A uniform action of the dihedral group $\mathbb{Z}_2 \times D_3$ on Littlewood–Richardson coefficients

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Overview

- Littlewood–Richardson (LR) coefficients as structure coefficients.
- The set $\mathcal{LR}$: LR tableaux, LR companion tableaux (Gelfand-Tsetlin patterns), Knutson-Tao (KT) puzzles and Hives.
- A presentation of the dihedral group $\mathbb{Z}_2 \times D_3$.
  - An index two subgroup action on $\mathcal{LR}$.
  - The involution to the other coset: Schützenberger-Lusztig involution.
- $\mathbb{Z}_2 \times D_3$ action on LR companion pairs or KT Hives.
  - The cocrytal of an LR tableau.

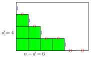
LR coefficients as structure coefficients and symmetries

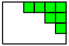
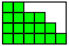
- Schur functions s_λ, λ runs over all Young shapes (partitions), form a linear $\mathbb{Z}$-basis for the ring Λ of symmetric functions with integer coefficients in countably many variables $x_1, x_2, \dots,$

  $$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\,\nu} s_\lambda,$$

  the structure constants $c^\lambda_{\mu\,\nu} \in \mathbb{Z}_{\geq 0}$ depending only on the three partitions μ, ν and λ, are the *Littlewood–Richardson (LR) coefficients*.

- Fix integers $0 \leq d < n$ and $D := d \times (n-d)$ the rectangle ambient.
- Given the Young shapes $\mu, \nu, \lambda \subseteq D$, $c_{\mu\,\nu\,\lambda^\vee} := c^\lambda_{\mu\,\nu}$. We refer to $(\mu, \nu, \lambda^\vee)$ as the *LR triple*.

λ 

rotate(λ) 
$\lambda^\vee$ 
$\lambda^\vee = \text{rotate}(D \setminus \lambda) = D \setminus \text{rotate}(\lambda)$

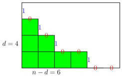
$$s_\mu s_\nu s_{\lambda^\vee} = \cdots + c_{\mu\,\nu\,\lambda^\vee} s_D + \cdots$$

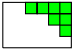
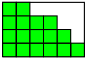
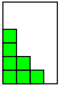
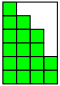
- $\mathfrak{S}_3$-symmetries: $c_{\mu,\nu,\lambda} = c_{\nu\,\mu\,\lambda} = c_{\mu\,\lambda\,\nu} = c_{\lambda\,\nu\,\mu}$, $\quad c_{\mu,\nu,\lambda} = c_{\lambda\,\mu\,\nu} = c_{\nu\,\lambda\,\mu}$.

LR coefficients as structure coefficients and symmetries

- λ^{tr} the *conjugate* or *transpose* of λ with rectangle ambient $D^{\text{tr}} = (n-d) \times d$,

$$s_{\mu^{\text{tr}}} s_{\nu^{\text{tr}}} s_{\lambda^{\vee\,\text{tr}}} = \cdots + c_{\mu^{\text{tr}}\ \nu^{\text{t}}\ \lambda^{\vee\,\text{tr}}} s_{D^{\text{tr}}} + \cdots$$

λ 

rotate(λ)  $\lambda^{\vee}$  λ^{tr}  $(\lambda^{\vee})^{\text{tr}}$ .

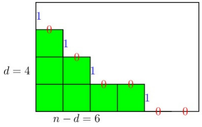
- The *conjugation symmetry* is not obvious from the Schur expansion

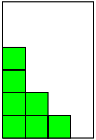
$$c_{\mu\ \nu\ \lambda} = c_{\mu^{\text{tr}}\ \nu^{\text{tr}}\ \lambda^{\text{tr}}}$$

It is shown *via* the involutive ring automorphism $\omega : \Lambda \to \Lambda$, $s_\lambda \mapsto s_{\lambda^{\text{tr}}}$.

The set $\mathcal{LR}$

- Let $\binom{[n]}{d}$ be the set of binary words consisting of d 1's and $n-d$ 0's. Partitions in the rectangle ambient D are also identified with the 01–words in $\binom{[n]}{d}$; and their transpose, in D^{tr}, with words in $\binom{[n]}{n-d}$

$\lambda =$ 

$\lambda^{\text{tr}} =$ 

$\lambda = 0010010101$ $\qquad$ $\lambda^{\text{tr}} = 0101011011$

- Given μ, ν, λ partitions $\subseteq D$, $\mathrm{LR}^{\lambda}_{\mu,\nu}$ is the set of LR tableaux of shape λ/μ and content ν, and $|\mathrm{LR}^{\lambda}_{\mu,\nu}| = \mathrm{c}^{\lambda}_{\mu\ \nu} = \mathrm{c}_{\mu,\nu,\lambda^\vee}$.

$T =$

		2	4	4											
			1	1	2	3	3								
							1	2	2	2					
										1	1	1	1	1	

$(\mu, \nu, \lambda^\vee)$.

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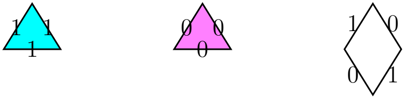
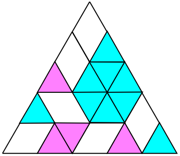
KT puzzles

A KT puzzle of size n is a tiling of an equilateral triangle $\triangle$ of side length n with three kind of puzzle pieces:



such that whenever two pieces share an edge, the labels on the edge must agree. Puzzle pieces may be **rotated in any orientation**, **rhombi can not be reflected**.

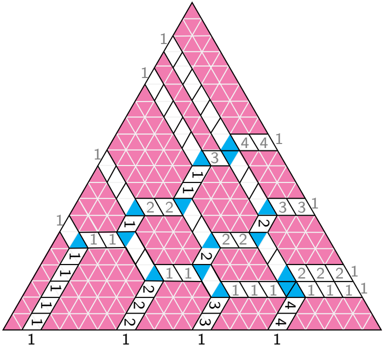
$n = 5, \; d = 3, \qquad \triangle_{\mu,\nu,\lambda}$



$\mu = 01011 = (1,0,0), \; \nu = 01101 = (1,1,0), \lambda = 10101 = (2,1,0)$

Tao's bijection between KT puzzles and LR tableaux

$n = 20, \qquad d = 4 \qquad\qquad \mathrm{LR}^{\lambda^\vee}_{\mu,\nu} \longleftrightarrow \mathrm{KT}_{\mu,\nu,\lambda} = \{\triangle_{\mu,\nu,\lambda}\}$



(μ, ν, λ)

$T =$

		2	4	4											
			1	1	2	3	3								
							1	2	2	2					
										1	1	1	1	1	

(μ, ν, λ).

The set $\mathcal{LR}$

- Let $\mathcal{LR}$ be either the set of all LR tableaux or KT puzzles, or LR companions or KT hives where the LR triple boundary (μ, ν, λ) fits the rectangle D or the rectangle D^{tr}:

$$\mathcal{LR} = \bigsqcup_{(\mu,\nu,\lambda)} \mathrm{LR}^{\lambda^\vee}_{\mu,\nu}\,(\mathrm{KT}_{\mu,\nu,\lambda})\,(\mathrm{LR}_{\nu,\lambda/\mu})\,(\mathrm{HIVE}^{\lambda^\vee}_{\mu,\nu})$$

where $(\mu, \nu, \lambda) \in \binom{[n]}{d}^3 \sqcup \binom{[n]}{n-d}^3$.

The dihedral group $\mathbb{Z}_2 \times D_3$ of order twelve

- $\mathbb{Z}_2 =< \tau | \tau^2 = 1 >$ and $D_3 = \langle \varsigma_1, \varsigma_2 \mid \varsigma_1^2 = \varsigma_2^2 = (\varsigma_1\varsigma_2)^3 = 1 \rangle$.

$$\mathbb{Z}_2 \times D_3 = \langle \tau, \varsigma_1, \varsigma_2 \mid \tau^2 = \varsigma_1^2 = \varsigma_2^2 = (\varsigma_1\varsigma_2)^3 = 1 = (\tau\varsigma_1)^2 = (\tau\varsigma_2)^2 \rangle.$$

- $\mathcal{H}$ the two index subgroup of $\mathbb{Z}_2 \times D_3$, defined by

$$\mathcal{H} := \quad \langle \tau\varsigma_1, \tau\varsigma \rangle \quad = \{1, \tau\varsigma_1, \tau\varsigma, \tau\varsigma_1\varsigma\varsigma_1, \varsigma_1\varsigma, \varsigma\varsigma_1\},$$

  where $\varsigma = \varsigma_1\varsigma_2\varsigma_1 = \varsigma_2\varsigma_1\varsigma_2$.

- $\mathbb{Z}_2 \times D_3 \simeq \mathbb{Z}_2 \times \mathcal{H}$

$$\langle \tau, \tau\varsigma_1, \tau\varsigma : \tau^2 = (\tau\varsigma_i)^2 = (\tau\varsigma)^2 = (\tau\varsigma_1\tau\varsigma)^3 = (\tau\varsigma_1\tau)^2 = (\tau\varsigma\tau)^2 = 1 \rangle.$$

  - As a set

$$\mathbb{Z}_2 \times D_3 = \mathcal{H} \sqcup \tau\mathcal{H}.$$

The $\mathcal{H}$-action on $\mathcal{LR}$

$$\mathbb{Z}_2 \times D_3 = \mathcal{H} \sqcup \tau\mathcal{H}.$$

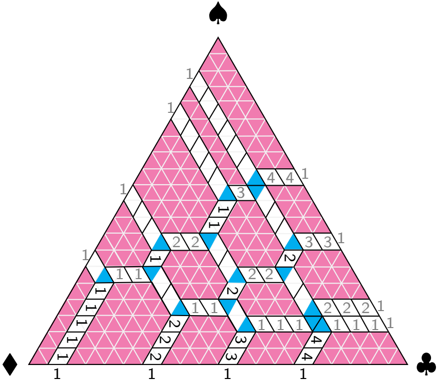
$$\mathcal{H} = \langle \tau\varsigma_1, \tau\varsigma \rangle = \{1, \tau\varsigma_1, \tau\varsigma, \tau\varsigma_1\varsigma\varsigma_1, \varsigma_1\varsigma, \varsigma\varsigma_1\}$$

$$\begin{array}{rcl} \varpi_{\mathcal{H}} : \mathcal{H} & \longrightarrow & \mathfrak{S}_{\mathcal{LR}} \\ \tau\varsigma & \mapsto & \blacklozenge \\ \tau\varsigma_1 & \mapsto & \spadesuit \end{array}$$

- $\mathcal{LR} = \bigsqcup_{(\mu,\nu,\lambda)} \mathrm{KT}_{\mu,\nu,\lambda}$, KT puzzles $\triangle_{\mu,\nu,\lambda}$.
- $\spadesuit$ vertical reflection while swapping 01 labels: $c_{\mu,\nu,\lambda} = c_{\nu^{tr},\mu^{tr},\lambda^{tr}}$.
- $\blacklozenge$ left reflection (NW-S) while swapping 01 labels: $c_{\mu,\nu,\lambda} = c_{\lambda^{tr},\nu^{tr},\mu^{tr}}$.
- $\clubsuit = \spadesuit\blacklozenge\spadesuit$ right reflection (NE-S) while swapping 01 labels: $c_{\mu,\nu,\lambda} = c_{\mu^{tr},\lambda^{tr},\nu^{tr}}$.
- $\blacklozenge\spadesuit$, $\spadesuit\blacklozenge$ clockwise central rotations $2\pi/3$, $4\pi/3$ radians: $c_{\mu,\nu,\lambda} = c_{\lambda,\mu,\nu} = c_{\nu,\lambda,\mu}$

$$\mathcal{H} \simeq \{1, \spadesuit, \blacklozenge, \clubsuit = \spadesuit\blacklozenge\spadesuit, \blacklozenge\spadesuit, \spadesuit\blacklozenge\}$$

$\mathcal{H}$-action on KT puzzles and LR tableaux



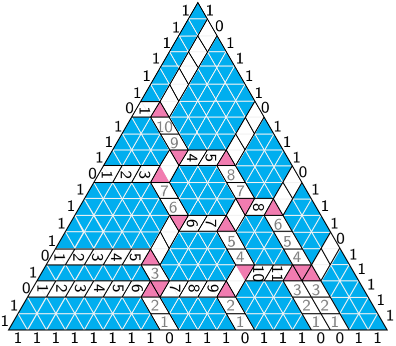
(μ, ν, λ)

$T =$

		2	4	4											
			1	1	2	3	3								
							1	2	2	2					
										1	1	1	1	1	

(μ, ν, λ).

♣



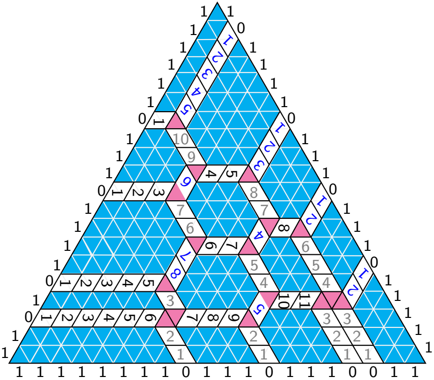
$(\mu^t, \lambda^t, \nu^t)$

$\clubsuit T =$

8	11		
5	10		
4	7		
1	6	9	
	3	8	
	2	7	
	1	5	6
		4	5
		3	4
		2	3
		1	2
			1

$(\mu^t, \lambda^t, \nu^t)$

♦



$(\mu^t, \lambda^t, \nu^t)$

♦ $T =$

5			
2	8		
1	7		
	4		
	2		
	1	6	
		3	
		2	
		1	5
			4
			3
			2
			1

$(\lambda^t, \nu^t, \mu^t)$

♦ Zaballa 95, A. 99

orthogonal transpose $(\mu, \nu, \lambda) \longleftrightarrow (\lambda^t, \nu^t, \mu^t)$

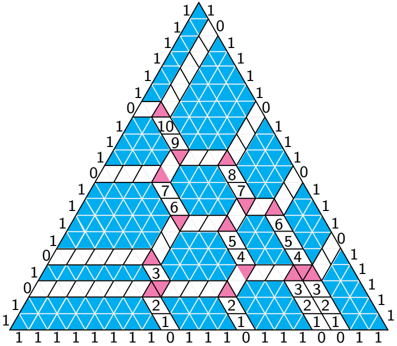
$T =$

		2	4	4											
			1	1	2	3	3								
							1	2	2	2					
										1	1	1	1	1	

♦$T =$

5			
2	8		
1	7		
	4		
	2		
	1	6	
		3	
		2	
		1	5
			4
			3
			2
			1

♠



$(\mu^t, \lambda^t, \nu^t)$

♠T =

10			
9			
7			
6			
3	8		
2	7		
1	5		
	4	6	
	2	5	
	1	4	
		3	3
		2	2
		1	1

$(\nu^t, \mu^t, \lambda^t)$

♠, ♣ puzzle reflections $\longleftrightarrow$ hybrid LR tableau switching

A., Conflitti, Mamede, 2010,

♠ : $(\mu, \nu, \lambda) \longleftrightarrow (\nu^{tr}, \mu^{tr}, \lambda^{tr}))$

$T(\mu, \nu, \lambda) =$

1	3					
	2	2	3			
		1	2	2		
				1	1	1

$\to ([Y(\mu)]^{tr}, T) =$

1	3					
a	2	2	3			
a	*b*	1	2	2		
a	*b*	*c*	*d*	1	1	1

$\to$

1			
1			
1	2		
d	2	3	
c	1	2	
b	*b*	2	3
a	*a*	*a*	1

$= (Y(\mu^{tr}), T^{tr}) \to$

1			
1			
1	2		
d	2	3	
1	2	*c*	
b	2	3	*b*
1	*a*	*a*	*a*

$\to$

d			
b			
1	*a*		
1	2	*c*	
1	2	*a*	
1	2	3	*b*
1	2	3	*a*

$= ([Y(\nu)]^{tr}, ♠T)$

$T(\mu, \nu, \lambda) =$

1	3					
	2	2	3			
		1	2	2		
				1	1	1

$\longleftrightarrow$

d			
b			
	a		
		c	
		a	
			b
			a

$= ♠\mathrm{T}(\nu^{tr}, \mu^{tr}, \lambda^{tr}))$

The $\mathcal{H}$-action on $\mathcal{LR}$ (LR tableaux, KT puzzles)

$$\mathbb{Z}_2 \times D_3 = \mathcal{H} \sqcup \tau\mathcal{H}$$

$$\begin{array}{rcl} \varpi_{\mathcal{H}} : \mathcal{H} & \longrightarrow & \mathfrak{S}_{\mathcal{LR}} \\ \tau\varsigma & \mapsto & \blacklozenge \\ \tau\varsigma_1 & \mapsto & \spadesuit \end{array}$$

$$\mathcal{H} = \langle \tau\varsigma_1, \tau\varsigma \rangle = \{1, \tau\varsigma_1, \tau\varsigma, \tau\varsigma_1\varsigma\varsigma_1, \varsigma_1\varsigma, \varsigma\varsigma_1\} \simeq \{1, \spadesuit, \blacklozenge, \clubsuit = \spadesuit\blacklozenge\spadesuit, \blacklozenge\spadesuit, \spadesuit\blacklozenge\}$$

The $\mathcal{H}$-action on $\mathcal{LR}$ exhibit the symmetries

$$c_{\mu,\nu,\lambda} = c_{\nu^{\text{tr}},\mu^{\text{tr}},\lambda^{\text{tr}}} = c_{\mu^{\text{tr}},\lambda^{\text{tr}},\nu^{\text{tr}}} = c_{\lambda^{\text{tr}},\nu^{\text{tr}},\mu^{\text{tr}}} = c_{\lambda,\mu,\nu} = c_{\nu,\lambda,\mu}$$

Theorem (A.,Conflitti,Mamede 2009, 2010)

The involutions $\spadesuit$, $\blacklozenge$, and $\clubsuit$ on $\mathcal{LR}$ have linear cost.

The involution from $\mathcal{H}$ to other coset of $\mathcal{H}$

- For $\mu \subseteq \lambda \subseteq D$, let $B(\lambda/\mu, d)$ be the $\mathfrak{gl}_d$-crystal of all semi-standard tableaux of shape λ/μ on the alphabet $[d]$.

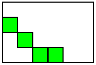
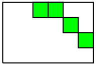
$$B(\lambda/\mu, d) = \bigsqcup_{\substack{\nu \\ T \in \mathrm{LR}^{\lambda}_{\mu,\nu}}} B(T) \simeq \bigsqcup_{\nu} B(\nu, d)^{c^{\lambda}_{\mu,\nu}},$$

  where $B(T)$ is the crystal connected component of $B(\lambda/\mu, d)$ with highest weight element $T^{\text{high}} = T \in \mathrm{LR}^{\lambda}_{\mu,\nu}$ of weight ν.

-
  - $\mathrm{LR}^{\lambda}_{\mu,\nu}$ the set of highest weight elements of $B(\lambda/\mu, d)$ of weight ν.
  - $\mathrm{LR}^{\lambda}_{\mu,\mathtt{rotate}\,\nu} = \mathtt{reversal}\mathrm{LR}^{\lambda}_{\mu,\nu}$ the set of lowest weight elements of $B(\lambda/\mu, d)$ of weight $\mathtt{rotate}\,\nu$.

- $\mathbb{Z}_2 \times D_3 = \langle \tau, \tau\varsigma_1, \tau\varsigma \rangle = \mathcal{H} \sqcup \tau\mathcal{H}$.
- $\mathcal{H} \simeq \{1, \spadesuit, \blacklozenge, \clubsuit = \spadesuit\blacklozenge\spadesuit, \blacklozenge\spadesuit, \spadesuit\blacklozenge\} \mapsto \varrho\mathcal{H} = \mathcal{H}\varrho = \varrho\{1, \spadesuit, \blacklozenge, \clubsuit = \spadesuit\blacklozenge\spadesuit, \blacklozenge\spadesuit, \spadesuit\blacklozenge\}$
- $c_{\mu,\nu,\lambda} = c_{\nu\ \mu\ \lambda} = c_{\mu\ \lambda\ \nu} = c_{\lambda\ \nu\ \mu}$,
- $c_{\mu,\nu,\lambda} = c_{\mu^{\mathrm{tr}},\nu^{\mathrm{tr}},\lambda^{\mathrm{tr}}} = c_{\lambda^{\mathrm{tr}},\mu^{\mathrm{tr}},\nu^{\mathrm{tr}}} = c_{\nu^{\mathrm{tr}},\lambda^{\mathrm{tr}},\mu^{\mathrm{tr}}}$

rotate crystal

λ/μ  $\texttt{rotate}\ \lambda/\mu$  $\texttt{rotate}(\lambda/\mu) = \mu^\vee/\lambda^\vee$

- If $U \in B(\lambda/\mu, d)$, $\texttt{rotate}(U)$ is obtained from U under rotation of λ/μ by π radians while replacing the entry i with $\omega_0(i) = d - i + 1$ throughout, ω_0 the long element of $\mathfrak{S}_d$.

$$U = \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & 2 & & \\ \hline & & 1 & \\ \hline \end{array} \mapsto \texttt{rotate}(U) = \begin{array}{|c|c|c|c|}\hline & 3 & & \\ \hline & & 2 & \\ \hline & & & 3 \\ \hline \end{array}$$

- The $\texttt{rotate}$ crystal $B(\lambda/\mu, d)^{\texttt{rotate}}$ is obtained from $B(\lambda/\mu, d)$ by $\texttt{rotating}$ the vertices, reversing each arrow $i \in [d-1]$ and relabeling it with $d - i$.
  - $B(\lambda/\mu, d)^{\texttt{rotate}} = B(\texttt{rotate}(\lambda/\mu), d) = B(\mu^\vee/\lambda^\vee, d)$.
  - $e_i\,\text{rotate}(U) = \text{rotate}\, f_{d-i}(U),\ \ f_i\,\text{rotate}(U) = \text{rotate}\, e_{d-i}(U),\ \ i \in [d-1]$
  - $\texttt{wt}(\texttt{rotate}(U)) = \omega_0 \texttt{wt}(U)$, ω_0 the long element of $\mathfrak{S}_d$.
  - $\texttt{rotate}$ is not a crystal isomorphism.

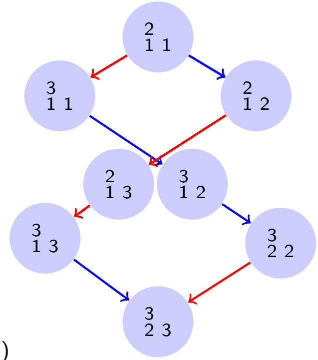
rotate crystal

- $B(\lambda/\mu, d)$ and $B(\mu^\vee/\lambda^\vee, d)$ have the same multiset of highest weights:

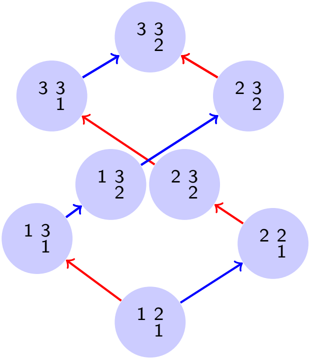
$$\bigsqcup_{\nu} B(\nu, d)^{c_{\mu,\nu,\lambda^\vee}} \simeq \bigsqcup_{\substack{\nu \\ T \in \mathrm{LR}^{\lambda}_{\mu,\nu}}} B(T) = B(\lambda/\mu, d) \simeq B(\mu^\vee/\lambda^\vee, d) = \bigsqcup_{\substack{\nu \\ T \in \mathrm{LR}^{\mu^\vee}_{\lambda^\vee,\nu}}} B(T)$$

$$\simeq \bigsqcup_{\nu} B(\nu, d)^{c_{\lambda^\vee,\nu,\mu}} \Rightarrow c_{\mu,\nu,\lambda^\vee} = c_{\lambda^\vee,\nu,\mu}$$

1 = — 2 = —



B((2,1,0),3)

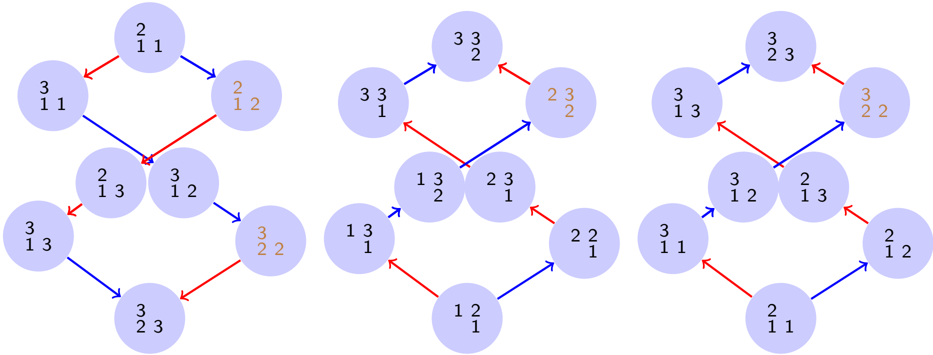


$B((2,2,0)/(1),3)$

Schützenberger–Lusztig involution

Schützenberger, 70' reversal/evacuation : $\begin{matrix}2 & \\ 1 & 1\end{matrix} \overset{\text{rotate}}{\longrightarrow} \begin{matrix}3 & 3\\ & 2\end{matrix} \overset{\text{rectification}}{\longrightarrow} \begin{matrix}3 & \\ 2 & 3\end{matrix}$

1 = — 2 = —



$$B((2,1,0),3) \underset{\text{rotate}}{\to} B((2,2,0)/(1),3) \underset{\text{rectification}}{\to} B((2,1,0),3)$$

$$\text{reversal}\, T^{\text{high}} = T^{\text{low}}$$

Schützenberger–Lusztig involution

- (*Lusztig, 90'*) There exists a unique involution of sets $\eta : B(\nu, d) \longrightarrow B(\nu, d)$ such that, for all $U \in B(\nu, d)$ and $i \in [d-1]$:
  1. $e_i \eta(U) = \eta f_{d-i}(U)$.
  2. $f_i \eta(U) = \eta e_{d-i}(U)$.
  3. $\mathrm{wt}(\eta(U)) = \omega_0 \mathrm{wt}(U)$, ω_0 the long element of $\mathfrak{S}_d$.

  η acts on $B(\lambda/\mu, d)$ via its action on the connected components and coplacity of crystal operators.
- $\eta = \texttt{reversal}/\texttt{evacuation}$.
- $\texttt{reversal}$ is a set involution on each connected component of $B(\lambda/\mu, d)$ that reverses all arrows and colors and weight. In particular, it interchanges the highest and lowest weight elements:

$$\texttt{reversal}(T^{\mathrm{high}}) = T^{\mathrm{low}}, \qquad \texttt{reversal}(T^{\mathrm{low}}) = T^{\mathrm{high}}.$$

LR commutor ρ for the symmetry $c_{\mu,\nu,\lambda} = c_{\lambda,\nu,\mu}$

$$
\begin{array}{ccc}
B(\lambda/\mu) & & B(\texttt{rotate}(\lambda/\mu)) \\
 & & \\
U & \overset{\texttt{rotate}}{\longleftrightarrow} & \texttt{rotate}(U) \\
\texttt{revers} \updownarrow & & \texttt{revers} \updownarrow \\
\texttt{reversal}(U) & \overset{\texttt{rotate}}{\longleftrightarrow} & \texttt{rotate reversal}(U)
\end{array}
$$

Proposition

- $(\mathit{reversal} \circ \mathit{rotate})^2 = 1.$
- $\mathit{reversal} \circ \mathit{rotate}(T^{high}) = \mathit{rotate} \circ \mathit{reversal}(T^{high}).$

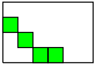
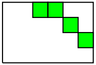
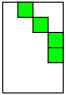
Theorem (A., Conflitti, Mamede 2009)

Let $\rho := \mathit{reversal} \circ \mathit{rotate} = \mathit{rotate} \circ \mathit{reversal}.$ *The involution*

$$\rho : \mathrm{LR}^{\lambda^\vee}_{\mu,\nu} \longrightarrow \mathrm{LR}^{\mu^\vee}_{\lambda,\nu},\ T \mapsto \rho(T) = \mathit{rotate} \circ \mathit{reversal}(T)$$

is an LR commutor that exhibits the symmetry $c_{\mu,\nu,\lambda} = c_{\lambda,\nu,\mu}.$

LR transposer ϱ for the symmetry $c_{\mu,\nu,\lambda} = c_{\mu^{\mathrm{tr}},\nu^{\mathrm{tr}},\lambda^{\mathrm{tr}}}$

λ/μ  $\texttt{rotate}\,\lambda/\mu$  $\texttt{tr rotate}\,\lambda/\mu$  $\texttt{tr rotate}\,\lambda/\mu = \mu^{\vee\mathrm{tr}}/\lambda^{\vee\mathrm{tr}}$

Proposition

$$\begin{array}{ccc} \mathrm{LR}^{\lambda}_{\mu,\nu} & \overset{\blacklozenge}{\longleftrightarrow} & \mathrm{LR}^{\mu^{\vee tr}}_{\lambda^{\vee tr},\nu^{tr}} \\ \mathit{revers}\updownarrow & & \mathit{revers}\updownarrow \\ \mathrm{LR}^{\lambda}_{\mu,\mathit{rotate}\,\nu} & \overset{\blacklozenge}{\longleftrightarrow} & \mathrm{LR}^{\mu^{\vee tr}}_{\lambda^{\vee tr},\mathit{rotate}\,\nu^{tr}} : \end{array}$$

- $B(\lambda/\mu, d) = \bigsqcup_{\substack{\nu \\ T\in\mathrm{LR}^{\lambda}_{\mu,\nu}}} B(T)$. $B(\mu^{\vee tr}/\lambda^{\vee tr}, n-d)) = \bigsqcup_{\substack{\nu \\ T\in\mathrm{LR}^{\lambda}_{\mu,\nu}}} B(\blacklozenge\, T)$.
- $(\blacklozenge\,\mathit{reversal})^2 = 1$.
- $(\blacklozenge\,\mathit{rotate})^2 = 1$.

♦ Zaballa 95, A. 99

orthogonal transpose $(\mu, \nu, \lambda) \longleftrightarrow (\lambda^t, \nu^t, \mu^t)$

$T =$

		2	4	4											
			1	1	2	3	3								
							1	2	2	2					
										1	1	1	1	1	

$\blacklozenge T =$

5			
2	8		
1	7		
	4		
	2		
	1	6	
		3	
		2	
		1	5
			4
			3
			2
			1

LR transposer ϱ

- $\rho :=$ `reversal` $\circ$ `rotate` $=$ `rotate` $\circ$ `reversal`
- $\blacklozenge\rho = \blacklozenge$ `rotate` $\circ$ `reversal` $=$ `rotate` $\blacklozenge$ `reversal` $=$ `rotate` $\circ$ `reversal` $\blacklozenge = \rho\blacklozenge$.
- $\varrho := \blacklozenge\rho = \rho\blacklozenge$.

Theorem (A., Conflitti, Mamede, 2009)

$$\varrho : \mathrm{LR}^{\lambda^\vee}_{\mu,\nu} \to \mathrm{LR}^{\lambda^{tr\vee}}_{\mu^{tr},\nu^{tr}},\ \mathrm{T} \mapsto \varrho(\mathrm{T}) = \blacklozenge\rho(\mathrm{T}) = \blacklozenge \textit{rotate} \circ \textit{reversal}(\mathrm{T}),$$

is an involution exhibiting the symmetry $c_{\mu\,\nu\,\lambda} = c_{\mu^{tr}\,\nu^{tr}\,\lambda^{tr}}$.

The three involutions *reversal*, ρ, *and* ϱ *are* $\mathcal{H} \sqcup \{rotate\}$*-reducible to each other and in particular to the* *reversal* *or Schützenberger-Lusztig involution.*

Theorem (A., Conflitti, Mamede, 2009)

The LR transposers ϱ^{WHS} *(White 90, Hanlon-Sundaram 92),* ϱ^{BSS} *(Benkart-Sottile-Stroomer 96),* ϱ^{A} *(A. 99) and* ϱ *are identical, and* $\mathcal{H} \sqcup \{rotate\}$*-reducible to the* *reversal* *or Schützenberger-Lusztig involution,*

$$\varrho = \varrho^{BSS} = \varrho^{WHS} = \blacklozenge\rho = \blacklozenge \textit{rotate} \circ \textit{reversal}$$

$\mathbb{Z}_2 \times \mathfrak{S}_3$-symmetries via `tableau-switching`

Theorem

$\mathbb{Z}_2 \times \mathfrak{S}_3 = \langle \tau, \varsigma_1, \varsigma_2 \mid \tau^2 = \varsigma_1^2 = \varsigma_2^2 = (\varsigma_1\varsigma_2)^3 = 1 = (\tau\varsigma_1)^2 = (\tau\varsigma_2)^2 \rangle$.

- *[Benkart, Sottile, Stroomer, 96] $\mathbb{Z}_2 \times \mathfrak{S}_3$-symmetries via `tableau-switching` LR commutors ρ_1^{BSS}, ρ_2^{BSS}, and LR transposer ϱ^{BSS}.*
- *$\rho_1^{BSS} : \mathrm{LR}^{\lambda^\vee}_{\mu,\nu} \to \mathrm{LR}^{\lambda^\vee}_{\nu,\mu}$ and $\rho_2^{BSS} : \mathrm{LR}^{\lambda}_{\mu,\nu} \to \mathrm{LR}^{\nu^\vee}_{\mu,\lambda}$ the `tableau-switching` LR commutors exhibit the symmetries $c_{\mu,\nu,\lambda} = c_{\nu,\mu,\lambda}$ and $c_{\mu,\nu,\lambda} = c_{\mu,\lambda,\nu}$ respectively.*

Theorem (Thomas-Yong, 2010)

$\mathfrak{S}_3$-symmetries and the carton rule

The carton rule exhibits uniformly the $\mathfrak{S}_3$-symmetries and is built upon Fomin's `jeu de taquin growth-diagrams` and the `infusion involution`, a particular case of Benkart-Sottille-Stroomer tableau-switching on pairs of standard tableaux.

The symmetries in the other coset of $\mathcal{H}$

The LR commutors ρ_1^{BSS}, ρ_2^{BSS}, $\rho = \texttt{reversal} \circ \texttt{rotate}$, and the LR transposer $\varrho = \varrho^{BSS}$ are related via $\mathcal{H}$-involutions $\spadesuit$, $\clubsuit$ and $\blacklozenge$.

Theorem

- $\rho = \blacklozenge\varrho = \varrho\blacklozenge, \quad c_{\mu,\nu,\lambda} = c_{\lambda,\nu,\mu}.$
- $\rho_1^{BSS} = \spadesuit\blacklozenge\,\rho = \clubsuit\spadesuit\,\rho = \blacklozenge\clubsuit\rho = \spadesuit\varrho = \varrho\spadesuit, \quad c_{\mu,\nu,\lambda} = c_{\nu,\mu,\lambda}.$
- $\rho_2^{BSS} = \blacklozenge\spadesuit\,\rho = \spadesuit\clubsuit\,\rho = \clubsuit\blacklozenge\,\rho = \clubsuit\varrho = \blacklozenge\spadesuit\blacklozenge\,\varrho = \varrho\clubsuit, \quad c_{\mu,\nu,\lambda} = c_{\mu,\lambda,\nu}.$
- *[A. 2017], All known LR commutors for the symmetry $c_{\mu,\nu,\lambda} = c_{\nu,\mu,\lambda}$ coincide with tableau switching involution ρ_1^{BSS}.*

All known LR commutors and LR transposers are $\mathcal{H}$-reducible to each other and to the Benkart-Sottile-Stroomer tableau switching involution.

In particular, they are $\mathcal{H} \sqcup \{\mathit{rotate}\}$-reducible to the involution reversal, equivalently, Schützenberger-Lusztig involution.

$$\varrho\mathcal{H} = \mathcal{H}\varrho = \{1, \spadesuit, \blacklozenge, \clubsuit = \spadesuit\blacklozenge\spadesuit, \blacklozenge\spadesuit, \spadesuit\blacklozenge\}\varrho = \{\varrho, \rho_1, \rho, \rho_2, \clubsuit\rho, \spadesuit\rho\} = \rho\mathcal{H} = \mathcal{H}\rho$$

A faithful representation of $\mathbb{Z}_2 \times D_3 \simeq \mathbb{Z}_2 \times \mathcal{H}$ in $\mathfrak{S}_{\mathcal{LR}}$

$$
\begin{array}{rcl}
\varpi : \mathbb{Z}_2 \times \mathcal{H} & \longrightarrow & \mathfrak{S}_{\mathcal{LR}} \\
\tau & \mapsto & \varrho \\
\tau\varsigma & \mapsto & \blacklozenge \\
\tau\varsigma_1 & \mapsto & \spadesuit
\end{array}
$$

$$\mathbb{Z}_2 \times D_3 \simeq \langle \spadesuit, \blacklozenge, \varrho \rangle := \langle \spadesuit, \blacklozenge, \varrho : \varrho^2 = \spadesuit^2 = \blacklozenge^2 = (\spadesuit\blacklozenge)^3 = (\spadesuit\varrho)^2 = (\blacklozenge\varrho)^2 = 1 \rangle.$$

Gelfand-Tsetlin patterns and LR companion pairs

Theorem (I.M. Gelfand, A.V. Zelevinsky 1986, A.D. Berenstein, A.V.Zelevinsky 1989)

The following statements are equivalent

- $T \in \mathrm{LR}^{\lambda}_{\mu,\nu}$
- *there exists a GT pattern* G_ν *of base* ν *and content* λ/μ *satisfying*

$$\varepsilon_{j-1}(G_\nu) \leq \mu_{j-1} - \mu_j, \quad \textit{for all } 1 < j \leq d.$$

- *there exists a GT pattern* L_μ *of base* μ *and weight the reverse of* λ/ν *satisfying*

$$\varphi_{d-j}(L_\mu) \leq \nu_j - \nu_{j+1}, \; \textit{for } 1 \leq j < d.$$

(L_μ, G_ν) *is said to be the LR companion pair of* T.

Definition

- $\mathrm{LR}^{\nu}_{\lambda/\mu}$ is the set of (right) LR companion tableaux of $\mathrm{LR}^{\lambda}_{\mu,\nu}$.
- $\mathrm{LR}^{\nu}_{\lambda/\mu}$ is the set of vertices G_ν in $B(\nu)$ and content λ/μ such that $\varepsilon_{j-1}(G_\nu) \leq \mu_{j-1} - \mu_j$, for all $1 < j \leq d$.

HIVES

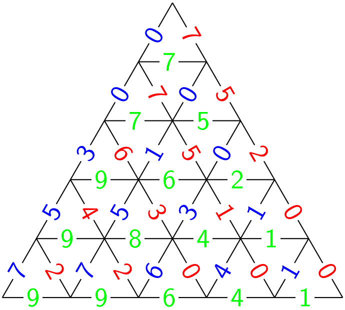
- $\mathrm{HIVE}^{\lambda}_{\mu,\nu}$ as the interlocking of the LR companion pairs in $\mathrm{LR}^{\lambda}_{\mu,\nu}$

$$L_\mu = \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 3 & & 1 & & 0 & & \\ & 5 & & 5 & & 3 & & 1 & \\ 7 & & 7 & & 6 & & 4 & & 1 \end{array}$$

$T =$

3								
1	2	2	3					
			1	1	2			
					1	1	2	2
							1	1

$$G_\nu = \begin{array}{ccccccccc} & & & & 7 & & & & \\ & & & 7 & & 5 & & & \\ & & 6 & & 5 & & 2 & & \\ & 4 & & 3 & & 1 & & 0 & \\ 2 & & 2 & & 0 & & 0 & & 0 \end{array}$$



The linear map bijection from LR tableaux to LR companions. Recording matrix of a tableau.

- $\iota : \mathrm{LR}^{\lambda}_{\mu,\nu} \to \mathrm{LR}_{\nu,\lambda/\mu},\ T \mapsto \iota(T) = G_\nu$ the `LR companion tableau` of T.

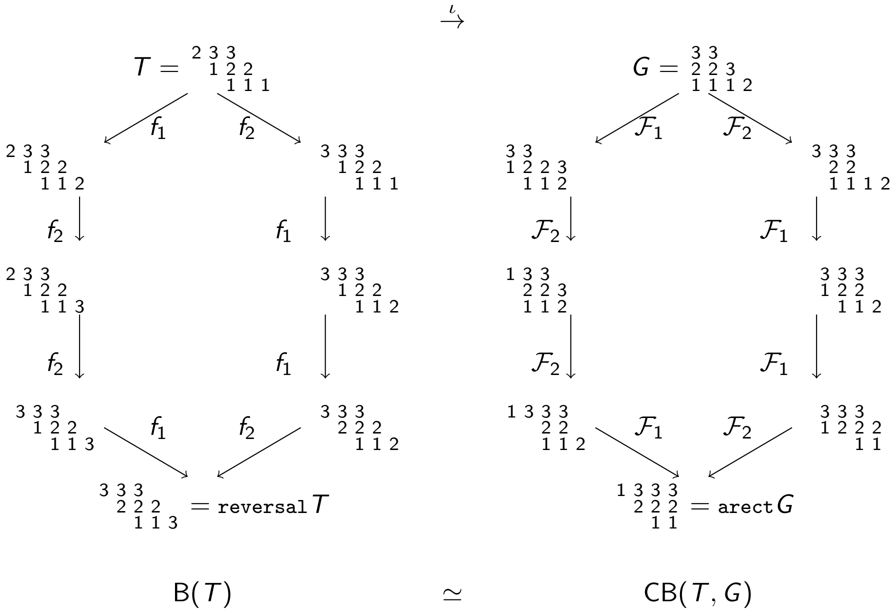
$$\iota : T = \begin{array}{|c|c|c|c|c|}\hline 2&3&3&&\\ \hline &1&2&2&\\ \hline &&1&1&1\\ \hline\end{array} \to M = \begin{pmatrix} 3&0&0\\ 1&2&0\\ 0&1&2 \end{pmatrix} \to M^t = \begin{pmatrix} 3&1&0\\ 0&2&1\\ 0&0&2 \end{pmatrix} \to G_{432} = \begin{array}{|c|c|c|c|c|}\hline 3&3&&&\\ \hline 2&2&3&&\\ \hline 1&1&1&2&\\ \hline\end{array}$$

- We may extend in the same fashion ι to $B(T)$.

$$T^{\mathrm{low}} = \begin{array}{|c|c|c|c|c|}\hline 3&3&3&&\\ \hline &2&2&2&\\ \hline &&1&1&3\\ \hline\end{array} \overset{\iota}{\to} M = \begin{pmatrix} 2&0&1\\ 0&3&0\\ 0&0&3 \end{pmatrix} \to M^t = \begin{pmatrix} 2&0&0\\ 0&3&0\\ 1&0&3 \end{pmatrix} \to G_{234} = \begin{array}{|c|c|c|c|c|}\hline &1&3&3&3\\ \hline &&2&2&2\\ \hline &&&1&1\\ \hline\end{array}$$

- The map ι induces a $\mathfrak{gl}_d$ structure on its image $\iota B(T)$ via *jeu de taquin* operators on the consecutive rows of the (right) LR companion G of T. This is the cocrystal $CB(T,G)$ of $B(T)$ and $CB(T,G) \simeq B(T)$.

The cocrystal



LR companion symmetries via cocrystal

Theorem (A., Conflitti, Mamede, 2009)

Let $T \in \mathrm{LR}^{\lambda}_{\mu,\nu}$ and $G_\nu \in \mathrm{LR}_{\nu,\lambda/\mu}$ its right LR companion tableau. Then

- *the following commutative diagram holds:*

$$\begin{array}{ccccc} T & \overset{\mathtt{reversal}}{\longleftrightarrow} & \mathtt{reversal}(T) & \overset{\mathtt{rotate}}{\longleftrightarrow} & \rho(T) \\ \iota \updownarrow & & \iota \updownarrow & & \iota \updownarrow \\ G_\nu & \underset{\mathtt{arectification}}{\longleftrightarrow} & \mathtt{arectification}(G_\nu) & \underset{\mathtt{rotate}}{\longleftrightarrow} & \mathtt{evac}(G_\nu) \end{array}$$

- *LR commutor ρ translates to evacuation of right LR companion tableaux*

$$\rho : \mathrm{LR}_{\nu,\lambda/\mu} \longrightarrow \mathrm{LR}_{\nu,\,\mu^\vee/\lambda^\vee} : \mathrm{G}_\nu \mapsto \mathtt{evac}\,\mathrm{G}_\nu$$

such that $\mathtt{evac}\,\iota(T) = \iota(\rho(T))$.

LR companion symmetries via cocrystal

Theorem

- *The $\mathcal{H}$-symmetries for LR companions.*
  1. *[Lecouvey, Lenart, 2017] The $\mathcal{H}$-involution $\blacklozenge$ translates to right LR companion tableaux*
  $$\blacklozenge : \mathrm{LR}_{\nu,\lambda/\mu} \to \mathrm{LR}_{\nu^{tr},\mu^{\vee\, tr}/\lambda^{\vee\, tr}},\ \mathrm{G}_\nu \mapsto \blacklozenge\mathrm{G}_\nu$$
  such that $\iota\blacklozenge T = \blacklozenge\iota(T)$ *whenever* $T \in \mathrm{LR}^{\lambda}_{\mu,\nu}$.
  2. *The $\mathcal{H}$-involution $\spadesuit$ translates to right LR companion tableaux*
  $$\spadesuit : \mathrm{LR}_{\nu,\lambda/\mu} \to \mathrm{LR}_{\mu^{tr},\lambda^{tr}/\nu^{tr}} : \mathrm{G}_\nu \mapsto \spadesuit\mathrm{G}_\nu$$
  where $\iota\spadesuit T = \spadesuit\iota(T)$ *whenever* $T \in \mathrm{LR}^{\lambda}_{\mu,\nu}$.
- *Commutors and transposers for LR companions*
  - *Let* $T \in \mathrm{LR}^{\lambda}_{\mu,\nu}$ *and* $G_\nu \in \mathrm{LR}_{\nu,\lambda/\mu}$ *its right LR companion tableau. Then:*

$$\begin{array}{ccccccccc}
T & \overset{\mathtt{revers}}{\leftrightarrow} & \mathtt{revers}(T) & \overset{\mathtt{rotate}}{\leftrightarrow} & \rho(T) & \overset{\blacklozenge}{\leftrightarrow} & \varrho(T) & \overset{\spadesuit}{\leftrightarrow} & \rho_1(T) \\
\iota \updownarrow & & \iota \updownarrow & & \iota \updownarrow & & \iota \updownarrow & & \iota \updownarrow \\
G_\nu & \underset{\mathtt{arect}}{\leftrightarrow} & \mathtt{arect}(G_\nu) & \underset{\mathtt{rotate}}{\leftrightarrow} & \mathtt{evac}(G_\nu) & \underset{\blacklozenge}{\leftrightarrow} & \blacklozenge\mathtt{evac}(G_\nu) & \underset{\spadesuit}{\leftrightarrow} & \spadesuit\blacklozenge\ \mathtt{evac}(G).
\end{array}$$

Henriques-Kamnitzer LR commutor

Theorem (Henriques-Kamnitzer 2006)

(L_μ, G_ν) is the LR companion pair (left and right) of $T \in \mathrm{LR}^{\lambda}_{\mu,\nu}$ iff $(\mathrm{evac}\, G_\nu, \mathrm{evac}\, L_\mu)$ is the LR companion pair (left and right) of $\rho_1(T) \in \mathrm{LR}^{\lambda}_{\nu,\mu}$.

$$\begin{array}{rcl} B(\mu) \otimes B(\nu) & \to & B(\nu) \otimes B(\mu) \\ U \otimes V & \mapsto & \eta(V) \otimes \eta(U) \end{array}$$

$$\spadesuit\blacklozenge \mathtt{evac}\ G_\nu = \mathtt{evac} \spadesuit\blacklozenge\ G_\nu = \mathtt{evac} L_\mu$$

Corollary

The pair (L_μ, G_ν) is the companion pair of T if and only if $L_\mu = \blacklozenge\spadesuit G_\nu$ and L_μ is the left companion of T or G_ν is the right companion of T.

- Pak and Vallejo (2010) have provided a linear map between the right LR companion and the left LR companion.

Hive symmetries

Theorem

Let $H \in \mathrm{HIVE}^{\lambda}_{\mu,\nu}$ *be defined by the LR companion pair* $(L = \blacklozenge\spadesuit G, G)$. *Then we have the following LR companion pairs or Hives under the action of* $\mathbb{Z}_2 \times D_3$:

- $\spadesuit H = (\blacklozenge G, \spadesuit G)$
- $\blacklozenge H = (\spadesuit L = \blacklozenge\spadesuit\blacklozenge G, \blacklozenge G)$
- $\blacklozenge\spadesuit\blacklozenge H = (\spadesuit G, \blacklozenge\spadesuit\blacklozenge G = \spadesuit L)$
- $\blacklozenge\spadesuit H = (\blacklozenge\spadesuit L = \spadesuit\blacklozenge G, \blacklozenge\spadesuit G = L)$
- $\spadesuit\blacklozenge\, H = (G, \spadesuit\blacklozenge G)$
- $\rho H = (\blacklozenge\spadesuit evac\, G, evac\, G)$
- $\rho_1 H = (evac\, G, \spadesuit\blacklozenge evac\, G = evac\, L)$
- $\rho_2 H = (\spadesuit\blacklozenge evac\, G = evac\, L, \blacklozenge\spadesuit evac\, G)$
- $\varrho H = (\blacklozenge evac\, L, \blacklozenge evac\, G)$
- $\spadesuit\rho H = (\blacklozenge evac\, G, \spadesuit evac\, G)$
- $\blacklozenge\spadesuit\blacklozenge\rho H = (\spadesuit evac\, G, \blacklozenge\spadesuit\blacklozenge evac\, G) = (\spadesuit evac\, G, \blacklozenge evac\, L)$.

$\mathfrak{gl}_d$-crystal tensor products, LR tableaux, LR companions and Hives

- $B(\lambda/\mu, d) = \bigsqcup_{\substack{\nu \\ T \in \mathrm{LR}^{\lambda}_{\mu,\nu}}} B(T) \simeq \bigsqcup_{\nu} B(\nu, d)^{c^{\lambda}_{\mu,\nu}}.$

- [G. P. Thomas 78, Nakashima 93, Henriques-Kamnitzer 2006] The tensor product decomposition:

$$B(\mu, d) \otimes B(\nu, d) = \bigsqcup_{\substack{\lambda \\ G_\nu \in \mathrm{LR}_{\nu,\lambda/\mu}}} B(Y_\mu \otimes G_\nu) \simeq \bigsqcup_{\substack{\lambda \\ T \in \mathrm{LR}^{\lambda}_{\mu,\nu}}} B(\lambda, d) \times \{T\}$$

$$= \bigsqcup_{\substack{\lambda \\ H \in \mathtt{HIVE}^{\lambda}_{\mu,\nu}}} B(\lambda, d) \times \{H\} \simeq \bigsqcup_{\lambda} B(\lambda, d)^{c^{\lambda}_{\mu,\nu}},$$

where for each crystal connected component of $B(\mu, d) \otimes B(\nu, d)$

  - there exists $T \in \mathrm{LR}^{\lambda}_{\mu,\nu}$ such that
    - ★ the highest weight element $Y_\mu \otimes G_\nu$ satisfies $Y_\mu \otimes G_\nu \overset{RSK}{\mapsto} (Y_\lambda, T)$,
    - ★ the lowest weight element $L_\mu \otimes Y_{\omega_0\nu}$ satisfies $L_\mu \otimes Y_{\omega_0\nu} \overset{RSK}{\mapsto} (Y_{\omega_0\lambda}, T)$.
  - $Y_\mu \otimes G_\nu$ and $L_\mu \otimes Y_{\omega_0\nu}$ are the highest and lowest weight elements whenever (L_μ, G_ν) is an LR companion pair or $H = (L_\mu, G_\nu)$ is a hive in $\mathtt{HIVE}^{\lambda}_{\mu,\nu}$.