Shuffle Lattices and Bubble Lattices

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Séminaire Lotharingien de Combinatoire 86

joint work with

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Outline

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Shuffle Words

- $X = \{x_1, x_2, \ldots, x_m\}, Y = \{y_1, y_2, \ldots, y_n\}$
- $\mathbf{x} \stackrel{\text{def}}{=} x_1 x_2 \cdots x_m, \mathbf{y} \stackrel{\text{def}}{=} y_1 y_2 \cdots y_n$

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$m = 3, n = 4$

$\mathbf{u} \quad = x_2 y_1 x_3 x_1 y_4$

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- **shuffle word**: $\mathbf{u} \in (X \uplus Y)^*$ such that $\mathbf{u}_X$ is a subword of $\mathbf{x}$ and $\mathbf{u}_Y$ is a subword of $\mathbf{y}$ $\rightsquigarrow \mathsf{Shuf}(m,n)$

$m = 3, n = 4$

$\mathbf{u} = x_2y_1x_3x_1y_4 \notin \mathsf{Shuf}(3,4)$ $\mathbf{u} = x_1y_3x_2y_3x_3 \notin \mathsf{Shuf}(3,4)$

$\mathbf{u} = y_2x_1x_3y_3y_4 \in \mathsf{Shuf}(3,4)$

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$m = 3, n = 4$

$x_1x_2x_3 \leq_{\mathsf{shuf}} y_1y_2y_3y_4$ $\qquad$ $y_2x_1x_3y_4 \leq_{\mathsf{shuf}} y_1y_2x_1y_4$

$y_2x_1x_3y_4 \not\leq_{\mathsf{shuf}} y_2x_1y_4x_3$

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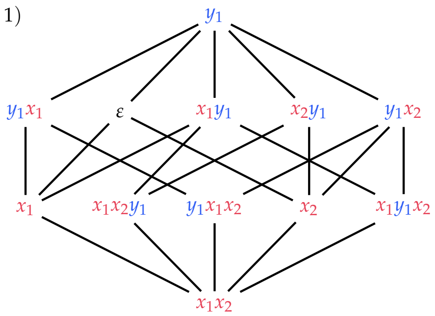
$\mathbf{Shuf}(2,1)$



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Theorem (C. Greene, 1988)

*For every $m, n \geq 0$, the poset $\mathbf{Shuf}(m,n) \stackrel{\text{def}}{=} (\mathsf{Shuf}(m,n), \leq_{\mathsf{shuf}})$ is a lattice; the **shuffle lattice**.*

An Order Extension of $\mathsf{Shuf}(n-1,1)$

Theorem (, 2020)

*For $n > 0$, the shuffle lattice $\mathsf{Shuf}(n-1,1)$ arises via a certain reordering (the **core label order**) from the **Hochschild lattice** $\mathsf{Hoch}(n)$.*

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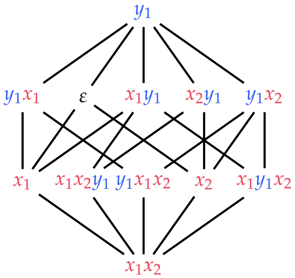
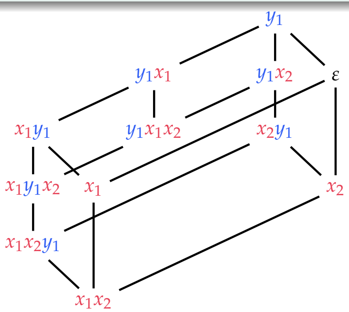


$\leftrightsquigarrow$



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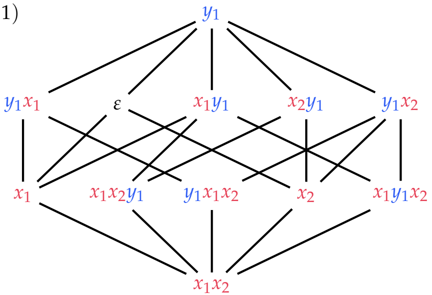
Outline

1. The Shuffle Lattice
2. The Bubble Lattice
3. Combinatorial Considerations
4. Enumerative Considerations

The Bubble Order

- **indel**: **in**serting y_t or **del**eting x_s $\rightsquigarrow$ $\hookrightarrow$

Shuf$(2,1)$



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Definition (C. Greene, 1988)

$\leq_{\mathsf{shuf}}$ is the reflexive and transitive closure of $\hookrightarrow$.

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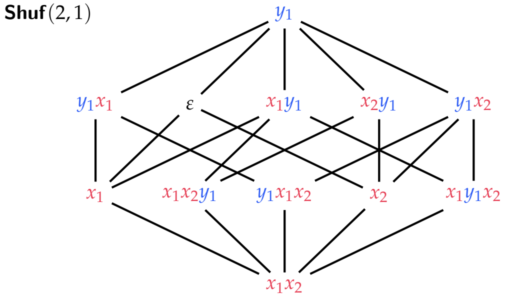

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- **indel**: **in**serting y_t or **del**eting x_s $\rightsquigarrow \hookrightarrow$
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Definition (T. McConville & , 2021)

The **bubble order** $\leq_{\mathsf{bub}}$ is the reflexive and transitive closure of $(\hookrightarrow\ \cup\ \Rightarrow)$.

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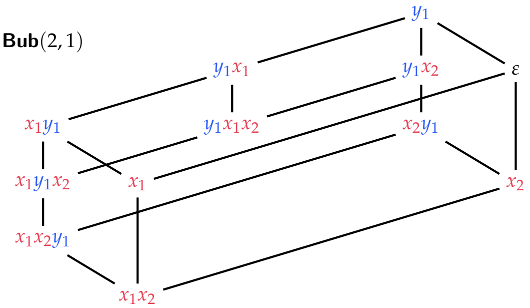

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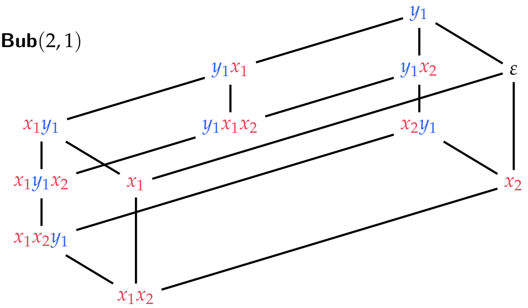

Outline

Cover Relations in $\mathbf{Bub}(m,n)$

- $\mathbf{u}' \lessdot_{\mathsf{bub}} \mathbf{u}$ is uniquely determined by either:
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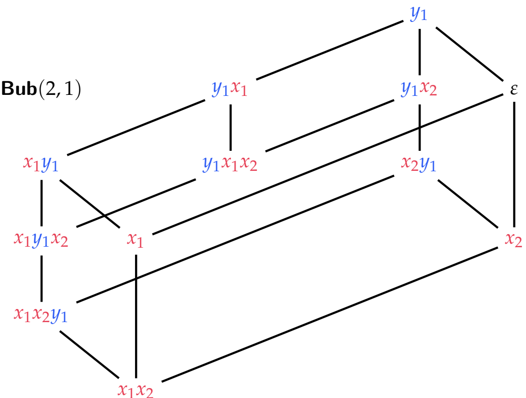

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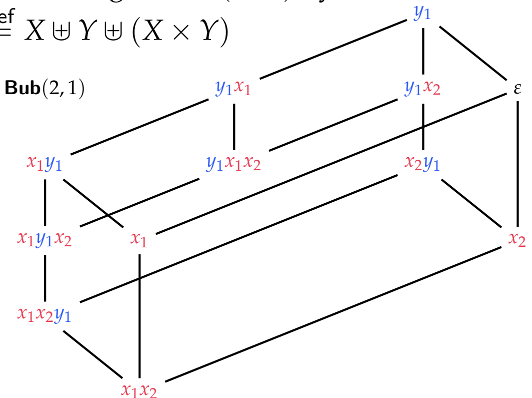

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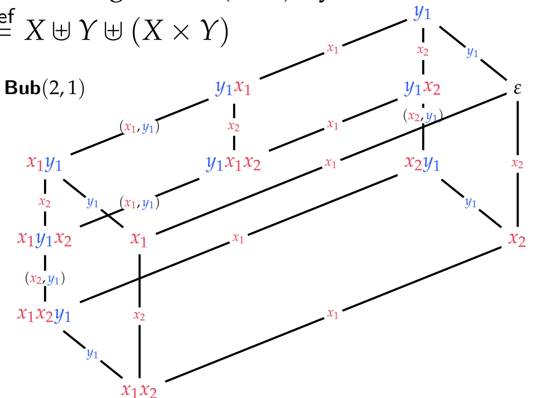

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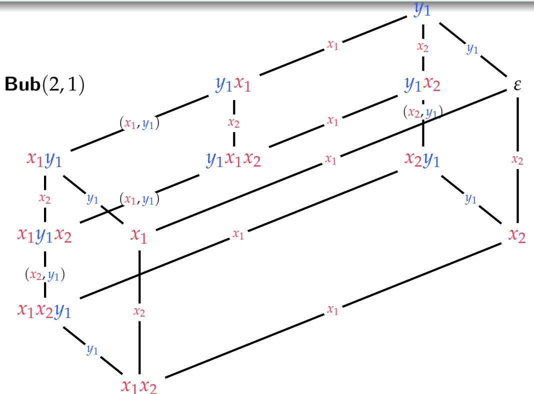
Proposition (T. McConville & , 2021)

Every $\mathbf{u} \in \mathsf{Shuf}(m,n)$ *is uniquely determined by*

$$\mathsf{Can}(\mathbf{u}) \overset{\mathsf{def}}{=} \{\lambda(\mathbf{u}',\mathbf{u}) : \mathbf{u}' \lessdot_{\mathsf{bub}} \mathbf{u}\}.$$



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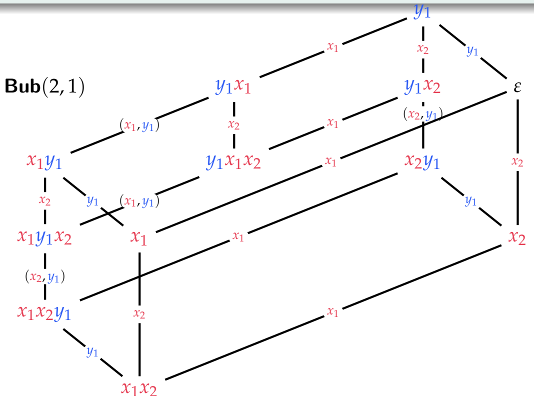

The Noncrossing Matching Complex

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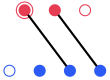

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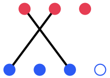

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Proposition (T. McConville & , 2021)

For $m, n \geq 0$,

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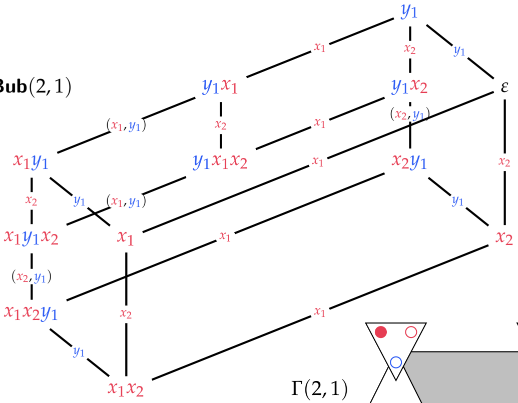
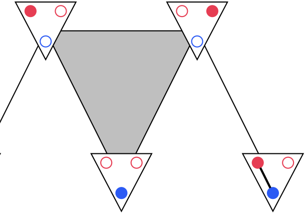
The Noncrossing Matching Complex

Bub$(2,1)$

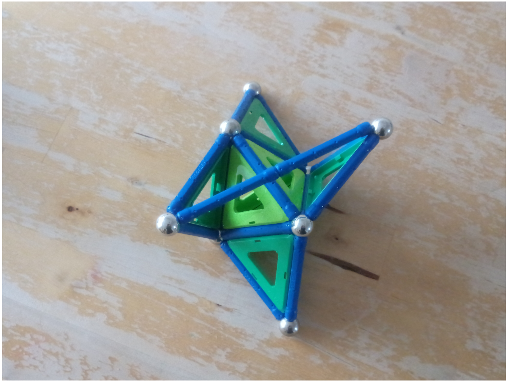


$\Gamma(2,1)$



The Noncrossing Matching Complex

$\Gamma(2,2)$



Outline

The H-Triangle of $\mathbf{Bub}(m,n)$

- $\mathbf{u} \in \mathsf{Shuf}(m,n)$
- **bubble-degree**: $\mathsf{in}_{\Rightarrow}(\mathbf{u}) \overset{\text{def}}{=} \bigl|\{\mathbf{u}' : \mathbf{u}' \lessdot \mathbf{u}, \mathbf{u}' \Rightarrow \mathbf{u}\}\bigr|$
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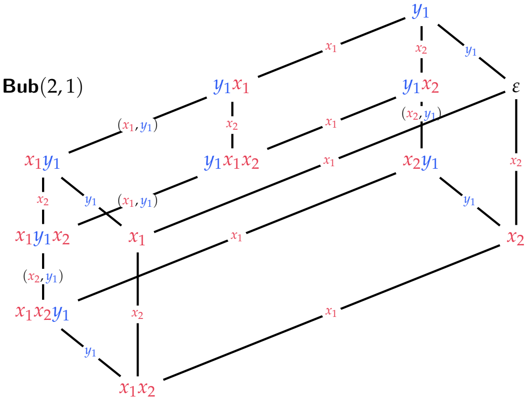

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$$\mathbf{u} = x_2x_3y_1y_2x_5x_6y_4y_5x_8 \in \mathsf{Shuf}(8,7)$$

The H-Triangle of $\mathbf{Bub}(m,n)$

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$$\mathbf{u} = x_2x_3y_1y_2x_5x_6y_4y_5x_8 \in \mathsf{Shuf}(8,7)$$

$$\mathsf{in}_{\Rightarrow}(\mathbf{u}) = 2$$

The H-Triangle of $\mathbf{Bub}(m,n)$

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$\mathsf{in}_{\Rightarrow}(\mathbf{u}) = 2$

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$$\mathbf{u} = x_2x_3y_1y_2x_5x_6y_4y_5x_8 \in \mathsf{Shuf}(8,7)$$

$$\mathsf{in}_{\Rightarrow}(\mathbf{u}) = 2$$

$$\mathsf{in}_{\hookrightarrow}(\mathbf{u}) = 3$$

The H-Triangle of $\mathbf{Bub}(m,n)$

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$$\mathbf{u} = x_2x_3y_1y_2x_5x_6y_4y_5x_8 \in \mathsf{Shuf}(8,7)$$

$$\mathsf{in}_{\Rightarrow}(\mathbf{u}) = 2$$

$$\mathsf{in}_{\hookrightarrow}(\mathbf{u}) = 3+$$

The H-Triangle of **Bub**(m,n)

- $\mathbf{u} \in \mathsf{Shuf}(m,n)$
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The H-Triangle of $\mathbf{Bub}(m,n)$

- $\mathbf{u} \in \mathsf{Shuf}(m,n)$
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$$\mathbf{u} = x_2x_3y_1y_2x_5x_6y_4y_5x_8 \in \mathsf{Shuf}(8,7)$$

$$\mathsf{in}_{\Rightarrow}(\mathbf{u}) = 2$$

$$\mathsf{in}_{\hookrightarrow}(\mathbf{u}) = 3 + 2$$

The H-Triangle of $\mathbf{Bub}(m,n)$

- $\mathbf{u} \in \mathsf{Shuf}(m,n)$
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$\mathsf{in}_{\hookrightarrow}(\mathbf{u}) = 5$

The H-Triangle of $\mathbf{Bub}(m,n)$

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Lemma (T. McConville & , 2021)

For $m,n \geq 0$, the number of $\mathbf{u} \in \mathsf{Shuf}(m,n)$ with $\mathsf{in}_{\Rightarrow} = a$ and $\mathsf{in}_{\hookrightarrow}(\mathbf{u}) = b$ is

$$\binom{m}{a}\binom{n}{a}\binom{m+n-2a}{b}.$$

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- ***H*-triangle**: $H_{m,n}(p,q) \overset{\mathsf{def}}{=} \sum_{\mathbf{u}\in\mathsf{Shuf}(m,n)} p^{\mathsf{in}(\mathbf{u})} q^{\mathsf{in}_{\hookrightarrow}(\mathbf{u})}$

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Proposition (T. McConville & Mühle, 2021)

For $m,n \geq 0$,

$$H_{m,n}(p,q) = \sum_{a \geq 0} \binom{m}{a}\binom{n}{a} p^a (1+pq)^{m+n-2a}.$$

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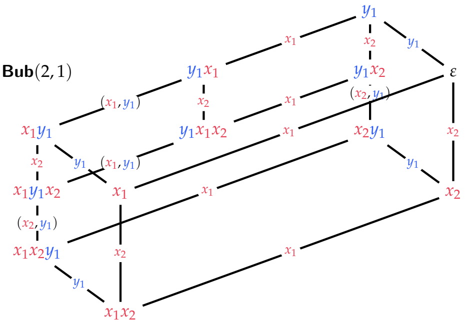
Corollary (C. Greene, 1988)

For $m,n \geq 0$, the rank-generating polynomial of $\mathbf{Shuf}(m,n)$ is

$$H_{m,n}(p,1) = \sum_{a\geq 0} \binom{m}{a}\binom{n}{a} p^a (1+p)^{m+n-2a}.$$

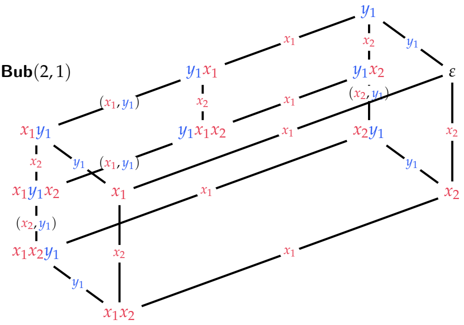
The H-Triangle of **Bub**(m,n)



$H_{2,1}(p,q) = p^3q^3+3p^2q^2+2p^2q+3pq+2p+1$

The H-Triangle of **Bub**(m, n)

Bub$(2, 1)$



$H_{2,1}(p, 1) = p^3+5p^2+5p+1$

The H-Triangle of **Bub**(m, n)

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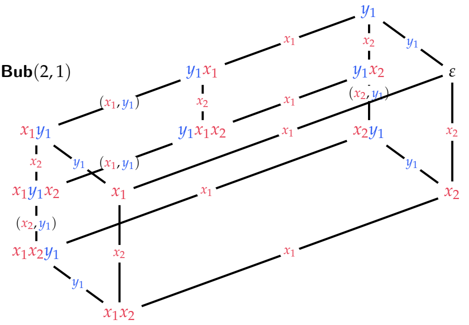

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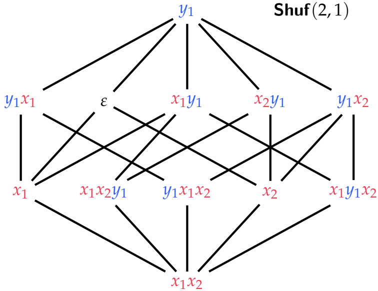
Shuf$(2,1)$



The H-Triangle of $\mathbf{Bub}(m,n)$

- **Delannoy numbers**:

$$\mathsf{Del}(m,n) \overset{\text{def}}{=} \sum_{a\geq 0}\binom{m}{a}\binom{n}{a}2^a$$

Proposition (T. McConville & [windmill symbol], 2021)

For $m,n\geq 0$,

$$H_{m,n}(p,q)=\sum_{a\geq 0}\binom{m}{a}\binom{n}{a}p^a(1+pq)^{m+n-2a}.$$

The H-Triangle of $\mathbf{Bub}(m,n)$

- **Delannoy numbers**:

$$\mathsf{Del}(m,n) \overset{\mathsf{def}}{=} \sum_{a\geq 0} \binom{m}{a}\binom{n}{a} 2^a = H_{m,n}(2,0)$$

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The H-Triangle of **Bub**(m,n)

- **Delannoy numbers**:

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- $\mathsf{Del}(m,n)$ counts lattice paths in $m\times n$-rectangle using north, northeast and east steps

Proposition (T. McConville & , 2021)

For $m,n\geq 0$,

$$H_{m,n}(p,q) = \sum_{a\geq 0}\binom{m}{a}\binom{n}{a}p^a(1+pq)^{m+n-2a}.$$

The $H{=}M$-Correspondence

- $\mu_{m,n}$ .. **Möbius function** of $\mathsf{Shuf}(m,n)$
- **M-triangle**:

$$M_{m,n}(p,q) \overset{\text{def}}{=} \sum_{u,v\in P} \mu_{m,n}(u,v)p^{\mathsf{rk}(u)}q^{\mathsf{rk}(v)}$$

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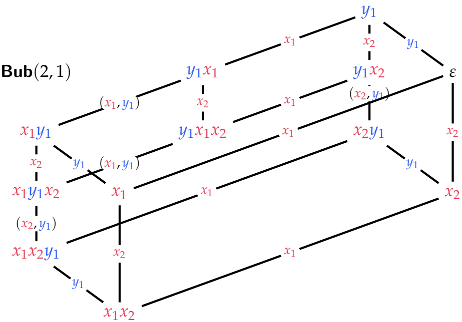
Conjecture (T. McConville & , 2021)

For $m,n \geq 0$,

$$M_{m,n}(p,q) = (1-q)^{m+n}H_{m,n}\left(\frac{q(p-1)}{1-q}, \frac{p}{p-1}\right).$$

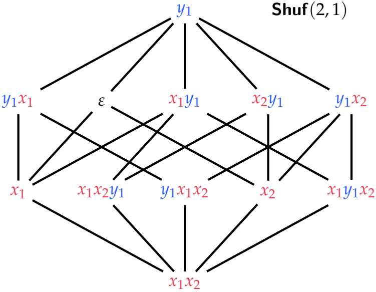
The $H{=}M$-Correspondence

Bub$(2,1)$



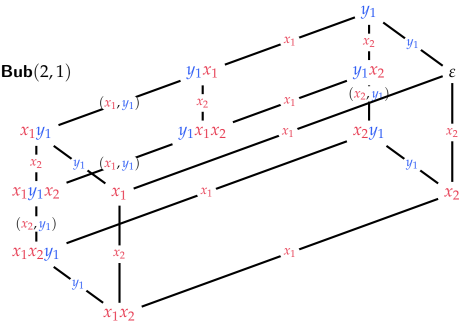
$$H_{2,1}(p,q) = p^3q^3+3p^2q^2+2p^2q+3pq+2p+1$$

Shuf$(2,1)$



The $H{=}M$-Correspondence

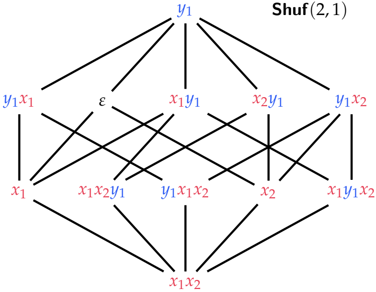
Bub$(2,1)$



$$H_{2,1}(p,q) = p^3q^3{+}3p^2q^2{+}2p^2q{+}3pq{+}2p{+}1$$

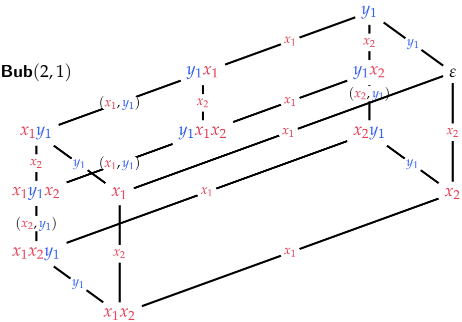
$$(1-q)^3H_{2,1}\left(\frac{q(p-1)}{1-q},\frac{p}{p-1}\right)$$
$$= p^3q^3{-}5p^2q^3{+}5p^2q^2{+}7pq^3$$
$$-12pq^2{-}3q^3{+}5pq{+}7q^2{-}5q{+}1$$

Shuf$(2,1)$

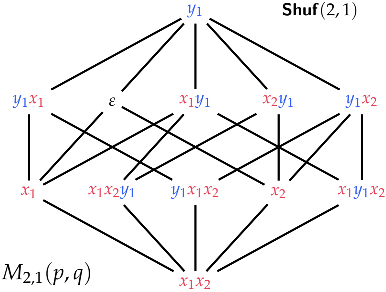


The $H{=}M$-Correspondence

Bub$(2,1)$



Shuf$(2,1)$



$$H_{2,1}(p,q) = p^3q^3+3p^2q^2+2p^2q+3pq+2p+1$$

$$(1-q)^3H_{2,1}\left(\frac{q(p-1)}{1-q},\frac{p}{p-1}\right)$$

$$= p^3q^3-5p^2q^3+5p^2q^2+7pq^3$$

$$-12pq^2-3q^3+5pq+7q^2-5q+1 = M_{2,1}(p,q)$$

Other Considerations

- some conjectures:
  - $\Gamma(m,n)$ is a vertex-decomposable, hence shellable, complex
  - $\mathsf{Bub}(m,n)$ orients the 1-skeleton of a simple polytope

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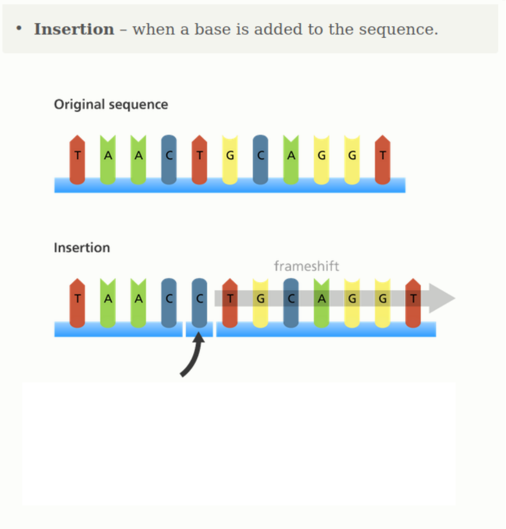
Other Considerations

- some conjectures:
  - $\Gamma(m,n)$ is a vertex-decomposable, hence shellable, complex
  - $\mathsf{Bub}(m,n)$ orients the 1-skeleton of a simple polytope $\rightsquigarrow$ this leads to the corresponding F-triangle
- some extensions:
  - word shuffles using repeated letters

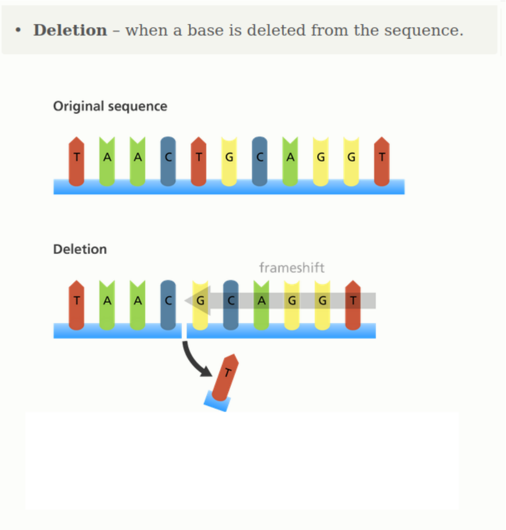
Thank You.

DNA Mutation



https://www.knowyourgenome.org/facts/what-types-of-mutation-are-there

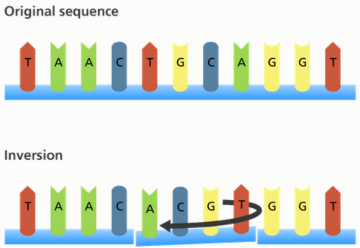
DNA Mutation



https://www.knowyourgenome.org/facts/what-types-of-mutation-are-there

DNA Mutation

- **Inversion** – when a segment of a chromosome is reversed end to end.



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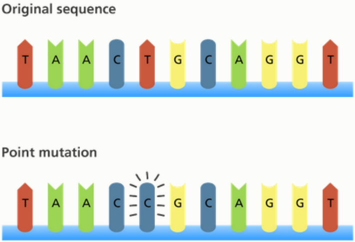
DNA Mutation

- **Point mutation**? – a change in one base? in the DNA? sequence.



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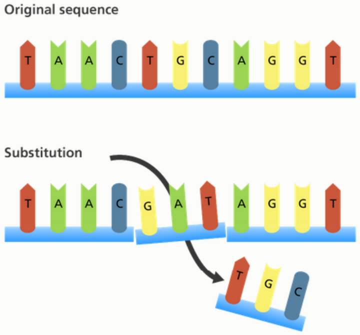
DNA Mutation

- **Substitution** – when one or more bases in the sequence is replaced by the same number of bases (for example, a cytosine? substituted for an adenine?).



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Möbius Polynomials

- $\mathbf{P} = (P, \leq)$ .. (finite) poset

Back

Möbius Polynomials

Back

- $\mathbf{P} = (P, \leq)$ .. (finite) poset
- **Möbius function**:

$$\mu_{\mathbf{P}}(u,v) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } u = v\\ -\sum_{u\leq w<v} \mu_{\mathbf{P}}(u,w), & \text{if } u < v\\ 0, & \text{otherwise}\end{cases}$$

Möbius Polynomials

- $\mathbf{P} = (P, \leq)$ .. graded (finite) poset with bounds $\hat{0}$ and $\hat{1}$

Back

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- (reverse) **characteristic polynomial**:

$$\chi_{\mathbf{P}}(p) \overset{\text{def}}{=} \sum_{u \in P} \mu_{\mathbf{P}}(\hat{0},u) p^{\mathsf{rk}(u)}$$

Möbius Polynomials

- $\mathbf{P} = (P, \leq)$ .. graded (finite) poset with bounds $\hat{0}$ and $\hat{1}$

Back

- **Möbius function**:

$$\mu_{\mathbf{P}}(u,v) \stackrel{\mathsf{def}}{=} \begin{cases} 1, & \text{if } u = v \\ -\sum_{u \leq w < v} \mu_{\mathbf{P}}(u,w), & \text{if } u < v \\ 0, & \text{otherwise} \end{cases}$$

- (reverse) **characteristic polynomial**:

$$\chi_{\mathbf{P}}(p) \stackrel{\mathsf{def}}{=} \sum_{u \in P} \mu_{\mathbf{P}}(\hat{0},u) p^{\mathsf{rk}(u)}$$

- **M-triangle**:

$$M_{\mathbf{P}}(p,q) \stackrel{\mathsf{def}}{=} \sum_{u,v \in P} \mu_{\mathbf{P}}(u,v) p^{\mathsf{rk}(u)} q^{\mathsf{rk}(v)}$$

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Lemma

- $M_{\mathbf{P}}(p, q) = \sum_{u \in P} (pq)^{\mathsf{rk}(u)} \chi_{[u,\hat{1}]}(q)$.
- $\chi_{\mathbf{P}}(p) = M_{\mathbf{P}}(0, p)$.

The Core Label Order

- $\mathbf{L} = (L, \leq)$ .. (finite) lattice, $u \in L$

Back

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- $\mathsf{Pre}(u) \overset{\mathsf{def}}{=} \{u' \in L \mid u' \lessdot u\}$
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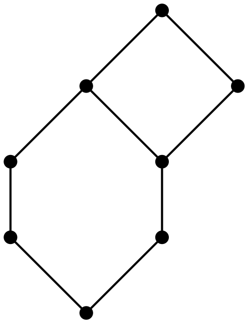

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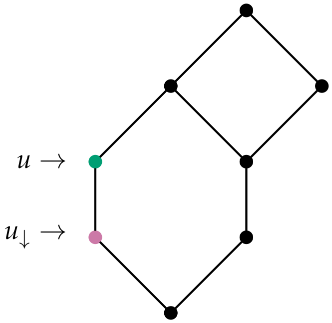

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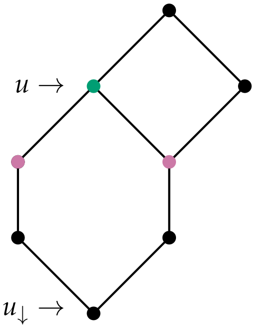

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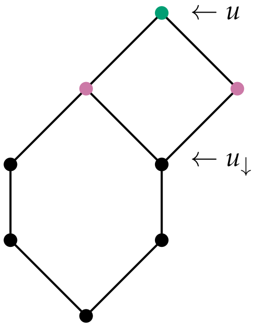

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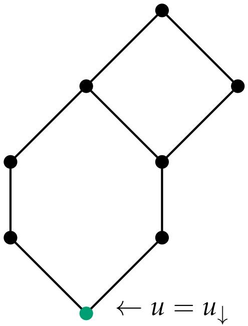

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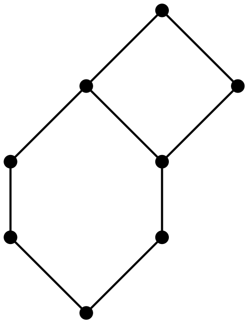

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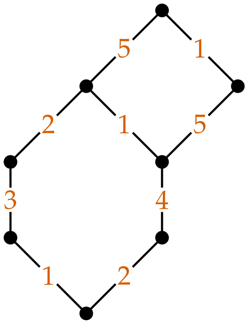

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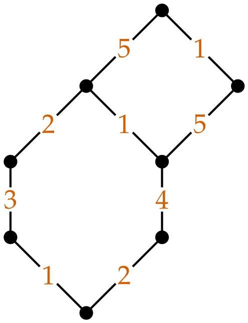

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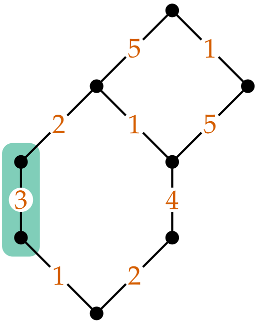
$\{3\} \rightarrow$



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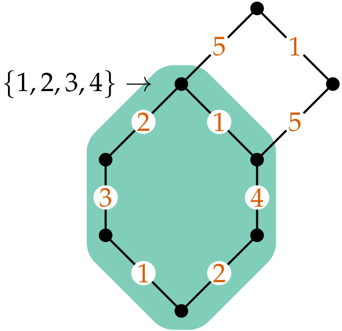

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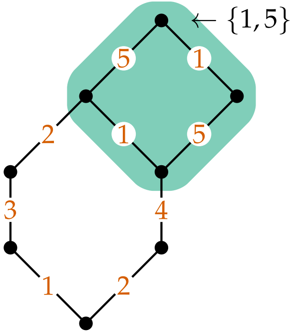

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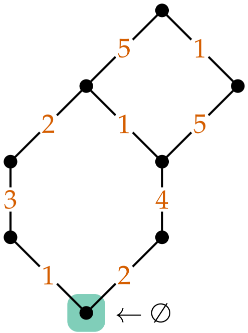

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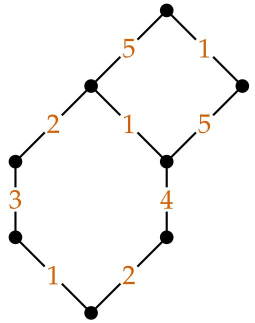

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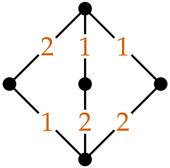


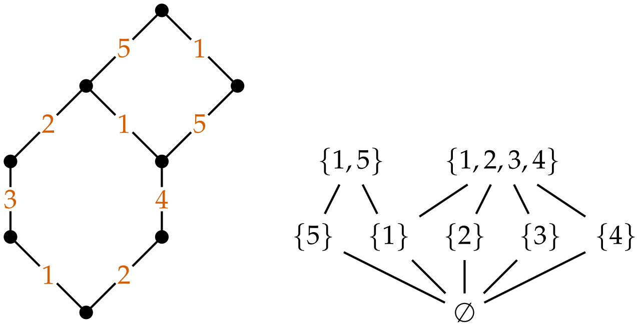
not a core labeling

The Core Label Order

- $\mathbf{L} = (L, \leq)$ .. (finite) lattice, λ .. edge labeling

Back

- **core label order**: $\mathsf{CLO}(\mathbf{L}) \overset{\text{def}}{=} (L, \leq_{\mathsf{clo}})$,
  where $u \leq_{\mathsf{clo}} v$ if and only if $\Psi(u) \subseteq \Psi(v)$



The Hochschild Lattice

Back

- **triword**: an integer tuple $(u_1, u_2, \ldots, u_n)$ such that
  - $u_i \in \{0, 1, 2\}$ $\rightsquigarrow \mathsf{Tri}(n)$
  - $u_1 \neq 2$
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$(0,0,0)$, $(0,0,2)$, $(0,2,0)$, $(0,2,2)$, $(1,0,0)$, $(1,0,2)$,
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For $n > 0$, the cardinality of $\mathsf{Tri}(n)$ is $2^{n-2}(n+3)$.

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$1, 2, 5, 12, 28, 64, 144, 320, 704, \ldots$ (A045623 in OEIS)

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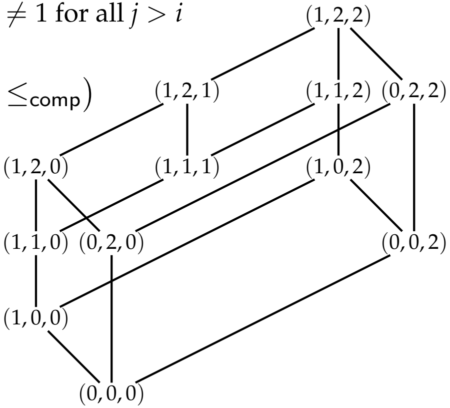

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Theorem (C. Combe, 2020)

For $n > 0$, $\mathbf{Hoch}(n)$ is a lattice.

The Bipartite Noncrossing Complex

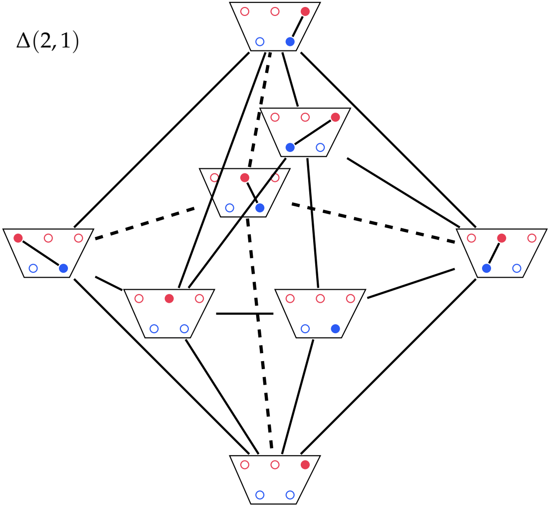
Back

- $\tilde{X} \overset{\text{def}}{=} X \uplus \{x_0\}$, $\tilde{Y} \overset{\text{def}}{=} Y \uplus \{y_0\}$
- $\tilde{\mathcal{T}} \overset{\text{def}}{=} X \uplus Y \uplus \big((\tilde{X} \times \tilde{Y}) \setminus \{(x_0, y_0)\}\big)$
- $\sigma \in \tilde{\mathcal{T}}$ is **noncrossing** if
  - $x_0, y_0 \notin \sigma$
  - each letter x_s or y_t appears at most once in σ
  - if $(x_{s_1}, y_{t_1}), (x_{s_2}, y_{t_2}) \in \sigma$ and $s_1 < s_2$, then $t_1 < t_2$
- **bipartite noncrossing complex**: simplicial complex of noncrossing subsets of $\tilde{T}$ $\rightsquigarrow \Delta(m, n)$

The Bipartite Noncrossing Complex

Back

$\Delta(2,1)$



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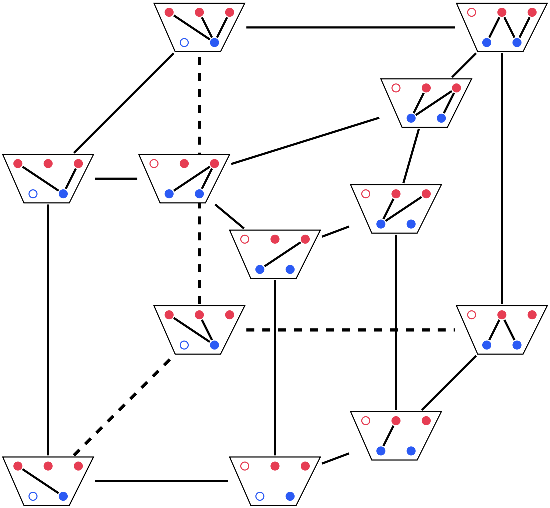

The Bipartite Noncrossing Complex

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- for $\sigma \in \tilde{\mathcal{T}}$ let
  - $\mathsf{lp}(\sigma) \stackrel{\mathsf{def}}{=} \bigl\lvert \{a \in \sigma : a \in X \uplus Y\} \bigr\rvert$
  - $\mathsf{ed}(\sigma) \stackrel{\mathsf{def}}{=} \bigl\lvert \{a \in \sigma : a \in (\tilde{X} \times \tilde{Y}) \setminus \{(x_0, y_0)\}\} \bigr\rvert$

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- ***F*-triangle**: $F_{m,n}(p,q) \overset{\text{def}}{=} \sum_{\sigma \in \tilde{T}} p^{\mathsf{ed}(\sigma)} q^{\mathsf{lp}(\sigma)}$

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Conjecture (T. McConville & , 2021)

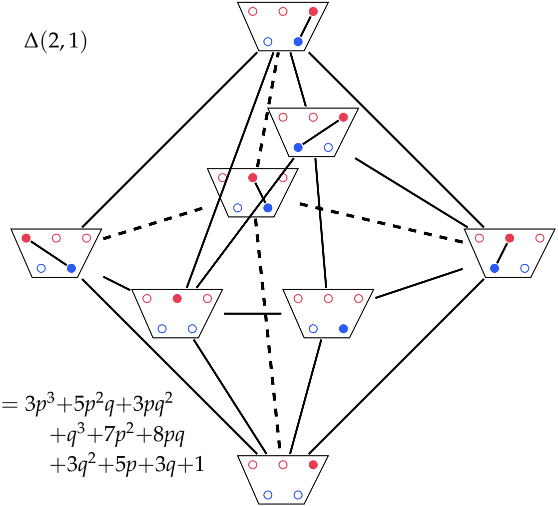
For $m, n \geq 0$,

$$F_{m,n}(p,q) = p^{m+n} H_{m,n}\left(\frac{p+1}{p}, \frac{q+1}{p+1}\right).$$

The Bipartite Noncrossing Complex

Back

$\Delta(2,1)$



$$F_{2,1}(p,q) = 3p^3+5p^2q+3pq^2 +q^3+7p^2+8pq +3q^2+5p+3q+1$$