An inequality for plane partitions

(joint work with B. Heim and R. Tröger)

Markus Neuhauser
markus.neuhauser@kiu.edu.ge

KUTAISI INTERNATIONAL UNIVERSITY

86th Séminaire Lotharingien de Combinatoire
September 7, 2021

Plane partitions

- A *plane partition of a non-negative integer* n is a rectangular array

$$\begin{array}{ccccccc} \pi_{1,1} & \pi_{1,2} & \cdots & \cdots & \cdots & \cdots & \pi_{1,\lambda_1} \\ \pi_{2,1} & \pi_{2,2} & \cdots & \cdots & \cdots & \pi_{2,\lambda_2} & \\ \vdots & \vdots & & & & & \\ \pi_{r,1} & \pi_{r,2} & \cdots & \pi_{r,\lambda_r} & & & \end{array}$$

  of non-negative integers weakly decreasing in rows and columns $\pi_{j+1,k} \leq \pi_{j,k} \geq \pi_{j,k+1}$ and $\sum_{j=1}^{r} \sum_{k=1}^{\lambda_j} \pi_{j,k} = n$.
- We denote the number of different plane partitions of a non-negative integer n by $\mathrm{pp}(n)$.
- We denote the number of different partitions of a non-negative integer n by $\mathrm{p}(n)$.

Plane partitions

Example

$$3 = 2 + 1 = 1 + 1 + 1 = \begin{matrix} 2 \\ + \\ 1 \end{matrix} = \begin{matrix} 1 \\ + \\ 1 \end{matrix} + 1 = \begin{matrix} 1 \\ + \\ 1 \\ + \\ 1 \end{matrix}$$

- $\mathrm{pp}(3) = 6$.
- $\mathrm{p}(3) = 3$.

Generating function (MacMahon 1897):

$$\prod_{m=1}^{\infty} \left(1 - q^m\right)^{-m} = \sum_{n=1}^{\infty} \mathrm{pp}(n)\, q^n.$$

An inequality by Ch. Bessenrodt and K. Ono

Theorem (Bessenrodt, Ono 2016)

If a, b *are integers with* $a, b \geq 2$ *and* $a + b \geq 9$, *then* $\mathrm{p}(a)\,\mathrm{p}(b) \geq \mathrm{p}(a+b)$, *with equality holding only for* $\{a, b\} = \{2, 7\}$.

Their proof uses an asymptotic formula by Lehmer (1939)

$$\mathrm{p}(n) \sim \frac{\sqrt{12}}{24n-1}\left[\left(1-\frac{1}{\mu}\right)e^{\mu} + \left(1+\frac{1}{\mu}\right)e^{-\mu}\right]$$

where $\mu = \mu(n) = \frac{\pi}{6}\sqrt{24n-1}$.

Remark

Combinatorial proof of the theorem by Alanazi, Gagola, and Munagi (2017).

Values of $\mathrm{p}(a)\,\mathrm{p}(b) - \mathrm{p}(a+b)$ for $1 \le a, b \le 9$

$a \backslash b$	1	2	3	4	5	6	7	8	9
1	−1	−1	−2	−2	−4	−4	−7	−8	−12
2	−1	−1	−1	−1	−1	0	0	2	4
3	−2	−1	−2	0	−1	3	3	10	13
4	−2	−1	0	3	5	13	19	33	49
5	−4	−1	−1	5	7	21	28	53	75
6	−4	0	3	13	21	44	64	107	154
7	−7	0	3	19	28	64	90	154	219
8	−8	2	10	33	53	107	154	253	363
9	−12	4	13	49	75	154	219	363	515

The inequality for integer plane partition numbers

Theorem (Heim, N., Tröger 2021)

Let a and b be positive integers.
Let $a, b \geq 2$ and $a + b \geq 12$. Then

$$\mathrm{pp}(a)\,\mathrm{pp}(b) > \mathrm{pp}(a+b).$$

Equality is never satisfied.

Similar to the asymptotic formula by Lehmer there is an asymptotic formula by Wright (1931)

$$\mathrm{pp}\,(n) \sim \frac{\zeta(3)^{\frac{7}{36}}}{\sqrt{12\pi}} \left(\frac{2}{n}\right)^{\frac{25}{36}} \exp\left(3\,\zeta\,(3)^{\frac{1}{3}} \left(\frac{n}{2}\right)^{\frac{2}{3}} + \zeta'\,(-1)\right).$$

Some values of $\mathrm{pp}(a)\,\mathrm{pp}(b) - \mathrm{pp}(a+b)$ for $2 \le a \le 13$ and $2 \le b \le 9$

$a \backslash b$	2	3	4	5	6	7	8	9
2	−4	−6	−9	−14	−16	−24	−20	−13
3	−6	−12	−8	−16	6	16	101	213
4	−9	−8	9	30	124	259	601	
5	−14	−16	30	76	293	585		
6	−16	6	124	293	825			
7	−24	16	259	585				
8	−20	101	601					
9	−13	213						
10	21	515						
11	92	987						
12	270							
13	576							

Main ingredients for the proof

- A recurrence relation for plane partitions

$$\mathrm{pp}\,(n) = \sum_{k=1}^{n} \sigma_2\,(k)\,\frac{\mathrm{pp}\,(n-k)}{n}$$

where $\sigma_2\,(k) = \sum_{t|k} t^2$.

- The function σ_2 can be estimated by

$$\sigma_2\,(k) = k^2 \sum_{t|k} \frac{1}{t^2} \leq k^2 \cdot \left(1 + \int_1^k t^{-2}\,\mathrm{d}t\right) < 2k^2.$$

- An estimate

$$\mathrm{pp}\,(n) \geq \mathrm{p}\,(n) \geq \sum_{j=1}^{9} \frac{1}{j!}\binom{n-1}{j-1}$$

by a polynomial in n (of degree 8).

An auxiliary result: comparing successive values

Table: Values of $\operatorname{pp}(n)$ for $0 \leq n \leq 11$.

n	0	1	2	3	4	5	6	7	8	9	10	11
$\operatorname{pp}(n)$	1	1	3	6	13	24	48	86	160	282	500	859

Lemma

We have $\operatorname{pp}(n+1) \leq 3\operatorname{pp}(n)$.

Proof of $\mathrm{pp}\,(n+1) \le 3\,\mathrm{pp}\,(n)$

Recurrence relation yields

$$\begin{aligned} & 3\,\mathrm{pp}\,(n) - \mathrm{pp}\,(n+1) \\ = \; & 3\sum_{k=1}^{n} \sigma_2\,(k)\,\frac{\mathrm{pp}\,(n-k)}{n} - \sum_{k=1}^{n+1} \sigma_2\,(k)\,\frac{\mathrm{pp}\,(n+1-k)}{n+1}. \end{aligned}$$

For $1 \le k \le n$ we can proceed by induction hypothesis

$$\frac{3\,\mathrm{pp}\,(n-k)}{n} - \frac{\mathrm{pp}\,(n-k+1)}{n+1} \ge \left(\frac{3}{n} - \frac{3}{n+1}\right)\mathrm{pp}\,(n-k) > 0.$$

Using only the dominant term for $k = 1$

$$\begin{aligned} 3\,\mathrm{pp}\,(n) - \mathrm{pp}\,(n+1) \; & \ge \; \frac{3\,\mathrm{pp}\,(n-1)}{(n+1)\,n} - \frac{\sigma_2\,(n+1)}{n+1} \\ & > \; \frac{3}{(n+1)\,n}\sum_{j=1}^{9}\binom{n-2}{j-1}\frac{1}{j!} - 2\,(n+1) > 0 \end{aligned}$$

for n sufficiently large (≥ 100).

The log-concave inequality

A sequence of positive numbers $(\alpha(n))_{n \geq 0}$ is *log-concave* at $n \in \mathbb{N}$ if: $(\alpha(n))^2 > \alpha(n-1)\,\alpha(n+1)$.

Theorem (Nicolas 1978)

$(\mathrm{p}(n))_{n \geq 0}$ *is log-concave at every* n *except for odd* $n \in [1, 25]$.

Proposition (Heim, N. 2021)

There is an $N > 0$ *such that* $(\mathrm{pp}(n))_{n \geq 0}$ *is log-concave at every* $n > N$.

Verified also for all $n \leq 10^5$ except odd $n \in [1, 11]$.
The proof uses Wright's formula that tells us more precisely that there are $A, B > 0$ and γ_1, γ_2 such that

$$\mathrm{pp}(n) = A n^{-\frac{25}{36}} e^{B n^{2/3}} \left(1 + \frac{\gamma_1}{n^{\frac{2}{3}}} + \frac{\gamma_2}{n^{\frac{4}{3}}} + O\left(\frac{1}{n^2}\right)\right).$$

Polynomization

A similar recurrence relation can be used to define polynomials in the following way.

Definition

Let $P_0(x) = 1$ and

$$P_n(x) = x \sum_{k=1}^{n} \sigma_2(k) \, \frac{P_{n-k}(x)}{n}.$$

Remark

(1) $\mathrm{pp}(n) = P_n(1)$.

(2) Leading coefficient $\dfrac{1}{n!}$. Therefore $\lim_{x\to\infty} (P_a(x) P_b(x) - P_{a+b}(x)) = +\infty$ for $a, b \geq 1$.

Approximative largest real zeros of $x \mapsto P_a(x)\,P_b(x) - P_{a+b}(x)$ for $1 \leq a \leq 12$, $1 \leq b \leq 11$.

$a \backslash b$	1	2	3	4	5	6	7	8	9	10	11
1	5.0	3.2	3.0	2.6	2.5	2.3	2.2	2.1	2.1	2.0	2.0
2	3.2	1.9	1.6	1.4	1.3	1.2	1.1	1.1	1.0	1.0	1.0
3	3.0	1.6	1.5	1.2	1.1	1.0	1.0	0.9	0.9	0.8	0.8
4	2.6	1.4	1.2	0.9	0.9	0.8	0.7	0.7	0.6	0.6	0.6
5	2.5	1.3	1.1	0.9	0.9	0.7	0.7	0.6	0.6	0.6	0.6
6	2.3	1.2	1.0	0.8	0.7	0.6	0.6	0.5	0.5	0.5	0.5
7	2.2	1.1	1.0	0.7	0.7	0.6	0.6	0.5	0.5	0.5	0.5
8	2.1	1.1	0.9	0.7	0.6	0.5	0.5	0.4	0.4	0.4	0.4
9	2.1	1.0	0.9	0.6	0.6	0.5	0.5	0.4	0.4	0.4	0.4
10	2.0	1.0	0.8	0.6	0.6	0.5	0.5	0.4	0.4	0.4	0.3
11	2.0	1.0	0.8	0.6	0.6	0.5	0.5	0.4	0.4	0.3	0.3
12	1.9	0.9	0.8	0.6	0.5	0.4	0.4	0.4	0.3	0.3	0.3

Inequalities for polynomials

Theorem (Heim, N., Tröger 2021)

Let $x > 5$. *Then*

$$P_a(x)\ P_b(x) > P_{a+b}(x)$$

for all positive integers a *and* b.

Theorem (Heim, N., Tröger 2021)

Let $x \geq 2$. *Then*

$$P_a(x)\ P_b(x) > P_{a+b}(x)$$

for all positive integers a *and* b *satisfying* $a + b \geq 12$.

References

B. Heim, M. Neuhauser.
Proof of the Bessenrodt–Ono inequality by induction.
Submitted, 2021, arXiv:2108.00191v1 [math.CO].

B. Heim, M. Neuhauser, R.Tröger.
Polynomization of the Bessenrodt–Ono inequality.
Ann. Comb. **24**, No. 4, 697–709 (2020).

Thank you for your attention!