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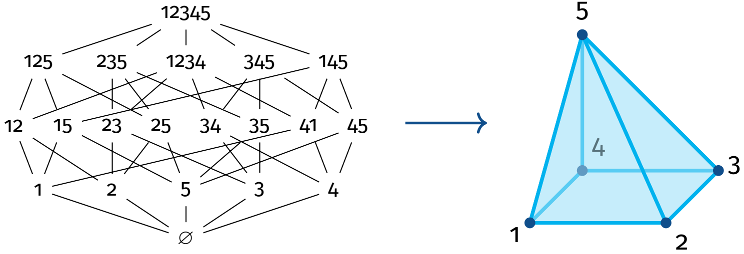

Slack realization spaces and realizability of polytopes

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joint work with **João Gouveia** (U Coimbra) and **Amy Wiebe** (FU Berlin)

Realizability of abstract polytopes

A classical problem in Polytope Theory concerns the existence of a convex realization of a given abstract polytope (lattice).

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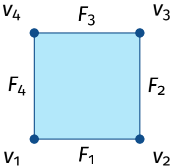
Question *Can we find certificates without any prescribed structure?*

The Slack Variety

P abstract d-dim polytope with vertices $v_1, \dots, v_n$ and facets $F_1, \dots, F_m$.

Symbolic slack matrix $S_P(\boldsymbol{x})$: sparse generic matrix whose (i,j) entry is zero if $v_i \in F_j$ and a variable otherwise.



$$S_P(\boldsymbol{x}) = \begin{bmatrix} 0 & x_0 & x_1 & 0 \\ 0 & 0 & x_2 & x_3 \\ x_4 & 0 & 0 & x_5 \\ x_6 & x_7 & 0 & 0 \end{bmatrix}$$

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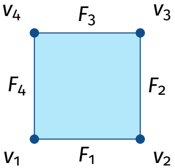

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Slack variety $\mathcal{V}_P := \overline{\{\xi \in \mathbb{R}^N_* : \operatorname{rank}(S_P(\xi)) \leq d+1\}} = \mathcal{V}(I_P)$ (slack ideal).

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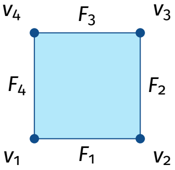

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Theorem (Gouveia, M., Thomas, Wiebe, 2019)

$\{$Realizations of polytope P up to projective equivalence$\}$ $\overset{1:1}{\longleftrightarrow}$ $\mathcal{V}_P \cap \mathbb{R}^N_{++}$ up to column and row scalings by positive scalars.

In particular, P is not realizable if and only if $\mathcal{V}_P \cap \mathbb{R}^N_{++} = \varnothing$.

The Grassmannian Variety

$\mathrm{Gr}(d+1,n)$ be the **Grassmannian variety** coordinatized by Plücker coordinates $\{p_J : J \subseteq \{1,\dots,n\}, |J| = d+1\}$.

$$\Gamma_0 = \{p_J : J \subseteq F_i \text{ facet of } P\} \text{ and } \Gamma_1 = \{p_J : J \text{ facet extension of } P\}.$$

Facet extension: $J = \{j_0,\dots,j_d\}$, where $j_0,\dots,j_{d-1}$ span a facet of P and j_d is not on that facet.

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Grassmannian variety of P:

$$\mathrm{Gr}(P) := \overline{\Pi_{\Gamma_1}\left(\left\{\xi \in \mathrm{Gr}(d+1, n) : \begin{array}{ll} p_J(\xi) = 0 & \text{if } J \in \Gamma_0 \\ p_J(\xi) \neq 0 & \text{if } J \in \Gamma_1 \end{array}\right\}\right)},$$

where Π_{Γ_1} is the signed projection onto the coordinates of Γ_1.

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Theorem (Gouveia, M., Wiebe, 2020+) *The Grassmannian variety* $\text{Gr}(P)$ *and the slack variety* $\mathcal{V}_P$ *are essentially equivalent.*

In particular, P is **not realizable** if and only if $\text{Gr}(P) \cap \mathbb{R}^N_{++} = \varnothing$.

Realizability and positive points

P is not realizable if a certain variety has **no positive points**.

Theorem (Becker, 1986) *A real variety $\mathcal{V}(I) \subseteq \mathbb{R}^N$ has no positive point if and only if there is an element of I of the form*

$$\boldsymbol{x}^\alpha + \sum \boldsymbol{x}^{\beta_i} \sigma_i(\boldsymbol{x}),$$

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In the case of $\mathrm{Gr}(P)$, the witness polynomials are called **final polynomials** and were used for certifying non-realizability [Bokowski, Sturmfels, 1989], [Bokowski, Richter, 1990].

Realizability and positive polynomials

- Sums of squares are hard to interpret and semidefinite programming has scalability issues.
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An alternative is to consider scalar σ_i. We say that a polynomial is positive if it is non-zero and has non-negative coefficients.

Proposition *Given a real variety $\mathcal{V}(I) \subseteq \mathbb{R}^N$, if I contains a positive polynomial, then $\mathcal{V}(I)$ has no positive points.*

Grassmannian vs Slack model

Grassmannian model: ideal generated by quadratic polynomials, full set of Plücker coordinates limits brute force search.

- Bi-quadratic final polynomials [Bokowski, Richter, 1990]
- Positive Plücker tree certificates [Pfeifle, 2020+]

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Slack model: slack ideal generated in higher degree, lower number of variables that can be further reduced by parametrizing the variety.

The Algorithm

Let P be an abstract d-dimensional polytope.

Inputs:

- F: list of facets of abstract polytope P as lists of vertex labels
- d: dimension of P
- k: maximum number of factors in the products of constraints
- l: maximum degree of constraints to consider

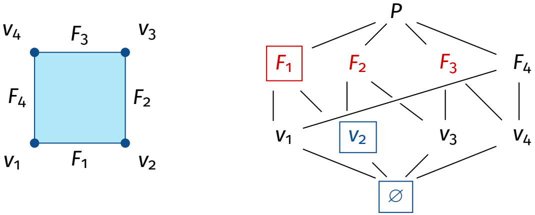
Aim: Find a (k, l)-positive polynomial in $\mathcal{V}_P$.

The algorithm is implemented in Macaulay2 and Gurobi.

Step 1: Construct parametrization of $S_P(\boldsymbol{x})$

- Consider or construct an **orientation** of the facets of P.
- Choose a **flag** $\mathcal{F} = \{F_1, \dots, F_{d+1}\}$ of facets of P:

$$F_1 \supsetneq (F_1 \cap F_2) \supsetneq \cdots \supsetneq (F_1 \cap \cdots \cap F_{d+1}) = \varnothing.$$



- Consider the **reduced symbolic slack matrix** $S_{\mathcal{F}}(\boldsymbol{x})$, whose transpose rows are $S_{\mathcal{F}}^{v_i}$.
- Construct the **parametrized slack matrix** entries

$$S_P(\boldsymbol{x}_{\mathcal{F}})_{i,j} := \det[S_{\mathcal{F}}^{v_{j_1}} \cdots S_{\mathcal{F}}^{v_{j_d}} \; S_{\mathcal{F}}^{v_i}] = p_{\{v_i\} \cup F_j}(S_{\mathcal{F}}(\boldsymbol{x})),$$

where $v_{j_1}, \dots, v_{j_d}$ span facet F_j.

Step 2: Construct constraints for linear program

The set of entries of $S_P(\boldsymbol{x}_{\mathcal{F}})$ and any products thereof gives us a collection of positive polynomials.

- Let $G_{k,l} = \{g_1, \dots, g_M\}$ be the set of entries of $S_P(\boldsymbol{x}_{\mathcal{F}})$ of degree $\leq l$ and of products of $\leq k$ of these entries, where $g_i = \sum_\alpha a_i^\alpha \boldsymbol{x}^\alpha$.
- We store the coefficients of g_i in a matrix

$$\mathcal{M}_{k,l} = \begin{array}{c} \\ g_1 \\ g_2 \\ \vdots \\ g_M \end{array} \begin{array}{c} \begin{array}{cccc} \boldsymbol{x}^{\alpha_1} & \boldsymbol{x}^{\alpha_2} & \cdots & \boldsymbol{x}^{\alpha_t} \end{array} \\ \begin{bmatrix} a_1^{\alpha_1} & a_1^{\alpha_2} & \cdots & a_1^{\alpha_t} \\ a_2^{\alpha_1} & a_2^{\alpha_2} & \cdots & a_2^{\alpha_t} \\ \vdots & \vdots & \ddots & \vdots \\ a_M^{\alpha_1} & a_M^{\alpha_2} & \cdots & a_M^{\alpha_t} \end{bmatrix} \end{array}$$

Notice that $\mathcal{M}_{k,l}$ records the linearization of the polynomials in $G_{k,l}$, that will be the constraints of our linear program.

Step 3: Solve linear program

We solve the following dual program whose coefficient matrix is the transpose of $\mathcal{M}_{k,l}$ using **Gurobi**:

$$\min 1 - \sum_{i=1}^{M} c_i \quad \text{subject to} \quad \begin{aligned} &\textstyle\sum_i c_i a_i^{\alpha} = 0 \text{ for all } \alpha \\ &\textstyle\sum_{i=1}^{M} c_i \leq 1 \\ &c_i \geq 0 \text{ for all } i = 1, \dots, M. \end{aligned}$$

This is always feasible and its optimal values are 0 or 1. If it is 0, then we have a non-realizability certificate for P as $\sum g_i = 0$.

Example 1: Doolittle's sphere

[Doolittle, 2020+] constructed some 4-dimensional simplicial spheres whose realizability was not known.

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One of them has 11 vertices and 44 facets. We consider the flag with positively oriented facets

$$\mathcal{F} = \{F_1 = \{5,6,7,8\}, F_2 = \{6,7,8,9\},$$
$$F_3 = \{7,8,9,10\}, F_4 = \{1,3,11,7\}, F_5 = \{4,8,11,9\}\}.$$

A reduced slack matrix $S_{\mathcal{F}}(\boldsymbol{x})$ is

$$\begin{bmatrix} x_{(1,1)} & x_{(1,2)} & x_{(1,3)} & 0 & x_{(1,5)} \\ x_{(2,1)} & x_{(2,2)} & x_{(2,3)} & x_{(2,4)} & x_{(2,5)} \\ x_{(3,1)} & x_{(3,2)} & x_{(3,3)} & 0 & x_{(3,5)} \\ x_{(4,1)} & x_{(4,2)} & x_{(4,3)} & x_{(4,4)} & 0 \\ 0 & x_{(5,2)} & x_{(5,3)} & x_{(5,4)} & x_{(5,5)} \\ 0 & 0 & x_{(6,3)} & x_{(6,4)} & x_{(6,5)} \\ 0 & 0 & 0 & 0 & x_{(7,5)} \\ 0 & 0 & 0 & x_{(8,4)} & 0 \\ x_{(9,1)} & 0 & 0 & x_{(9,4)} & 0 \\ x_{(10,1)} & x_{(10,2)} & 0 & x_{(10,4)} & x_{(10,5)} \\ x_{(11,1)} & x_{(11,2)} & x_{(11,3)} & 0 & 0 \end{bmatrix}$$

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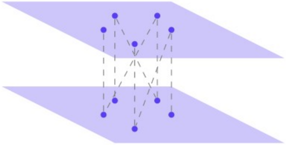
Proposition (Gouveia, M., Wiebe, 2021+) *A* (4, 2)-positive polynomial *certificate of non-realizability is:*

$$S_{(10,6)}S_{(8,11)}S_{(5,2)}S_{(11,2)} + S_{(9,12)}S_{(10,7)}S_{(3,2)}S_{(11,2)} + S_{(4,7)}S_{(8,8)}S_{(3,2)}S_{(11,2)}$$
$$+S_{(9,13)}S_{(6,5)}S_{(3,2)}S_{(10,2)} + S_{(10,6)}S_{(8,10)}S_{(2,2)}S_{(11,2)} + S_{(9,13)}S_{(6,9)}S_{(3,2)}S_{(4,2)} = 0,$$

where we use some additional facets $F_6, \dots, F_{13}$.

Example 2: Combinatorial Prismatoids

Prismatoid: all vertices lie in two parallel hyperplanes.
Used by Francisco Santos to construct a counterexample to the Hirsch Conjecture.



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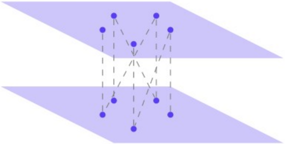

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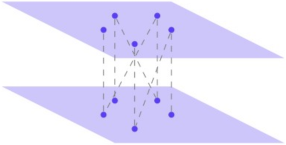

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Theorem (Gouveia, M., Wiebe, 2021+) *The 40 Criado-Santos prismatoids with 14 and 15 vertices are not realizable as convex polytopes.*

Prismatoid #3513

Prismatoid #3513 has dimension 5, 14 vertices and 94 facets. We consider the flag:

$$\mathcal{F} = \{F_1 = \{1,2,6,8,14\}, F_2 = \{1,6,8,14,9\}, F_3 = \{1,5,8,9,14\},$$
$$F_4 = \{\mathbf{8,9,10,11,12}, 13, 14\}, F_5 = \{6,8,9,12,11\}, F_6 = \{\mathbf{1,2,3,4,5}, 6, 7\}\}.$$

and remove the vertices forming the triangle $\{2,4,7\}$.

We found a $(3,3)$-positive polynomial certificate of non-realizability:

$$S_{(8,7)}S_{(12,8)}S_{(13,5)} + S_{(12,8)}S_{(11,9)}S_{(10,5)} + S_{(3,5)}S_{(13,8)}S_{(10,5)} = 0,$$

where we use some additional facets F_7, F_8, F_9.

Pfeifle found a certificate with 5 terms of degree 4.

Conclusions

- The **slack model** provides a simple method for studying realizability problems through the search for positive polynomials.
- Brute force: we do not search for certificates of a certain form (such as bi-quadratic final polynomials).
- This approach is doable and effective, but the complexity grows fast.

What's next:

- Implement the method in a **unique system** optimized for computations with matrices and polynomials. This would make previously intractable realizability problems possible.
- Develop new strategies to reduce the size of computations (e.g., vertex selection, exploit symmetry).

Bibliography

[1] J. Gouveia, A. Macchia, A. Wiebe, Slack realization spaces and realizability of polytopes, in preparation.

[2] J. Gouveia, A. Macchia, A. Wiebe, Combining realization space models of polytopes, preprint (2020), [arXiv:2001.11999].

[3] J. Gouveia, A. Macchia, R. R. Thomas, A. Wiebe, Projectively unique polytopes and toric slack ideals, J. Pure Appl. Algebra **224** (2020), 5, paper 106229.

[4] J. Gouveia, A. Macchia, R. R. Thomas, A. Wiebe, The slack realization space of a polytope, SIAM J. Discrete Math. **33** (2019), 3, 1637–1653.

[5] A. Macchia, A. Wiebe, Slack Ideals in Macaulay2, *Mathematical software – ICMS 2020*, 7th international conference, Braunschweig, Germany, July 13–16, 2020, Proceedings.

Thank you for listening!