Crystal graph theory and some of its generalizations I: basics on crystals

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Plan of the mini-course

1. Basics on crystals
2. Crystals and Kostka polynomials
3. Crystals and random processes

Some classical problems in representation theory

Let $\mathfrak{g}$ be a simple Lie algebra of rank n over $\mathbb{C}$.
The following questions are classical.

1. How to compute the character of a representation ?
2. How to decompose the tensor product of two representations into irreducible components ?
3. How to define and compute a "canonical" basis of representations ?

Problem

Find a general frame to answer problems 1,2,3

The prototype of a Lie algebra

We have

$$\mathfrak{sl}_2 = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f.$$

It can be embedded in its universal envelopping algebra $U(\mathfrak{sl}_2)$, the **associative** $\mathbb{C}$-algebra generated by $\mathbf{e}, \mathbf{f}, \mathbf{h}$ with the relations

$$[\mathbf{h}, \mathbf{e}] = \mathbf{he} - \mathbf{eh} = 2\mathbf{e}$$
$$[\mathbf{h}, \mathbf{f}] = \mathbf{hf} - \mathbf{fh} = -2\mathbf{f}$$
$$[\mathbf{e}, \mathbf{f}] = \mathbf{ef} - \mathbf{fe} = \mathbf{h}$$

Linear algebras and their root systems

Example

$\mathfrak{sl}_{n+1} = \{M \in \mathfrak{gl}_{n+1} \mid \mathrm{tr}(M) = 0\}$ satifies

$$\mathfrak{sl}_{n+1} = \mathfrak{t}_+ \oplus \mathfrak{h} \oplus \mathfrak{t}_-$$

with root system in

$$E = \mathfrak{h}^*_{\mathbb{R}} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\}$$

such that

$$S = \{\varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n\} \quad R_+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n+1\}$$
$$W \simeq \mathfrak{S}_{n+1} = \langle s_{\varepsilon_i - \varepsilon_{i+1}} \mid 1, \dots, n\}.$$

Its Chevalley generators are

$$e_i = E_{i,i+1}, \quad h_i = E_{i,i} - E_{i+1,i+1}, \quad f_i = E_{i+1,i}$$

Root system

Each simple Lie algebra over $\mathbb{C}$ has a triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$$

where $\mathfrak{h}$ is the cartan subalgebra of $\mathfrak{g}$.
Its root system (S, R, Q, P, W) is realized in the Euclidean space $E = \mathfrak{h}_{\mathbb{R}}^*$:

- $S = \{\alpha_1, \ldots, \alpha_n\}$ is the set of simple roots
- $R = R_+ \sqcup -R_+$ is the set of positive roots of $\mathfrak{g}$
- $Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ is the root lattice
- $P = \oplus_{i=1}^n \mathbb{Z}\omega_i$ is the weight lattice such that $(\omega_i, \alpha_j^\vee) = \delta_{i,j}$ where $\alpha_j^\vee = \frac{2\alpha_j}{(\alpha_j, \alpha_j)}$.
- $W = \langle s_i = s_{\alpha_i^\perp} \mid i = 1, \ldots, n\}$ is the Weyl group

To each $\alpha \in R_+$ corresponds a triple $(e_\alpha, h_\alpha, f_\alpha)$ in $\mathfrak{g}$ such that

$$\mathbb{C}e_\alpha \oplus \mathbb{C}h_\alpha \oplus \mathbb{C}f_\alpha \simeq \mathfrak{sl}_2(\mathbb{C}).$$

We have

$$\mathfrak{g}_+ = \bigoplus_{\alpha \in R_+} \mathbb{C}e_\alpha \quad \mathfrak{h} = \bigoplus_{\alpha \in R_+} \mathbb{C}h_\alpha \quad \mathfrak{g}_- = \bigoplus_{\alpha \in R_+} \mathbb{C}f_\alpha.$$

$\mathfrak{g}$ has a presentation in terms of its Chevalley generators

$$\{e_i = e_{\alpha_i}, f_i = f_{\alpha_i}, h_i = h_{\alpha_i} \mid i \in I\}$$

and relations depending of the Cartan matrix

$$A = (a_{i,j})_{(1 \leq i,j \leq n} \text{ where } a_{i,j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

Also $\{\omega_i, i = 1, \dots, n\} \subset \mathfrak{h}_{\mathbb{R}}^*$ is the dual basis of $\{h_i, i = 1, \dots, n\} \subset \mathfrak{h}_{\mathbb{R}}$.

Representation theory

Since $(\omega_i, \alpha_j^\vee) = \delta_{i,j}$ for any $(i,j) \in \{1, \ldots, n\}^2$, we have

$$P_+ = \{\lambda \in P \mid (\lambda, \alpha_i^\vee) \geq 0, i = 1, \ldots, n\} = \oplus_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i.$$

This is the cone of dominant weights.

The f.d. irreducible representations of $\mathfrak{g}$ are parametrized by P_+ .

Let $V(\lambda)$ be the f.d. irr. rep associated to $\lambda \in P_+$. Then for any $\beta \in P$

$$V(\lambda)_\beta = \{v \in V(\lambda) \mid h(v) = \beta(h)v \text{ for any } h \in \mathfrak{h}\}$$

is the subspace of weight β in $V(\lambda)$. We have

$$V(\lambda) = \bigoplus_{\beta \in P} V(\lambda)_\beta$$

By writing $\mathbb{Z}[P] = \{e^\beta \mid \beta \in P\}$ and setting $\dim V(\lambda)_\beta = K_{\lambda,\beta}$, we get the character of $V(\lambda)$

$$s_\lambda = \sum_{\beta \in P} K_{\lambda,\beta} e^\beta \in \mathbb{Z}[P]$$

In fact, $K_{\lambda,w(\beta)} = K_{\lambda,\beta}$ for any $w \in W$ so that $s_\lambda \in \mathbb{Z}^W[P]$.

The Weyl character formula gives

$$s_\lambda = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}}{\sum_{w \in W} \varepsilon(w) e^{w(\rho)}} \text{ where } \rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha.$$

The type A case

One identifies the weight lattice with

$$P = \{\beta \in \mathbb{Z}^n \mid \beta_1 + \cdots + \beta_n = 0\}$$

and the dominant weights of $\mathfrak{sl}_n$ with the partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0)$

$$\lambda = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\omega_i$$

and by setting

$$x_1 = e^{\omega_1}, \quad x_i = e^{\omega_i - \omega_{i-1}}, i = 2, \ldots, n-1, \quad x_n = e^{-\omega_{n-1}}$$

the character s_λ is the image of the Schur function $s_\lambda(x_1, \ldots, x_n)$ in $\mathrm{Sym}[x_1, \ldots, x_n]/(x_1 \cdots x_n = 1)$.

Semisimplicity

Each f.d. representation of $\mathfrak{g}$ (or $\mathfrak{g}$-module) M decomposes on the form

$$M \simeq \bigoplus_{\lambda \in P_+} V(\lambda)^{\oplus m_\lambda}.$$

The module M is irreducible (or simple) and isomorphic to $V(\lambda)$ when its highest weight vectors space

$$M^h := \{v \in M \mid \text{weight vectors } v \text{ s.t. } e_i \cdot v = 0 \text{ for any } i = 1, \ldots, n\}$$

has dimension 1 and coincides with the weight space

$$M_\lambda = \{v \in M \mid h_i \cdot v = \lambda(h_i) v \text{ for any } i = 1, \ldots, n\}$$

Quantum groups

Each simple Lie algebra $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$ is embedded in its universal envelopping algebra $U(\mathfrak{g})$.
The quantum group $U_q(\mathfrak{g})$ is obtained by deforming $U(\mathfrak{g})$ with a formal parameter q.

Example

$U_q(\mathfrak{sl}_2)$ is the associative algebra over $\mathbb{C}(q)$ generated by E, F, T et T^{-1} and the relations

$$\begin{cases} T = q^h \\ TET^{-1} = q^2 E \\ TFT^{-1} = q^{-2} F \\ [E,F] = EF - FE = \frac{T - T^{-1}}{q - q^{-1}} \end{cases} \quad \text{versus} \quad \begin{cases} [\mathbf{h},\mathbf{e}] = \mathbf{he} - \mathbf{eh} = 2\mathbf{e} \\ [\mathbf{h},\mathbf{f}] = \mathbf{hf} - \mathbf{fh} = -2\mathbf{f} \\ [\mathbf{e},\mathbf{f}] = \mathbf{ef} - \mathbf{fe} = \mathbf{h} \end{cases}$$

The Cartan subalgebra is now $U_q(\mathfrak{g}) = \{q^h \mid h \in \mathfrak{h}\}$.

F.d. $U_q(\mathfrak{g})$-modules are just q-deformations of $U(\mathfrak{g})$-modules. Their weights yet belong to P

A weight vector of M_q of weight $\mu \in \mathfrak{h}^*$ is a vector v s.t.$q^h.v = q^{\mu(h)}v$ for any $q^h \in U_q(\mathfrak{h})$.

The highest weight vectors of M_q is a weight vector v s.t. $E_i \cdot v = 0$ for any $i = 1, \ldots, n$. Its weight is dominant.

Theorem

- *The irreducible f.d. $U_q(\mathfrak{g})$-modules are indexed by the dominant weights λ.*
- $V_q(\lambda) = U_q(\mathfrak{g}) \cdot v_\lambda$ *where v_λ is of h.w. λ.*
- ‘$\lim_{q \to 1} V_q(\lambda) = V(\lambda)$.”
- M_q *is irreducible i.i.f it admits up to a constant only one h.w.v.*

An example

The irreducible f.d. rep. of $U_q(\mathfrak{sl}_n)$ are indexed by the partitions

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0) \in \mathbb{Z}_{\geq 0}^{n-1}.$$

The Chevalley generators $E_i, H_i, F_i, i = 1, ..., n-1$ of $U_q(\mathfrak{sl}_n)$ act on

$$V_q(\square) = \bigoplus_{j=1}^{n} \mathbb{C}(q) v_j$$

by

$$E_i(v_{j+1}) = \delta_{i,j} v_i \qquad H_i(v_j) = q^{(\delta_{i,j} - \delta_{i,i+1})} v_j \qquad F_i(v_j) = \delta_{i,j} v_{i+1}.$$

This can be encoded by the graph:

$$v_1 \overset{F_1}{\longrightarrow} v_2 \overset{F_2}{\longrightarrow} v_3 \overset{F_3}{\longrightarrow} \cdots \overset{F_{n-2}}{\longrightarrow} v_{n-1} \overset{F_{n-1}}{\longrightarrow} v_n.$$

Tensor product of two modules

Given M_q and N_q two $U_q(\mathfrak{g})$-modules, the Chevalley generators act on $M_q \otimes N_q$ by

$$
\begin{aligned}
H_i(u \otimes v) &= H_i(u) \otimes H_i(v) \\
E_i(u \otimes v) &= E_i(u) \otimes H_i^{-1}(v) + u \otimes E_i(v) \\
F_i(u \otimes v) &= F_i(u) \otimes v + H_i(u) \otimes F_i(v)
\end{aligned}
$$

Example

The $U_q(\mathfrak{sl}_3)$-module $V_q(\square)^{\otimes 3}$ has four irreducible components of h.w. v.

1. $v_1 \otimes v_1 \otimes v_1$ of weight $(3,0)$
2. $v_1 \otimes (v_1 \otimes v_2 - qv_2 \otimes v_1)$ and $(v_1 \otimes v_2 - qv_2 \otimes v_1) \otimes v_1$ of weight $(2,1)$
3. $\sum_{\sigma \in S_3} (-q)^{l(\sigma)} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}$ of weight $(0,0)$.

So

$$V_q(\square)^{\otimes 3} \simeq V(3,0) \oplus V(2,1)^{\oplus 2} \oplus V(0,0).$$

Crystals in rank 1

For $U_q(\mathfrak{sl}_2)$, we have $P_+ = \mathbb{Z}_{\geq 0}\omega_1$.
Set

$$[a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}$$

Given $k \in \mathbb{Z}_{\geq 0}$, the irr. rep. $V_q(k)$ is

$$V_q(k) = \bigoplus_{a=0}^{k} \mathbb{C}(q)v_a$$

where

$$F(v_a) = [a+1]_q v_{a+1}, \quad E(v_{a+1}) = [k-a]_q v_a \text{ and } \quad H(v_a) = q^{k-2a} v_i$$

This suggests to introduce the graph

$$B(k) : v_0 \to v_1 \to \cdots \to v_k$$

by "renormalizing" the actions of F and E.

Crystal graph of a simple module

The crystal graph $B(\lambda)$ of $V_q(\lambda)$ is a colored oriented graph: an arrow $b \xrightarrow{i} b'$ means

$$b' = \widetilde{F}_i(b) \text{ i.e. } \text{"}b' = F_i(b) \text{ at } q = 0\text{"}$$

Each $b \in B(\lambda)$ belongs to an i-chain starting at $s(b)$ and ending at $e(b)$. Set

$$\varepsilon_i(b) = d(b, b_s) \text{ and } \varphi_i(b) = d(b, b_e).$$

$$b_s \underbrace{\xrightarrow{i} \cdots \xrightarrow{i}}_{\varepsilon_i(b)} b \underbrace{\xrightarrow{i} \cdots \xrightarrow{i}}_{\varphi_i(b)} b_e$$

Example

The crystal of $V(1)$ for $U_q(\mathfrak{sl}_n)$ is simply

$$B(1) : 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n.$$

Theorem (Kashiwara-Littelmann 1992)

For any dominant weight λ

$$\mathrm{char}\, V(\lambda) = \sum_{b \in B(\lambda)} e^{\mathrm{wt}(b)} \textit{ where } \mathrm{wt}(b) = \sum_{i=1}^{n} (\varphi_i(b) - \varepsilon_i(b))\omega_i.$$

This answers to Problem 1.

Example

With $B(1)$ for $U_q(\mathfrak{sl}_n)$, we get $\mathrm{wt}(i) = \omega_i - \omega_{i-1}$ and with $x_i = e^{\omega_i - \omega_{i-1}}$
$\mathrm{char}\, V(1) = x_1 + \cdots + x_n$.

The Weyl group W also acts on $B(\lambda)$ by symmetrizing each i-chain:

$$s_i(b_e) \underbrace{\overset{i}{\to} \cdots \overset{i}{\to}}_{\varepsilon_i(s_i(b)) = \varphi_i(b)} s_i(b) \underbrace{\overset{i}{\to} \cdots \overset{i}{\to}}_{\varphi_i(s_i(b)) = \varepsilon_i(b)} s_i(b_s).$$

Crystal of a tensor product

The crystal graph $B(\lambda) \otimes B(\mu)$ of $V_q(\lambda) \otimes V_q(\mu)$ has vertices

$$b \otimes b' \text{ s.t. } b \in B(\lambda), b' \in B(\mu)$$

and the action of the $\widetilde{F}_i$ is

$$\widetilde{F}_i(b \otimes b') = \begin{cases} \widetilde{F}_i(b) \otimes b' \text{ if } \varphi_i(b) > \varepsilon_i(b') \\ b \otimes \widetilde{F}_i(b') \text{ otherwise} \end{cases} .$$

Theorem (Kashiwara 1992)

The decomposition of $B(\lambda) \otimes B(\mu)$ into its connected components gives that of $V_q(\lambda) \otimes V_q(\mu)$ into its irreducible components.

This answers to Problem 2.

An example

Crystal of the defining rep. of $U_q(\mathfrak{sl}_3)$.

$$1 \quad \overset{1}{\longrightarrow} \quad 2 \quad \overset{2}{\longrightarrow} \quad 3$$

and its tensor square

$$\begin{array}{cccccc}
 & 1 & \overset{1}{\longrightarrow} & 2 & \overset{2}{\longrightarrow} & 3 \\
1 & 1\otimes 1 & \overset{1}{\longrightarrow} & 2\otimes 1 & \overset{2}{\longrightarrow} & 3\otimes 1 \\
{}_1\downarrow & & & {}_1\downarrow & & {}_1\downarrow \\
2 & 1\otimes 2 & & 2\otimes 2 & \overset{2}{\longrightarrow} & 3\otimes 2 \\
{}_2\downarrow & {}_2\downarrow & & & & {}_2\downarrow \\
3 & 1\otimes 3 & \overset{1}{\longrightarrow} & 2\otimes 3 & & 3\otimes 3
\end{array}$$

We get

$$V^{\otimes 2} \simeq V(2,0) \bigoplus V(1,1)$$

The crystal structure on words

Each vertex $b \in B(1)^{\otimes \ell}$ can identified with the word $w = x_1 \cdots x_\ell$ on the alphabet $\{1 < \cdots < n\}$.

For each $i = 1, \ldots, n-1$, form w_i the subword of w containing only the letters i and $i+1$.

$w_i^{\mathrm{red}} = (i+1)^{\varepsilon_i(w)} i^{\varphi_i(w)}$ is obtained by recursive deletion of factors $i(i+1)$ in w_i.

Example: $w = 212111322313$ with $n = 3$

1. $w_1 = 2(12)1[1(12)2]1$ and $w_1^{\mathrm{red}} = 211$. Thus $\varepsilon_1(w) = 1$ and $\varphi_1(w) = 2$
2. $w_2 = 2(23)[2(23)3]$ and $w_2^{\mathrm{red}} = 2$. Thus $\varepsilon_2(w) = 0$ and $\varphi_3(w) = 1$.

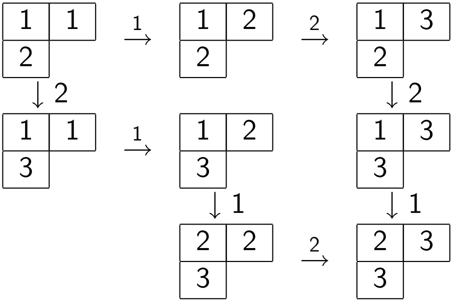
Fact

$\tilde{f}_i$ (resp. $\tilde{e}_i$) is obtained by modifying the leftmost surviving i (resp. rightmost $i+1$) into $i+1$ (resp. i).

Example: $\tilde{f}_1(w) = 212\mathbf{2}11322313$

Tableaux and crystals

By identifying each tableau with its row reading, $B(\lambda)$ can be labelled by semistandard tableaux of shape λ.



where $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} = 2 \otimes 1 \otimes 3.$

The crystal $B(2,1,0)$ for $U_q(\mathfrak{sl}_3)$ is labelled by the tableaux of shape $\lambda = (2,1,0)$.

$$\mathrm{char}\, V_q(\lambda) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2 x_1 x_2 x_3.$$

Littelmann path model

Littelmann defined operators $\tilde{f}_i, \tilde{e}_i, i = 1, \ldots, n$ on paths $\pi : [0,1] \to P$.

Each vertex $b \in B(\lambda)$ is now regared as a path s.t. $\pi(1) = \mathrm{wt}(b)$. There are many realizations of $B(\lambda)$ by

1. first choosing a h.w path π_λ such that $\mathrm{Im}\,\pi_\lambda \subset P_+$,

Example

By identifying each word $w = x_1 \cdots x_\ell$ with the piecewise paths $\pi[0,1] \to \mathbb{Z}^n$ s.t.

$$\pi\left(\frac{k}{\ell}\right) = \varepsilon_1 + \cdots + \varepsilon_k$$

and projecting on $P = \{x \in \mathbb{Z}^n \mid x_1 + \cdots + x_n = 0\}$, we get a Littelmann path model in which the h.w.v are the reverse lattice words.

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There are many realizations of $B(\lambda)$ by

1. first choosing a h.w path π_λ such that $\mathrm{Im}\,\pi_\lambda \subset P_+$,
2. next appying to π_λ the Littelmann operators $\tilde{f}_i, i = 1, \ldots, n-1$

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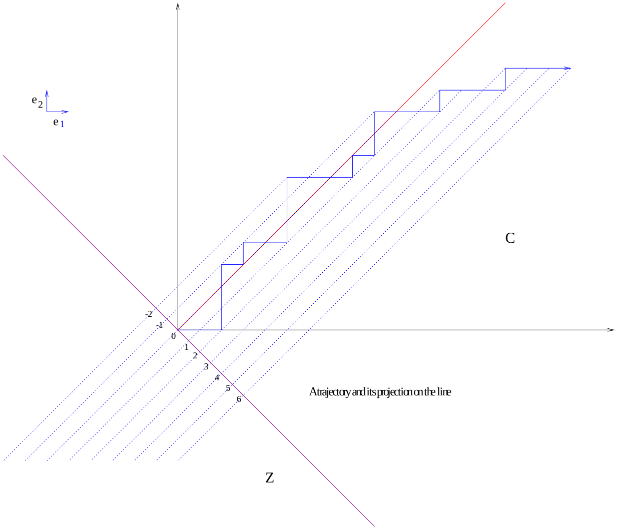

Figure: A path corresponding to a word on letters 1, 2 and its projection in P for $\mathfrak{sl}_2$.

Canonical bases

Theorem (Kashiwara-Luszig)

The canonical basis is the unique basis $\{G(b) \mid b \in B(\lambda)\}$ *of* $V_q(\lambda)$ *such that*

1. $G(b) = b$ *when* $q = 0$,
2. $G(b) = \overline{G(b)}$ *where* $\overline{}$ *is a simple involution defined on* $V_q(\lambda)$.

This answers to Problem 3.