Multiplication-addition theorems for self-conjugate partitions

Séminaire Lotharigien de Combinatoire n°86
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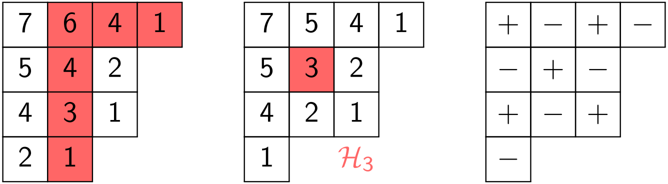
Summary

Ferrers diagram and hooks of partitions



(a) $(4,3,3,2) \in \mathcal{P}$ (b) $(4,3,3,1) \in \mathcal{SC}$ (c) BG-rank $= -1$

- $\mathcal{H}(\lambda) :=$ {hook-length}
- for $t \in \mathbb{N}$, $\mathcal{H}_t(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod t\}$
- BG-rank of Berkovich-Garvan (2008): sum of signs

Ferrers diagram and hooks of partitions

7	6	4	1
5	4	2	
4	3	1	
2	1		

(a) $(4,3,3,2) \in \mathcal{P}$

7	5	4	1
5	3	2	
4	2	1	
1			

$\mathcal{H}_3$

(b) $(4,3,3,1) \in \mathcal{SC}$

$+$	$-$	$+$	$-$
$-$	$+$	$-$	
$+$	$-$	$+$	
$-$			

(c) BG-rank $= -1$

- $\mathcal{H}(\lambda) :=$ {hook-length}
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- BG-rank of Berkovich-Garvan (2008): sum of signs
- Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = (q;q)_\infty^{z-1}$$

where $(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$

Littlewood decomposition

Set $\mathcal{A} \subseteq \mathcal{P}$, $\mathcal{A}_{(t)} := \{\omega_t \in \mathcal{A} \mid \mathcal{H}_t(\omega_t) = \emptyset\}$

1. partitions: $\lambda \in \mathcal{P} \mapsto (\omega_t, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$

$$\mathcal{H}_t(\lambda) = t \bigcup_{i=0}^{t-1} \mathcal{H}(\nu^{(i)}),$$

$$|\lambda| = |\omega_t| + t \sum_{i=0}^{t-1} |\nu^{(i)}|$$

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2. Self-conjugate partitions:
   (a) for t even: $\lambda \in \mathcal{SC} \mapsto (\omega_t, \underline{\nu}) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{t/2}$
   (b) for t odd: $\lambda \in \mathcal{SC} \mapsto (\omega_t, \underline{\nu}, \mu) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{(t-1)/2} \times \mathcal{SC}$

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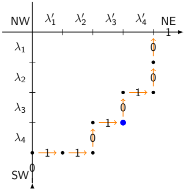
Cho–Huh–Sohn (2019) $\lambda \in \mathcal{SC}^{(\mathrm{BG})} \mapsto \kappa \in \mathcal{P}$ bijection such that $|\lambda| = 4|\kappa| + \mathrm{BG}(\lambda)(2\,\mathrm{BG}(\lambda) - 1)$

An example $\lambda = (4, 4, 3, 2)$ and $t = 4$



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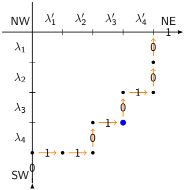

$$s(\lambda) = \cdots 00001101|01001111 \cdots$$

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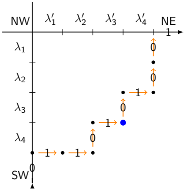

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$$\begin{array}{l} s\left(\nu^{(0)}\right) = \cdots 001|011 \cdots \\ s\left(\nu^{(1)}\right) = \cdots 001|111 \cdots \\ s\left(\nu^{(2)}\right) = \cdots 000|011 \cdots \\ s\left(\nu^{(3)}\right) = \cdots 001|011 \cdots \end{array} \longmapsto \left| \begin{array}{l} s\left(w_0\right) = \cdots 000|111 \cdots \\ s\left(w_1\right) = \cdots 001|111 \cdots \\ s\left(w_2\right) = \cdots 000|011 \cdots \\ s\left(w_3\right) = \cdots 000|111 \cdots \end{array} \right.$$

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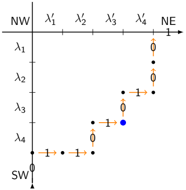

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$$s\left(\nu^{(0)}\right) = \cdots 001|011\cdots \qquad s(w_0) = \cdots 000|111\cdots$$
$$s\left(\nu^{(1)}\right) = \cdots 001|111\cdots \qquad s(w_1) = \cdots 001|111\cdots$$
$$\longmapsto$$
$$s\left(\nu^{(2)}\right) = \cdots 000|011\cdots \qquad s(w_2) = \cdots 000|011\cdots$$
$$s\left(\nu^{(3)}\right) = \cdots 001|011\cdots \qquad s(w_3) = \cdots 000|111\cdots$$

$$s(\omega_t) = \cdots 00000100|11011111\cdots \to \omega_t = (3,1,1) \in \mathcal{SC}_{(4)}$$
$$\left(\nu^{(0)}, \nu^{(1)}, \nu^{(2)}, \nu^{(3)}\right) = ((1), \emptyset, \emptyset, (1))$$

Multiplication-addition theorem for partitions

Theorem [Han–Ji (2009)]

Set $t \in \mathbb{N}$ and let ρ_1, ρ_2 be two functions defined over $\mathbb{N}$

$$f_t(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th)$$

$$g_t(q) := \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \rho_1(th) \sum_{h \in \mathcal{H}(\lambda)} \rho_2(th)$$

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Then

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h) = t \frac{(q^t; q^t)_\infty^t}{(q; q)_\infty} \left(f_t(xq^t)\right)^{t-1} g_t(xq^t)$$

Multiplication-addition theorem for $\mathcal{SC}$ and t even

Theorem [W. (2021)]

Set $t \in 2\mathbb{N}$ and let ρ_1, ρ_2 be two functions defined over $\mathbb{N}$

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Then

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \rho_1(h) \sum_{h \in \mathcal{H}_t(\lambda)} \rho_2(h)$$

$$= t \left(f_t(x^2 q^{2t})\right)^{t/2-1} g_t(x^2 q^{2t}) \left(q^{2t}; q^{2t}\right)_\infty^{t/2} \times \left(-bq; q^4\right)_\infty \left(-q^3/b; q^4\right)_\infty$$

Applications for t even (1)

1. $\rho_1(h) = \rho_2(h) = 1$: trivariate generating function of $\mathcal{SC}$

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} = \frac{\phi(q,b,t)}{(x^2q^{2t};x^2q^{2t})_\infty^{t/2}}$$

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2. $\rho_1(h) = 1/\sqrt{h}$ and $\rho_2(h) = 1$: hook-length formula

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{\sqrt{h}}$$

$$= \phi(q,b,t) \exp\left(\frac{x^2 q^{2t}}{2} + \frac{x^4 q^{4t}}{4t} \right)$$

Applications for t even (2)

1. $\rho_1(h) = \sqrt{1 - z/h^2}$ and $\rho_2(h) = 1$: modular Nekrasov–Okounkov

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2. $\rho_1(h) = 1/h$ and $\rho_2(h) = h^{2k}$: modular Stanley–Panova

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} \sum_{h \in \mathcal{H}_t(\lambda)} h^{2k} = \phi(q, b, t)$$

$$\times t^{2k+1} \exp\left(\frac{x^2 q^{2t}}{2t}\right) \sum_{i=0}^{k} T(k+1, i+1) C(i) \left(\frac{x^2 q^{2t}}{t^2}\right)^{k+1}$$

A signed multiplication theorem for $\mathcal{SC}$ and t even

$\varepsilon_u = (-1)^{c_{(i,j)}}$ and $\delta_\lambda = (-1)^d$: King (1989), Pétréolle (2016)

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Theorem [W. (2021)]

Set $t \in 2\mathbb{N}$ and let $\tilde{\rho}_1$ be a function defined over $\mathbb{Z} \times \{-1, 1\}$

$$f_t(q) := \sum_{\nu \in \mathcal{P}} q^{|\nu|} \prod_{h \in \mathcal{H}(\nu)} \tilde{\rho}_1(th, 1)\tilde{\rho}_1(th, -1),$$

Then

$$\sum_{\lambda \in \mathcal{SC}} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} b^{\mathrm{BG}(\lambda)} \prod_{\substack{u \in \lambda \\ h_u \in \mathcal{H}_t(\lambda)}} \tilde{\rho}_1(h_u, \varepsilon_u)$$

$$= \left(q^{2t}; q^{2t}\right)_\infty^{t/2} \left(-bq; q^4\right)_\infty \left(-q^3/b; q^4\right)_\infty \left(f_t(x^2 q^{2t})\right)^{t/2}$$

A similar signed multiplication theorem for BG_t and t odd

- Littlewood decomposition for t odd:
  $\lambda \in \mathcal{SC} \mapsto (\omega_t, \underline{\nu}, \mu) \in \mathcal{SC}_{(t)} \times \mathcal{P}^{(t-1)/2} \times \mathcal{SC}.$

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- t odd prime, $\mathrm{BG}_t := \{\lambda \in \mathcal{SC} \mid \forall i \in \{1, \ldots, d\}, t \nmid h_{(i,i)}\}$ [Bessenrodt (1991), Brunat–Gramain (2010), Bernal (2019)] is equivalent to $\mu = \emptyset$ in Littlewood decomposition

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Theorem [W. (2021)]

Set $t \in 2\mathbb{N}+1$ and let $\tilde{\rho}_1$ be a function defined on $\mathbb{Z} \times \{-1, 1\}$ Then

$$\sum_{\lambda \in \mathrm{BG}_t} q^{|\lambda|} x^{|\mathcal{H}_t(\lambda)|} \prod_{\substack{u \in \lambda \\ h_u \in \mathcal{H}_t(\lambda)}} \tilde{\rho}_1(h_u, \varepsilon_u)$$

$$= \frac{(q^{2t}; q^{2t})_\infty^{(t-1)/2} (-q; q^2)_\infty}{(-q^t; q^{2t})_\infty} \left(f_t(x^2 q^{2t})\right)^{(t-1)/2}$$