Cones of Hyperplane Arrangements through the Varchenko-Gel'fand Ring

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Hyperplane Arrangements & Cones

Hyperplane Arrangements

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ is a central hyperplane arrangement in a real vector space V. This talk concerns chambers

$$\mathcal{C}(\mathcal{A}) = \binom{\text{open, connected}}{\text{components of } V \backslash \mathcal{A}}$$

and intersections

$$\mathcal{L}(\mathcal{A}) = \left\{ \bigcap_{i \in B} H_i \neq \emptyset \;\middle|\; B \subseteq [n] \right\}$$

of this arrangement.

Hyperplane Arrangements

The elements of $\mathcal{L}(\mathcal{A})$ form a poset under reverse inclusion. This poset is

- a geometric lattice,
- ranked by codimension, and
- equipped with a Möbius function $\mu(X, Y)$ for $X \supseteq Y$ in $\mathcal{L}(\mathcal{A})$.

The *Whitney numbers* of $\mathcal{A}$ are

$$c_k(\mathcal{A}) = \sum_{\substack{X \in \mathcal{L}(\mathcal{A}) \\ rk(X) = k}} |\mu(V, X)|.$$

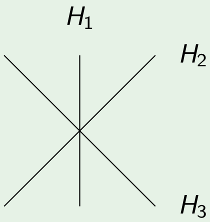
We collect these into the *Poincaré polynomial* of the arrangement

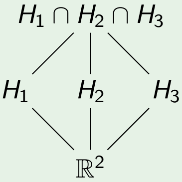
$$\text{Poin}(\mathcal{A}, t) = \sum_{k \geq 0} c_k(\mathcal{A})\ t^k.$$

Hyperplane Arrangements

Example

An arrangement $\mathcal{A} = \{H_1, H_2, H_3\} \subseteq \mathbb{R}^2$ (left) together with the Hasse diagram of its intersection poset $\mathcal{L}(\mathcal{A})$ (right).





The Poincaré polynomial of this arrangement is

$$\text{Poin}(\mathcal{A}, t) = 1 + 3t + 2t^2.$$

Cones of Hyperplane Arrangements

Each hyperplane $H \in \mathcal{A}$ defines a pair of open halfpsaces.

Definition

A *cone* $\mathcal{K}$ of an arrangement $\mathcal{A}$ is an intersection of (open) half spaces defined by some of the hyperplanes of $\mathcal{A}$.

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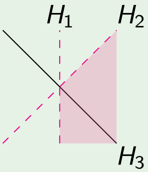
One cone defined by H_1 and H_2.



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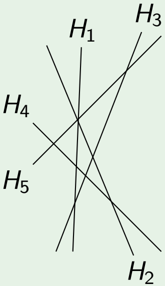
Let's consider a cone $\mathcal{K}$ defined by H_4 and H_5 in the following three-dimensional arrangement of which I've drawn an affine slice.



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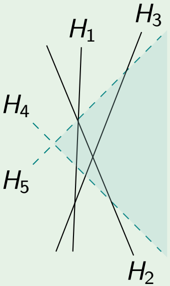

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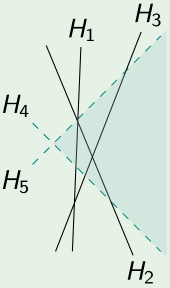

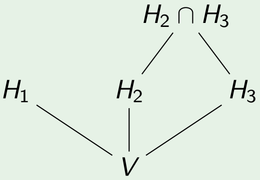

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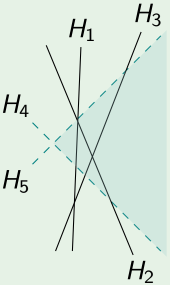
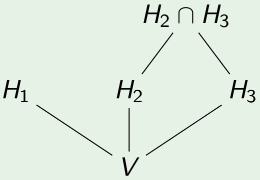
1. The *chambers of* $\mathcal{K}$ are the chambers $\mathcal{C}(\mathcal{K}) \subseteq \mathcal{C}(\mathcal{A})$ strictly contained in $\mathcal{K}$.
2. The nonempty intersections $\mathcal{L}^{int}(\mathcal{K}) \subseteq \mathcal{L}(\mathcal{A})$ whose intersection with $\mathcal{K}$ is nonempty are called *interior intersections* of $\mathcal{K}$, i.e.

$$X \in \mathcal{L}^{int}(\mathcal{K}) \qquad \text{if} \qquad X \cap \mathcal{K} \neq \varnothing.$$

Example





Cones in an Arrangement

The elements of $\mathcal{L}^{int}(\mathcal{K})$ form a poset under reverse inclusion. This poset is

- a meet semi-lattice,
- ranked by codimension, and
- for all $X \in \mathcal{L}^{int}(\mathcal{K})$, the lower interval $[V, X]$ is isomorphic to the corresponding interval in $\mathcal{L}(\mathcal{A})$.

The *(unsigned) Whitney numbers* of $\mathcal{K}$ are

$$c_k(\mathcal{K}) = \sum_{\substack{X \in \mathcal{L}^{int}(\mathcal{K}) \\ rk(X)=k}} |\mu(V, X)|.$$

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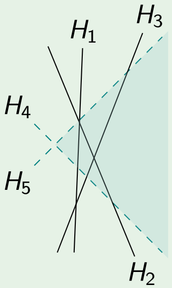

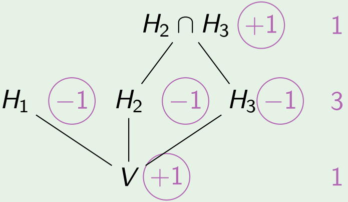

Thus $\texttt{Poin}(\mathcal{K}, t) = 1 + 3t + t^2$.

Zaslavsky's Theorem for cones

Theorem (Zaslavsky, '77)

For a cone $\mathcal{K}$ of an arrangement $\mathcal{A}$ with intersection poset $\mathcal{L}^{int}(\mathcal{K})$, we have

$$\#\mathcal{C}(\mathcal{K}) = \sum_{X \in \mathcal{L}^{int}(\mathcal{K})} |\mu(V, X)| = \sum_{k=0}^{n} c_k(\mathcal{K})$$

where $\mu(V, X)$ denotes the Möbius function of $\mathcal{L}^{int}(\mathcal{K})$ and $\{ c_k(\mathcal{K}) \}$ are the (unsigned) Whitney numbers of the cone $\mathcal{K}$.

In other words $\#\mathcal{C}(\mathcal{K}) = [\mathtt{Poin}(\mathcal{K}, t)]_{t=1}$.

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This result is well-known when we take $\mathcal{K}$ to be the full arrangement.

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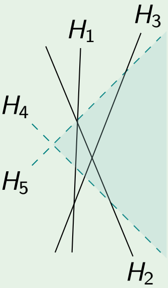

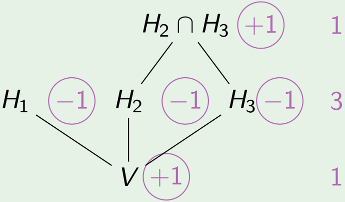

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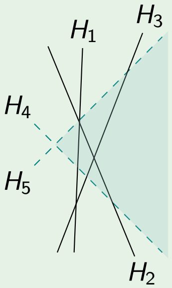

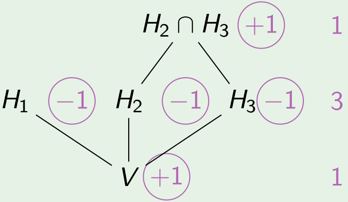

Zaslavsky says: there are $1 + 3 + 1 = 5$ chambers in this cone.

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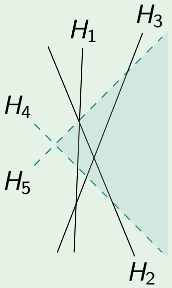

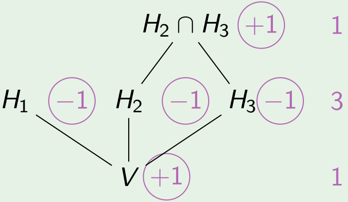

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Goal: Construct a ring from $\mathcal{K}$ whose Hilbert Series is $\texttt{Poin}(\mathcal{K}, t)$.

The Varchenko-Gel'fand Ring

The Varchenko-Gel'fand Ring of a Cone

Definition

The *Varchenko-Gel'fand ring* of a cone $\mathcal{K}$ is the collection of maps $VG(\mathcal{K}) = \{f : \mathcal{C}(\mathcal{K}) \to \mathbb{Z}\}$ under pointwise addition and multiplication.

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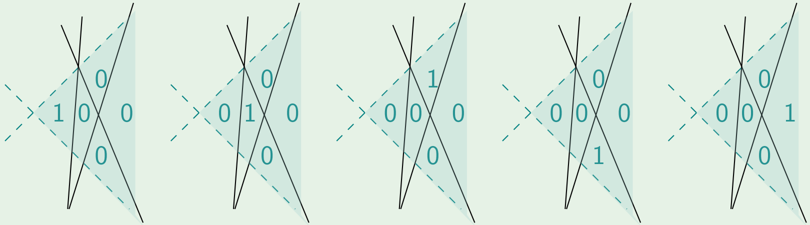
For every cone $\mathcal{K}$, $VG(\mathcal{K})$ has a $\mathbb{Z}$-basis of indicator functions of chambers in $\mathcal{C}(\mathcal{K})$, as shown in the example.

Example



The Varchenko-Gel'fand Ring of a Cone

Pick an orientation of $\mathcal{A}$. It's easy to see that the Varchenko-Gel'fand ring $VG(\mathcal{K})$ of a cone $\mathcal{K}$ is generated by Heaviside functions

$$x_i(C) = \begin{cases} 1 & \text{if } v \in H_i^+ \cap \mathcal{K} \\ 0 & \text{else} \end{cases} \qquad \text{for } C \in \mathcal{C}(\mathcal{K})$$

for each hyperplane $H_i \in \mathcal{L}^{\text{int}}(\mathcal{K})$.

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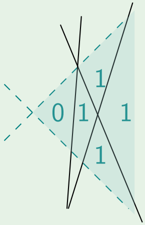
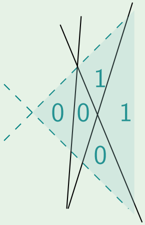
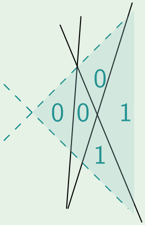
$x_1 =$ 

$x_2 =$ 

$x_3 =$ 

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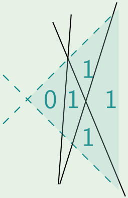

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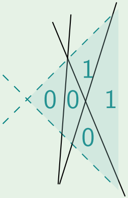

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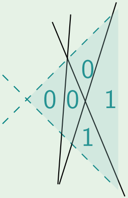
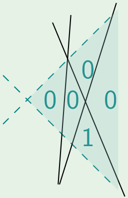


We can write the basis element corresponding to any chamber as a product of Heavisde functions for its walls.



$$= (1 - x_2)x_3x_4 = (1 - x_2)x_3$$

Define a map

$$\varphi : \mathbb{Z}[e_1, \ldots, e_n] \to VG(\mathcal{K})$$
$$e_i \mapsto x_i.$$

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- By the previous observation, this map is surjective.
- $I_{\mathcal{K}} := \ker \varphi$ has a nice description.

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- A *circuit* of a collection of vectors is a minimal dependent set, i.e. a set of linear dependent vectors such that if any single vector is removed, the set is independent. We will view circuits as sets of indices, so that $C \subseteq \{1, 2, \ldots, n\}$.

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- We can choose a set of normal vectors for the hyperplanes of $\mathcal{A}$ so that v_i is normal to H_i.
- A *circuit* of a collection of vectors is a minimal dependent set, i.e. a set of linear dependent vectors such that if any single vector is removed, the set is independent. We will view circuits as sets of indices, so that $C \subseteq \{1, 2, \ldots, n\}$.
- We'll keep track of *signed circuits* where we write down the explicit linear relations

$$\sum_{c \in C} \alpha_c v_c = 0 \qquad \text{for } \alpha_i \in \mathbb{R}$$

and we sort the elements of C into C^+ and C^-, depending on whether $\alpha_c > 0$ or $\alpha_c < 0$.

Presenting the Varchenko-Gel'fand Ring

Theorem (D.-B., '21)

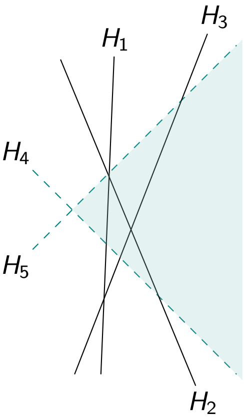
Let $\mathcal{K}$ be a cone of a central arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$. Then $VG(\mathcal{K}) \cong \mathbb{Z}[e_1, \ldots, e_n]/I_{\mathcal{K}}$ where $I_{\mathcal{K}}$ is generated by

1. *(`Idempotent`) $e_i^2 - e_i$ for $i \in [n]$,*
2. *(`Unit`) $e_i - 1$ for $i \in [n]$ such that H_i is a wall of $\mathcal{K}$,*
3. *(`Circuit`) $\prod_{i \in C^+} e_i \prod_{j \in C^-} (e_j - 1) - \prod_{i \in C^+} (e_i - 1) \prod_{j \in C^-} e_j$ for signed circuits $C = C^+ \cup C^-$,*

This was proved in 1987 by Varchenko and Gel'fand for $\mathcal{K} = V$.

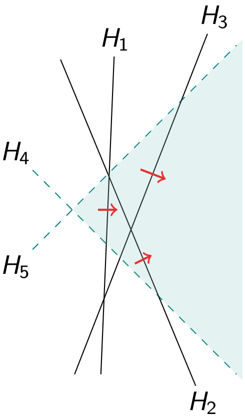
Example

Consider the following cone



Example

Consider the following cone with the orientation given by the red arrows



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$$e_4 - 1, \qquad e_5 - 1$$

Some of the signed circuits are on the left and their corresponding `Circuit` relation are on the right:

$$\begin{aligned}
\{2,5\} \cup \{1\} &\to e_2 e_5 (e_1 - 1) - (e_2 - 1)(e_5 - 1) e_1 \\
\{1,3\} \cup \{2,4\} &\to e_1 e_3 (e_2 - 1)(e_4 - 1) - (e_1 - 1)(e_3 - 1) e_2 e_4 \\
\{3,4,5\} \cup \{1\} &\to e_3 e_4 e_5 (e_1 - 1) - (e_3 - 1)(e_4 - 1)(e_5 - 1) e_1 \\
\{2,4\} \cup \{3,5\} &\to e_2 e_4 (e_3 - 1)(e_5 - 1) - (e_2 - 1)(e_4 - 1) e_3 e_5
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\{2,4\} \cup \{3,5\} &\to e_2 e_4(e_3 - 1)(e_5 - 1) - (e_2 - 1)(e_4 - 1)e_3 e_5
\end{aligned}$$

By combining the `Unit` and `Circuit` relations, we can write down a more refined set of generators.

Presenting the Varchenko-Gel'fand Ring

Theorem (D.-B., '21)

Let $W = \{i \in [n] \mid H_i \text{ is a wall of } \mathcal{K}\}$. For any graded monomial ordering on $\mathbb{Z}[e_1, \ldots, e_n]$, $I_\mathcal{K}$ has Gröbner basis[a]:

1. *(`Idempotent`)* $e_i^2 - e_i$ *for* $i \in [n]$,
2. *(`Unit`)* $e_i - 1$ *for* $i \in [n]$ *such that* $i \in W$
3. *(`Combination Circuit`) Let* $C = C^+ \cup C^-$ *be a signed circuit.*
   - *If* $W \cap C^\pm \neq \varnothing$ *but* $W \cap C^\mp = \varnothing$, *then*

$$\prod_{i \in C^+ \setminus W} e_i \prod_{j \in C^-} (e_j - 1) = \prod_{i \in C \setminus W} e_i - \pm l.o.t.$$

   - *If* $W \cap C = \varnothing$, *then*

$$\prod_{i \in C^+} e_i \prod_{j \in C^-} (e_j - 1) - \prod_{i \in C^+} (e_i - 1) \prod_{j \in C^-} e_j = \sum_{j \in C} \pm \prod_{i \in C - \{j\}} e_i \pm l.o.t.$$

[a] The leading term of any polynomial in $I_\mathcal{K}$ is divisible by the leading term of some polynomial in the Gröbner basis.

Goal Theorem

The Associated Graded Ring

- For $d \geq 0$, define $F_d := \mathbb{Z} \cdot \{\text{monomials of degree} \leq d\} \subseteq VG(\mathcal{K})$.

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- From this filtration, we define the *associated graded ring* of $VG(\mathcal{K})$:

$$\mathfrak{gr}_{\mathcal{F}}(VG(\mathcal{K})) := \bigoplus_{d \geq 0} F_d / F_{d-1}$$

where we set $F_{-1} = 0$.

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- This yields a filtration $\mathcal{F}$ of $VG(\mathcal{K})$: $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$
- From this filtration, we define the *associated graded ring* of $VG(\mathcal{K})$:

$$\mathfrak{gr}_{\mathcal{F}}(VG(\mathcal{K})) := \bigoplus_{d \geq 0} F_d / F_{d-1}$$

  where we set $F_{-1} = 0$.
- The Hilbert series (or Hilbert-Poincaré Series) of $\mathfrak{gr}_{\mathcal{F}}(VG(\mathcal{K}))$ is the formal power series

$$\sum_{d \geq 0} \mathrm{rk}_{\mathbb{Z}}(F_d / F_{d-1}) t^d$$

The Hilbert Series of $\mathfrak{gr}_{\mathcal{F}}(VG(\mathcal{K}))$

Theorem (D.-B., '21)

The Hilbert series of $\mathfrak{gr}_{\mathcal{F}}(VG(\mathcal{K}))$ *is* $\mathcal{Poin}(\mathcal{K}, t)$.

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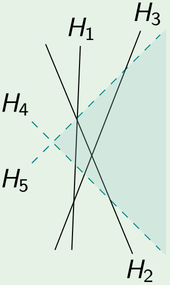
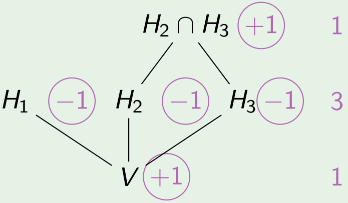
Example





The theorem says that the Hilbert series of $\mathfrak{gr}_{\mathcal{F}}(VG(\mathcal{K}))$ is $1 + 3t + t^2$.

The NBC basis (time permitting)

No Broken Circuit Sets

Recall

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- Let C be a circuit of $\mathcal{A}$. We can *break* C by removing the smallest index i contained in C. We call $C - \{i\}$ the *broken circuit* corresponding to C.

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Definition

A set $N \in NBC(\mathcal{A})$ is a $\mathcal{K}$-*NBC set* if

$$\bigcap_{i \in N} H_i \in \mathcal{L}^{\text{int}}(\mathcal{K}).$$

Denote the set of $\mathcal{K}$-NBC sets by $NBC(\mathcal{K})$.

A Basis for the Varchenko-Gel'fand Ring

Theorem (D.-B., '21)

Let $\mathcal{K}$ be a cone of a central arrangement $\mathcal{A}$. Then $VG(\mathcal{K})$ has

$$\left\{ \prod_{i \in N} e_i \;\middle|\; N \in NBC(\mathcal{K}) \right\}$$

as a $\mathbb{Z}$-basis.

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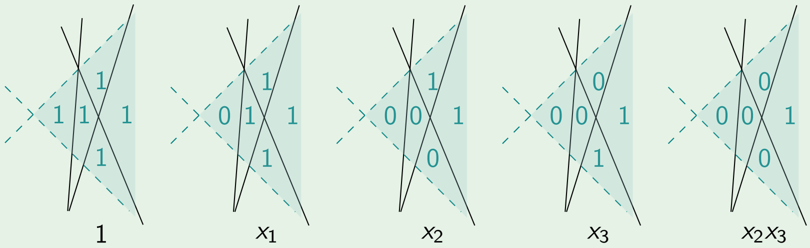

Thank you!

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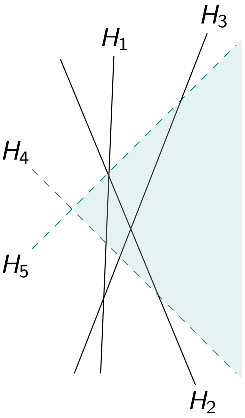
Advances in Math., 25(3):267–285, 1977.

A worked example of the Theorem

Example Computation I

Consider the following cone



The cone has 5 chambers, so $VG(\mathcal{K}) \cong \mathbb{Z}^5$. Earlier we computed its Whitney numbers, which are $(1, 3, 1)$.

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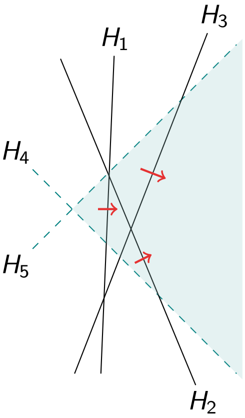

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Example Computation II

Let's write down the Gröbner basis for $I_{\mathcal{K}}$. The `Idempotent` and `Unit` relations are

$$e_1^2 - e_1, e_2^2 - e_2, e_3^2 - e_3, e_4^2 - e_4, e_5^2 - e_5$$

and $e_4 - 1, e_5 - 1$ respectively.

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and $e_4 - 1, e_5 - 1$ respectively. In order to write down the `Combination Circuit` relations, we need to do some work. The signed circuits are on the left and the relation is on the right:

$$\begin{aligned} \{2,5\} \cup \{1\} &\to e_2(e_1 - 1) = e_1 e_2 - e_2 \\ \{1,3\} \cup \{2,4\} &\to e_1 e_3 (e_2 - 1) = e_1 e_2 e_3 - e_1 e_3 \\ \{3,4,5\} \cup \{1\} &\to (e_1 - 1) e_3 = e_1 e_3 - e_3 \\ \{2,4\} \cup \{3,5\} &\to 0 \end{aligned}$$

Example Computation III

From this we can write down the NBC-basis of $VG(\mathcal{K})$ itself. The circuits are on the left and the broken circuits are on the right:

$$125 \to 25$$
$$1234 \to 234$$
$$1345 \to 345$$
$$2345 \to 345$$

The no broken circuit sets associated to $\mathcal{A}$ are:

$$\varnothing,$$
$$1, 2, 3, 4, 5,$$
$$12, 13, 14, 15, 23, 24, 34, 35, 45,$$
$$123, 124, 134, 135, 145$$

Example Computation III

From this we can write down the NBC-basis of $VG(\mathcal{K})$ itself. The circuits are on the left and the broken circuits are on the right:

$$125 \to 25$$
$$1234 \to 234$$
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The $\mathcal{K}$-NBC sets are:

$\varnothing$,
$1, 2, 3,$ 4, 5,
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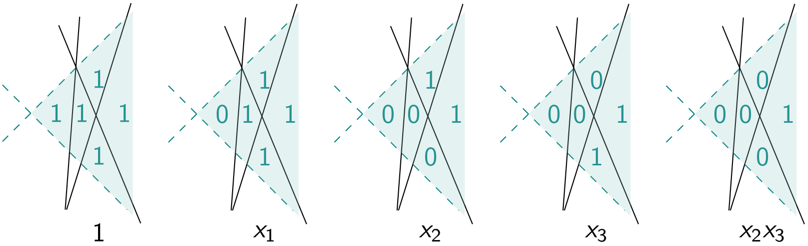
123, 124, 134, 135, 145

Example Computation IV

The NBC-basis for $VG(\mathcal{K})$ is



So the associated graded ring is

$$\mathfrak{gr}_{\mathcal{F}}(VG(\mathcal{K})) \cong \mathbb{Z} \cdot \{1\} \oplus \mathbb{Z} \cdot \{x_1, x_2, x_3\} \oplus \mathbb{Z} \cdot \{x_2x_3\}$$

and has Hilbert series $1 + 3t + t^2$.

Supersolvable Arrangements

What is a supersolvable arrangement?

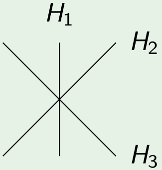
Definition

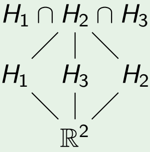
An arrangement is *supersolvable* if there is a maximal chain Δ of the intersection lattice $\mathcal{L}(\mathcal{A})$ such that for every chain K, the sublattice generated by Δ and K is *distributive*[a].

[a] A lattice L is distributive if for all $x, y \in L$, we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Example

The $(n-1)$st braid arrangement is supersolvable and consists of hyperplanes $H_{ij} = \{\mathbf{x} \in \mathbb{R}^d \mid x_i = x_j\}$ for $i, j \in [n]$. A linearly equivalent picture of the $(3-1)st$ braid arrangement is below (left) together with its intersection poset $\mathcal{L}(\mathcal{A})$ (right).





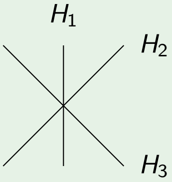
What is a supersolvable arrangement?

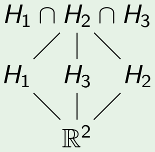
Theorem (Björner-Ziegler, '91)

When we order the broken circuits of a supersolvable arrangement by inclusion, the minimal broken circuits have cardinality exactly 2.

Example

The $(3-1)$st braid arrangement.





There is one circuit consisting of all three hyperplanes $\{1, 2, 3\}$.
The broken circuit is $\{2, 3\}$.

* The $(n-1)$st braid arrangement is the *complete graph arrangement*. **Upshot**: We can write down the circuits of the braid arrangement from the

What does being supersolvable have to do with the Varchenko-Gel'fand ring?

Definition

The *Varchenko-Gel'fand ring* of a cone $\mathcal{K}$ over a field $\mathbb{F}$ is the collection of maps $VG_{\mathbb{F}}(\mathcal{K}) = \{f : \mathcal{C}(\mathcal{K}) \to \mathbb{F}\}$ under pointwise addition and multiplication.

* Our previous theorems still hold for $VG_{\mathbb{F}}(\mathcal{K})$
  (in fact they are easier because we're now over a field!)

Theorem (D.-B. '21)

If $\mathcal{A}$ is a supersolvable arrangement, then for every cone $\mathcal{K}$, the associated graded ring $\mathfrak{gr}(VG_{\mathbb{F}}(\mathcal{K}))$ is Koszul.

(!!) This theorem fits into a larger context.

Fitting this into a Larger Context: the Orlik-Solomon Algebra

The Orlik-Solomon algebra is a noncommutative analogue of the Varchenko-Gel'fand ring.

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Theorem (Peeva '02)

If $\mathcal{A}$ is a supersolvable arrangement, then the Orlik-Solomon algebra of $\mathcal{A}$ is supersolvable.