Hydrodynamic limit of RSK algorithm

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Robinson–Schensted–Knuth algorithm

input:

- sequence of numbers $s = (X_1, X_2, \ldots, X_n)$

output:

two tableau with the same shape

- semistandard tableau P
- standard tableau Q

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example

$s = (12, 3, 13, 8, 5, 19, 10, 15, 9)$

12	13		
8	10	19	
3	5	9	15

Insertion tableau P(s)

5	9		
2	4	7	
1	3	6	8

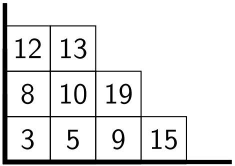
Recording tableau Q(s)

induction step of RSK algorithm

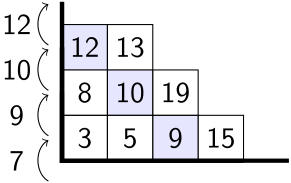
12	13		
8	10	19	
3	5	9	15

Insertion tableau P(s)

induction step of RSK algorithm

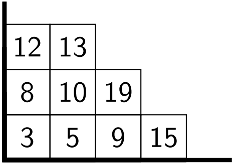


Insertion tableau P(s)

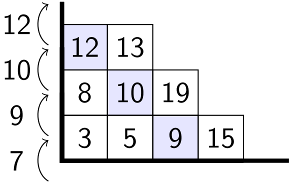


coming of a new number 7

induction step of RSK algorithm



Insertion tableau P(s)



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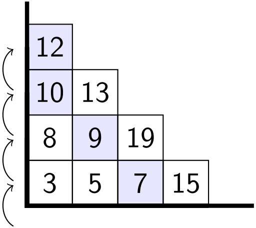
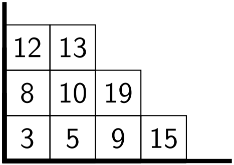
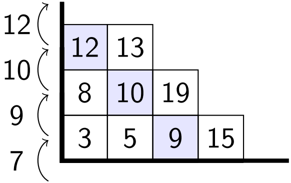


insertion of the number 7

induction step of RSK algorithm



Insertion tableau P(s)



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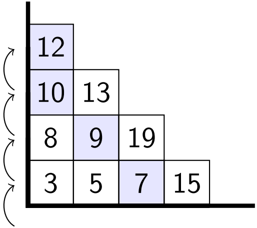
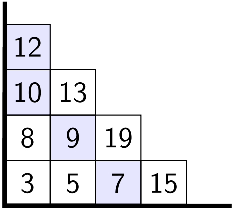


insertion of the number 7

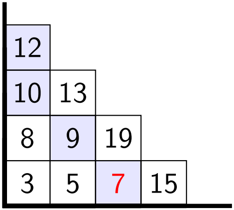


Insertion tableau P(s,7)

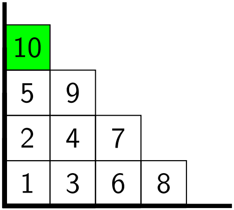
insertion step of RSK algorithm

example

$s = (12, 3, 13, 8, 5, 19, 10, 15, 9, 7)$



Insertion tableau



Recording tableau

- bumping route

- new box in Recording tableau

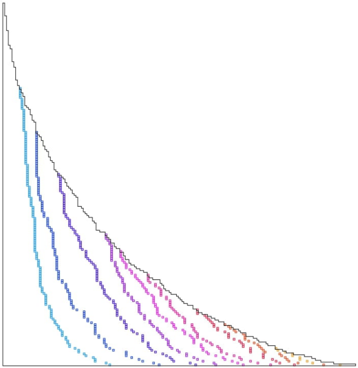
properties

- length of first row = length of longest increasing subsequence
- length of first column = length of longest decreasing subsequence
- for any permutation Π occurs $P(\Pi) = Q(\Pi^{-1})$

typical asymptotic problems

We apply the RSK algorithm to long random input. What is the:

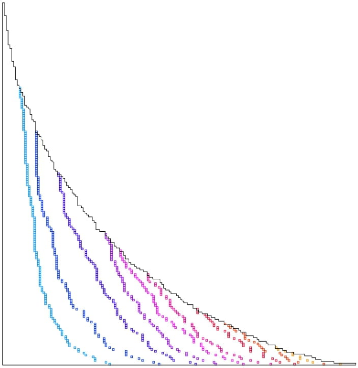
- lenght of the first row?
- shape of the tableau's?
- shape of bumping Route?

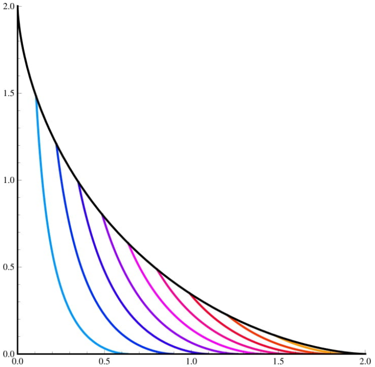


typical asymptotic problems

We apply the RSK algorithm to long random input. What is the:

- lenght of the first row? $\approx 2\sqrt{n}$
- shape of the tableau's? $\approx$ Logan–Shepp–Vershik–Kerov
- shape of bumping Route? $\approx$ Romik–Śniady





main result

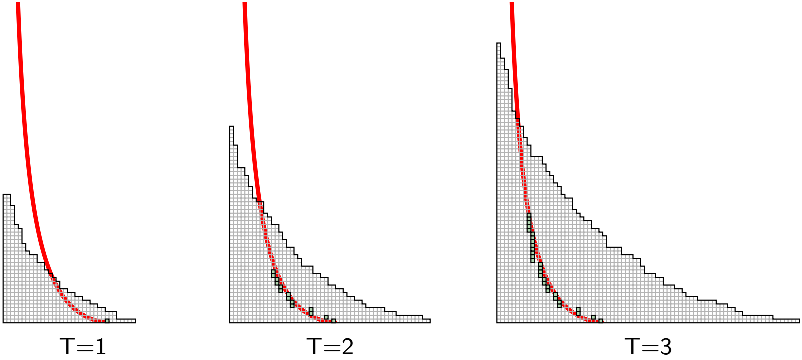
The boxes are slid by the RSK insertion step along the bumping routes.

Main question

We apply the RSK algorithm to a random input with a fixed number w in a particular moment.

- What can we say about the position of the box with the number w in the Insertion tableau?
- What can we say about the trajectory of the box with number w?

trajectory of a fixed number



Insertion tableau $P(X_1, \ldots, X_n, w, X_{n+1}, \ldots, X_{\lfloor Tn \rfloor})$ for $T = 1, 2, 3$
where $\{X_j\}_{j=1}^{n}$ — i.i.d. $U(0,1)$

Sketch of proof

requested number	tableau
w	$P(X_1, \dots, X_n, w, X_{n+1}, \dots, X_m)$

Sketch of proof

requested number		tableau	
w		$P(X_1, \ldots, X_n, w, X_{n+1}, \ldots, X_m)$	
	$\Updownarrow$		
w		$P(X'_1, \ldots, X'_{n'}, w, X'_{n'+1}, \ldots, X'_{m'})$	$X'_j < w$

Sketch of proof

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w		$P(X_1,\dots,X_n,w,X_{n+1},\dots,X_m)$	
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w		$P(X'_1,\dots,X'_{n'},w,X'_{n'+1},\dots,X'_{m'})$	$X'_j < w$
	$\Updownarrow$		
maximal		$P(\Pi)$	

Sketch of proof

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w		$P(X_1, \ldots, X_n, w, X_{n+1}, \ldots, X_m)$	
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	$\Updownarrow$		
maximal		$P(\Pi)$	
	$\Updownarrow$		
maximal		$Q(\Pi^{-1})$	

Sketch of proof

requested number		tableau	
w		$P(X_1, \dots, X_n, w, X_{n+1}, \dots, X_m)$	
	$\Updownarrow$		
w		$P(X'_1, \dots, X'_{n'}, w, X'_{n'+1}, \dots, X'_{m'})$	$X'_j < w$
	$\Updownarrow$		
maximal		$P(\Pi)$	
	$\Updownarrow$		
maximal		$Q(\Pi^{-1})$	
	$\Updownarrow$		
maximal		$Q(Y_1, \dots, Y_{m'}, \approx \frac{1}{T})$	

Sketch of proof

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w		$P(X_1, \dots, X_n, w, X_{n+1}, \dots, X_m)$	
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w		$P(X'_1, \dots, X'_{n'}, w, X'_{n'+1}, \dots, X'_{m'})$	$X'_j < w$
	$\Updownarrow$		
maximal		$P(\Pi)$	
	$\Updownarrow$		
maximal		$Q(\Pi^{-1})$	
	$\Updownarrow$		
maximal		$Q(Y_1, \dots, Y_{m'}, \approx \frac{1}{T})$	
	$\Updownarrow$		

position of new box during insertion step of RSK algorithm

Solution

- We can use the Romik–Śniady theorem.

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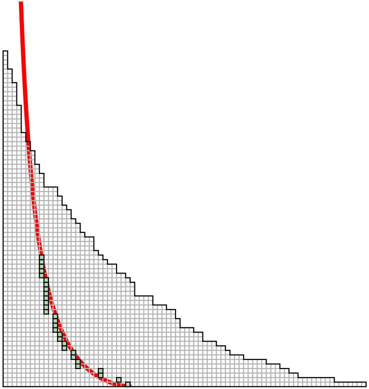
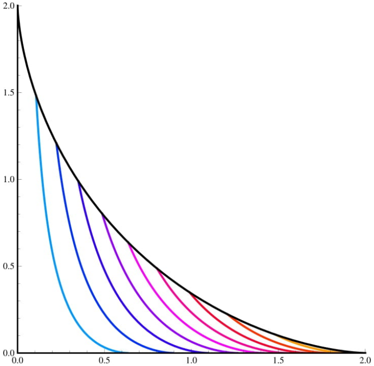
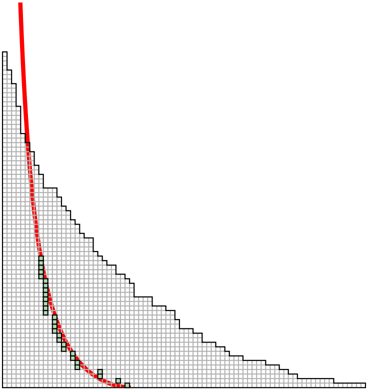
- We can use the Romik–Śniady theorem.
- The answer is the same curve as the limit shapes of the bumping route.





Further questions

- will the box with the number w slid into the first column?
- How long should we wait for this?
- What about more the boxes?



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Thank You