l$ we have $r_i - w_i \geq r_l - w_l = r_l > \lambda_k$. Let $J_1 \in \mathcal{J}(P_{\boldsymbol{\nu}'/I_1})$ be such that $\max(J_1) = \{(w_1, r_1 - \lambda_k), (w_2, r_2 - \lambda_k), \cdots, (w_l = 0, r_l - \lambda_k)\}$. Then $J_1 \in \mathcal{J}^0(P_{\boldsymbol{\nu}'/I_1})$. Here we observe in this way, that there is a bijection between the sets $\mathcal{J}^0(P_{\lambda'/I})$ and $\mathcal{J}^0(P_{\nu'/I_1})$.

Suppose K is a height zero ideal and $(0,\lambda_k) \notin \max(K)$. Let $K_1 \in \mathcal{J}(P_{\nu''})$ be the height zero ideal associated to K under the bijection $\mathcal{J}^0(P_{\lambda''}) \longleftrightarrow \mathcal{J}(P_{\nu''})$.

We have in this scenario,

$$\begin{aligned}
\left|X^{\lambda}_{I,J,K}\right| &= \frac{|\mathcal{A}_{\lambda'}|}{|\mathcal{A}_{\lambda'/I}|}\left|(\mathcal{A}_{\lambda'/I})^*_J\right|\left|(\mathcal{A}_{\lambda''})^*_K\right| \\
&= q^{\lambda_k}\frac{|\mathcal{A}_{\nu'}|}{|\mathcal{A}_{\nu'/I_1}|}q^{(s-1)\lambda_k}\left|(\mathcal{A}_{\nu'/I_1})^*_{J_1}\right|q^{(k-s)\lambda_k}\left|(\mathcal{A}_{\nu''})^*_{K_1}\right| \\
&= q^{k\lambda_k}\left|X^{\nu}_{I_1,J_1,K_1}\right|.
\end{aligned}$$

In addirion,

$$\begin{aligned}
\alpha^{\lambda}_{I,J,K} &= \left|(\mathcal{A}_{\lambda'})_{J\cup K}\right|\left|(\mathcal{A}_{\lambda''})^*_K + (\mathcal{A}_{\lambda''})_J\right| \\
&= q^{s\lambda_k}\left|(\mathcal{A}_{\nu'})_{J_1\cup K_1}\right|q^{(k-s)\lambda_k}\left|(\mathcal{A}_{\nu''})^*_{K_1} + (\mathcal{A}_{\nu''})_{J_1}\right| \\
&= q^{k\lambda_k}\alpha^{\nu}_{I_1,J_1,K_1}.
\end{aligned}$$

So we have

$$\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}} = \frac{\left|X^{\nu}_{I_1,J_1,K_1}\right|}{\alpha^{\nu}_{I_1,J_1,K_1}}.$$

Now suppose $(0,\lambda_k) \in \max(K)$. Let $K_1 \in \mathcal{J}(P_{\nu''})$ be the ideal of positive height associated to K under the bijection $\mathcal{J}^0(P_{\lambda''}) \longleftrightarrow \mathcal{J}(P_{\nu''})$. Let $K' \in \mathcal{J}(P_{\lambda})$ be such that $\max(K') = \max(K)\backslash\{(0,\lambda_k)\}$. Then K' is an ideal of positive height. Any positive height ideal $K_2 \in \mathcal{J}(P_{\lambda})$ satisfies the condition, $K' \subseteq K_2 \subsetneq K$ for some unique $K \in \mathcal{J}(P_{\lambda})$ such that $(0,\lambda_k) \in \max(K)$. The ideal K is obtained from K_2 as follows: $K = K_2 \cup [\langle\{(0,\lambda_k)\}\rangle]_{\lambda''}$. Now it is clear that for this ideal K we have $K' \subseteq K_2 \subsetneq K$ and K is uniquely determined by K_2.

Now we observe that for the ideal K_2 we have

$$\max(K_2)\backslash[J]_{\lambda''} = \max(K)\backslash[J]_{\lambda''} = \max(K')\backslash[J]_{\lambda''}.$$

Moreover we have the ideals $J \cup K_2 = J \cup K = J \cup K'$ in the big fundamental poset P as defined in 2.1. Hence we have

$$\begin{aligned}
\alpha^{\lambda}_{I,J,K} &= \alpha^{\lambda}_{I,J,K'} = \alpha^{\lambda}_{I,J,K_2} \\
\alpha^{\lambda}_{I,J,K} &= \left|(\mathcal{A}_{\lambda'})_{J\cup K}\right|\left|(\mathcal{A}_{\lambda''})^*_K + (\mathcal{A}_{\lambda''})_J\right| \\
&= q^{s\lambda_k}\left|(\mathcal{A}_{\nu'})_{J_1\cup K_1}\right|q^{(k-s)\lambda_k}\left|(\mathcal{A}_{\nu''})^*_{K_1} + (\mathcal{A}_{\nu''})_{J_1}\right| \\
&= q^{k\lambda_k}\alpha^{\nu}_{I_1,J_1,K_1}.
\end{aligned}$$

We also observe that

$$\left|\bigsqcup_{K'\subseteq K_2\subsetneq K}(\mathcal{A}_{\lambda''})^*_{K_2}\right| = q^{(k-s)\lambda_k}\left|(\mathcal{A}_{\nu''})^*_{K_1}\right|$$

$$\left|\bigsqcup_{K'\subseteq K_2\subsetneq K}X^{\lambda}_{I,J,K_2}\right| = q^{k\lambda_k}\left|X^{\nu}_{I_1,J_1,K_1}\right|.$$

This proves that

$$\sum_{K'\subseteq K_2\subseteq K}\frac{\left|X^{\lambda}_{I,J,K_2}\right|}{\alpha^{\lambda}_{I,J,K_2}} = \frac{\left|\bigsqcup_{K'\subseteq K_2\subseteq K}X^{\lambda}_{I,J,K_2}\right|}{\alpha^{\lambda}_{I,J,K}} = \frac{\left|X^{\nu}_{I_1,J_1,K_1}\right|}{\alpha^{\nu}_{I_1,J_1,K_1}}.$$

Hence we have proved the identity in Equation (4.8). This proves Theorem 4.12. ∎

4.2.2. **Proof of Theorem 4.13.**

Proof. Here we observe that $I \in \mathcal{J}^0(P_\lambda)$ is a height zero ideal, such that $t_s = \lambda_k$ occurs if and only if $I_1 \in \mathcal{J}(P_\nu)\backslash\mathcal{J}^0(P_\nu)$ is an ideal of positive height. The set $\mathcal{J}^0(P_{\lambda'/I})$ of ideals of height zero is in bijection with the set $\mathcal{J}^0(P_{\nu'/I_1})$ of height zero ideals. Here λ_k is a part of $\boldsymbol{\lambda}'$ and it is the last part of $\boldsymbol{\lambda}'$. So the last part of $\boldsymbol{\lambda}''$ is greater than λ_k. Let $\boldsymbol{\lambda}'''$ be the partition obtained by adding λ_k to the partition $\boldsymbol{\lambda}''$ at the end. The set $\mathcal{J}^0(P_{\lambda'''})$ of ideals of height zero is is in bijection with the set $\mathcal{J}(P_{\nu''})$ of all ideals.

Let $J \in \mathcal{J}^0(P_{\lambda'/I}), K \in \mathcal{J}(P_{\lambda''})$ and $\max(J) = \{(w_1,r_1),(w_2,r_2),\cdots,(w_l = 0,r_l)\}$. Then $r_l - w_l = r_l \geq v_{s-1} + t_s > \lambda_k$ and therefore for $1 \leq i \leq l$ we have $r_i - w_i \geq r_l - w_l = r_l > \lambda_k$. Let $J_1 \in \mathcal{J}(P_{\nu'/I_1})$ be such that $\max(J_1) = \{(w_1,r_1-\lambda_k),(w_2,r_2-\lambda_k),\cdots,(w_l = 0,r_l-\lambda_k)\}$. Then $J_1 \in \mathcal{J}^0(P_{\nu'/I_1})$. Here we observe in this way, that there is a bijection between the sets $\mathcal{J}^0(P_{\lambda'/I})$ and $\mathcal{J}^0(P_{\nu'/I_1})$.

Let $K' \in \mathcal{J}^0(P_{\lambda'''})$ be such that $(0,\lambda_k) \notin \max(K')$. Then $K = K' \cap P_{\lambda''} \in \mathcal{J}^0(P_{\lambda''})$. Let $\max(K') = \max(K) = \{(x_1,a_1),\cdots,(x_m = 0,a_m)\}$. Let $K_1 \in \mathcal{J}^0(P_{\nu''})$ be such that $\max(K_1) = \{(x_1,a_1-\lambda_k),(x_2,a_2-\lambda_k),\cdots,(x_m = 0,a_m-\lambda_k)\}$. Then it is clear that

$$\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}} = \frac{\left|X^{\nu}_{I_1,J_1,K_1}\right|}{\alpha^{\nu}_{I_1,J_1,K_1}}.$$

Let $K' \in \mathcal{J}^0(P_{\lambda'''})$ be such that $(0,\lambda_k) \in \max(K')$. Let $K \in \mathcal{J}(P_{\lambda''})$ be such that $\max(K) = \max(K')\backslash\{(0,\lambda_k)\}$. Let $K_1 \in \mathcal{J}(P_{\nu''})\backslash\mathcal{J}^0(P_{\nu''})$ be the ideal of positive height. Now given any ideal $K_2 \in \mathcal{J}(P_{\lambda''})$ of positive height, there is a unique ideal $K' \in \mathcal{J}^0(P_{\lambda'''})$ be such that $(0, \lambda_k) \in \max(K')$ satisfying $K \subseteq K_2 \subseteq K' \backslash P_{(\lambda_k)}$, where $K \in \mathcal{J}(P_{\lambda''})$ is such that $\max(K) = \max(K') \backslash \{(0, \lambda_k)\}$. Then it is clear that

$$\sum_{K \subseteq K_2 \subseteq K' \backslash P_{(\lambda_k)}} \frac{\left|X^{\lambda}_{I,J,K_2}\right|}{\alpha^{\lambda}_{I,J,K_2}} = \frac{\left|\bigsqcup_{K \subseteq K_2 \subseteq K' \backslash P_{(\lambda_k)}} X^{\lambda}_{I,J,K_2}\right|}{\alpha^{\lambda}_{I,J,K}} = \frac{\left|X^{\nu}_{I_1,J_1,K_1}\right|}{\alpha^{\nu}_{I_1,J_1,K_1}}.$$

This proves the identity in Equation (4.9). Hence the theorem follows. ■

4.2.3. **Proof of Theorem 4.14.**

Proof. Suppose $I \in \mathcal{J}^0(P_\lambda)$ is a height zero ideal, such that $t_s > \lambda_k$, which occurs if and only if $I_1 \in \mathcal{J}^0(P_\nu)$ is a height zero ideal, then λ_k is a part of $\boldsymbol{\lambda}''$ and it is the last part of $\boldsymbol{\lambda}''$. So the set of ideals of height zero, that is, the set $\mathcal{J}^0(P_{\lambda''})$ is in bijection with the set $\mathcal{J}(P_{\nu''})$ of all ideals. Let $\boldsymbol{\lambda}'''$ be the partition obtained by adding the part λ_k to $\boldsymbol{\lambda}'/I$. Then the set of all ideals $J' \in \mathcal{J}(P_{\lambda'''})$ of height zero such that $(0, \lambda_k) \in \max(J')$ is in bijection with $\mathcal{J}(P_{\nu'/I_1}) \backslash \mathcal{J}^0(P_{\nu'/I_1})$. Now the rest of the proof follows using standard arguments. This proves the theorem. ■

4.2.4. **Proof of Theorem 4.15.**

Proof. Here we observe that $I \in \mathcal{J}^0(P_\lambda)$ is a height zero ideal, such that $t_s = \lambda_k$ occurs if and only if $I_1 \in \mathcal{J}(P_\nu) \backslash \mathcal{J}^0(P_\nu)$ is an ideal of positive height. Here λ_k is a part of $\boldsymbol{\lambda}'$ and it is the last part of $\boldsymbol{\lambda}'$. So the last part of $\boldsymbol{\lambda}''$ is greater than λ_k. The set $\mathcal{J}^0(P_{\lambda''})$ of ideals of height zero is in bijection with the set $\mathcal{J}^0(P_{\nu''})$ of height zero ideals. Let $\boldsymbol{\lambda}'''$ be the partition obtained by adding λ_k to the partition $\boldsymbol{\lambda}''$ at the end. The set $\mathcal{J}^0(P_{\lambda'''})$ of ideals J' of height zero such that $(0, \lambda_k) \in \max(J')$ is in bijection with the set $\mathcal{J}(P_{\nu''})$ of all ideals. Now the rest of the proof follows using standard arguments. This proves the theorem. ■

4.3. **Proof of Theorem 4.9.**

Proof. We prove (1), (2) simultaneously now. First we have using Equation (4.1)

$$n^0_\lambda(q) = \sum_{I \in \mathcal{J}^0(P_\lambda)} |(\mathcal{G}_\lambda)_I \backslash \mathcal{A}_\lambda| = \sum_{I \in \mathcal{J}^0(P_\lambda)} \left(\sum_{J \in \mathcal{J}(P_{\lambda'/I}), K \in \mathcal{J}(P_{\lambda''})} \frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}} \right).$$

The above sum splits as two sums, one of which is given in the right-hand side of Equation (4.7) and the other one is given by

$$\sum_{I \in \mathcal{J}^0(P_\lambda)} \left(\sum_{(J,K) \in \mathcal{J}^0(P_{\lambda'/I}) \times \mathcal{J}(P_{\lambda''}) \cup \mathcal{J}(P_{\lambda'/I}) \times \mathcal{J}^0(P_{\lambda''})} \frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}} \right). \tag{4.16}$$

Theorem 4.11 then gives the first summand in Theorem 4.9 which is $qn^0_{\mu}(q)$ if $\lambda_k > 1$ and $qn_{\mu}(q)$ if $\lambda_k = 1$. We need to prove that the remaining sum in 4.16 is $n^0_{\nu}(q) + n^0_{\delta}(q)$ if $\lambda_{k-1} - \lambda_k > 1$ and $n^0_{\nu}(q) + n_{\delta}(q)$ if $\lambda_{k-1} - \lambda_k = 1$.

Now we break the subsum given in 4.16 into two subsums as:

$$\begin{aligned}
&\sum_{I\in\mathcal{J}^0(P_\lambda)}\left(\sum_{(J,K)\in\mathcal{J}^0(P_{\lambda'/I})\times\mathcal{J}(P_{\lambda''})\cup\mathcal{J}(P_{\lambda'/I})\times\mathcal{J}^0(P_{\lambda''})}\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}\right)\\
&\quad=\sum_{\substack{I\in\mathcal{J}^0(P_\lambda)\\(0,\lambda_k)\notin\max(I)}}\left(\sum_{(J,K)\in\mathcal{J}^0(P_{\lambda'/I})\times\mathcal{J}(P_{\lambda''})\cup\mathcal{J}(P_{\lambda'/I})\times\mathcal{J}^0(P_{\lambda''})}\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}\right)\\
&\qquad+\sum_{\substack{I\in\mathcal{J}^0(P_\lambda)\\(0,\lambda_k)\in\max(I)}}\left(\sum_{(J,K)\in\mathcal{J}^0(P_{\lambda'/I})\times\mathcal{J}(P_{\lambda''})\cup\mathcal{J}(P_{\lambda'/I})\times\mathcal{J}^0(P_{\lambda''})}\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}\right).
\end{aligned}$$

To complete the proof of Theorem 4.9, we need the identities given by Equation (4.17) and Equation (4.18). We have

$$\begin{aligned}
&\sum_{\substack{I\in\mathcal{J}^0(P_\lambda)\\(0,\lambda_k)\notin\max(I)}}\left(\sum_{(J,K)\in\mathcal{J}^0(P_{\lambda'/I})\times\mathcal{J}(P_{\lambda''})\cup\mathcal{J}(P_{\lambda'/I})\times\mathcal{J}^0(P_{\lambda''})}\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}\right)\\
&\quad=\sum_{I_1\in\mathcal{J}^0(P_\nu)}\left(\sum_{J_1\in\mathcal{J}(P_{\nu'/I_1}),K_1\in\mathcal{J}(P_{\nu''})}\frac{\left|X^{\nu}_{I_1,J_1,K_1}\right|}{\alpha^{\nu}_{I_1,J_1,K_1}}\right)=n^0_{\nu}(q).
\end{aligned}\tag{4.17}$$

Furthermore,

(4.18)

$$\begin{aligned}
&\sum_{\substack{I\in\mathcal{J}^0(P_\lambda)\\(0,\lambda_k)\in\max(I)}}\left(\sum_{(J,K)\in\mathcal{J}^0(P_{\lambda'/I})\times\mathcal{J}(P_{\lambda''})\cup\mathcal{J}(P_{\lambda'/I})\times\mathcal{J}^0(P_{\lambda''})}\frac{\left|X^{\lambda}_{I,J,K}\right|}{\alpha^{\lambda}_{I,J,K}}\right)\\
&\quad=\sum_{I_1\in\mathcal{J}(P_\nu)\backslash\mathcal{J}^0(P_\nu)}\left(\sum_{(J_1,K_1)\in\mathcal{J}^0(P_{\nu'/I_1})\times\mathcal{J}(P_{\nu''})\cup\mathcal{J}(P_{\nu'/I_1})\times\mathcal{J}^0(P_{\nu''})}\frac{\left|X^{\nu}_{I_1,J_1,K_1}\right|}{\alpha^{\nu}_{I_1,J_1,K_1}}\right)\\
&\quad=\left|\mathcal{G}_\nu\backslash(\pi\mathcal{A}_\nu\times\mathcal{H}^0_\nu)\right|.
\end{aligned}$$

Observe that the last part of ν is $\lambda_{k-1} - \lambda_k$. We have already observed that if $\lambda_{k-1} - \lambda_k > 1$ then $\left|\mathcal{G}_\nu\backslash(\pi\mathcal{A}_\nu\times\mathcal{H}^0_\nu)\right| = n^0_{\delta}(q)$ and if $\lambda_{k-1} - \lambda_k = 1$ then $\left|\mathcal{G}_\nu\backslash(\pi\mathcal{A}_\nu\times\mathcal{H}^0_\nu)\right| = n_{\delta}(q)$, where $\pi\mathcal{A}_\nu \cong \mathcal{A}_\delta$.

Equation (4.17) is obtained by combining Equation (4.8) and Equation (4.10). Equation (4.18) is obtained by combining Equation (4.9) and Equation (4.11).

We established the identities in Equations (4.8), (4.10), (4.9), (4.11) in Theorems 4.12, 4.14, 4.13 4.15, respectively.

Theorem 4.9(3) is easy. Therefore we have completed the proof of Theorem 4.9. ■

Remark 4.16. Theorem 4.9 can be generalized to partitions which need not have distinct parts.

Many more nice identities such as the following have been observed. Let $\boldsymbol{\lambda} = (\lambda_1 > \lambda_2 > \cdots > \lambda_k)$ be a partition with distinct parts.

- Suppose $k \geq 2, \lambda_k = 1, \lambda_1 - \lambda_2 > 1$. Let

$$\boldsymbol{\mu} = (\lambda_1 - 1 > \lambda_2 > \cdots > \lambda_k), \boldsymbol{\nu} = (\lambda_2 > \cdots > \lambda_k).$$

  Then we have

$$n^0_{\boldsymbol{\lambda}}(q) = qn^0_{\boldsymbol{\mu}}(q) + 2n^0_{\boldsymbol{\nu}}(q).$$

- Suppose $k \geq 2, \lambda_k = 1, \lambda_1 - \lambda_2 = 1$. Let

$$\begin{aligned}
\boldsymbol{\mu} &= (\lambda_2^2 > \lambda_3 > \cdots > \lambda_k), \\
\boldsymbol{\nu} &= (\lambda_2 > \lambda_3 > \cdots > \lambda_k), \\
\boldsymbol{\delta} &= \left((\lambda_2 - 1)^2 \geq \lambda_3 > \cdots > \lambda_k\right).
\end{aligned}$$

  Then we have

$$n^0_{\boldsymbol{\lambda}}(q) = qn^0_{\boldsymbol{\mu}}(q) + 2n^0_{\boldsymbol{\nu}}(q) + n^0_{\boldsymbol{\delta}}(q).$$

Theorem 4.17. *Let $\boldsymbol{\lambda} = (\lambda_1^{\rho_1} > \lambda_2^{\rho_2} > \lambda_3^{\rho_3} > \cdots > \lambda_k^{\rho_k}) \in \Lambda_0$ be a partition and $\mathcal{R}$ be a discrete valuation ring with maximal ideal generated by a uniformizing element π having finite residue field $\mathbf{k} = \mathcal{R}/\pi\mathcal{R} \cong \mathbb{F}_q$. Let $\mathcal{A}_{\boldsymbol{\lambda}} = \bigoplus_{i=1}^{k}(\mathcal{R}/\pi^{\lambda_i}\mathcal{R})^{\rho_i}$ be its associated finite $\mathcal{R}$-module and $\mathcal{G}_{\boldsymbol{\lambda}}$ be its automorphism group. Let $n_{\boldsymbol{\lambda}}(q)$ be the number of orbits of pairs in $\mathcal{A}_{\boldsymbol{\lambda}} \times \mathcal{A}_{\boldsymbol{\lambda}}$ for the diagonal action of $\mathcal{G}_{\boldsymbol{\lambda}}$ on $\mathcal{A}_{\boldsymbol{\lambda}} \times \mathcal{A}_{\boldsymbol{\lambda}}$. Then $n_{\boldsymbol{\lambda}}(q)$ is a polynomial of degree λ_1 with non-negative integer coefficients.*

Proof. This theorem follows from Theorems Δ, 4.2, 4.5, 4.9 and the fact that $n^0_{(1)}(q) = q, n_{\emptyset}(q) = n^0_{\emptyset}(q) = 1$. ■

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School of Mathematics, Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Prayagraj (Allahabad), 211 019, India. email: akcp1728@gmail.com