Crystal graph theory and some of its generalizations II: Kostka polynomials

Cédric Lecouvey

University of Tours

Saint-Paul en Jarez 2022

I. Kostka-Foulkes polynomials in type A

$\mathbb{Z}^n = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$.

Remind that the root lattice Q of $\mathfrak{sl}_n$ is the sublattice of $\mathbb{Z}^n$ generated by the vectors $\varepsilon_i - \varepsilon_{i+1}, 1 \leq i < n$.

For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$, set $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.
The q-Kostant partition function of type A_{n-1} is defined by

$$\prod_{1 \leq i < j \leq n} \frac{1}{1 - q\frac{x_i}{x_j}} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q(\beta) x^\beta.$$

- $\mathcal{P}_q(\beta) \in \mathbb{Z}_{\geq 0}[q]$

I. Kostka-Foulkes polynomials in type A

$\mathbb{Z}^n = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$.

Remind that the root lattice Q of $\mathfrak{sl}_n$ is the sublattice of $\mathbb{Z}^n$ generated by the vectors $\varepsilon_i - \varepsilon_{i+1}, 1 \leq i < n$.

For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$, set $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.
The q-Kostant partition function of type A_{n-1} is defined by

$$\prod_{1 \leq i < j \leq n} \frac{1}{1 - q\frac{x_i}{x_j}} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q(\beta) x^\beta.$$

- $\mathcal{P}_q(\beta) \in \mathbb{Z}_{\geq 0}[q]$
- $\mathcal{P}_1(\beta)$ gives the number of nonnegative decompositions of β as a sum of $\varepsilon_i - \varepsilon_j, 1 \leq i < j \leq n$.

I. Kostka-Foulkes polynomials in type A

$\mathbb{Z}^n = \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i$.

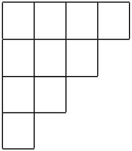
Remind that the root lattice Q of $\mathfrak{sl}_n$ is the sublattice of $\mathbb{Z}^n$ generated by the vectors $\varepsilon_i - \varepsilon_{i+1}, 1 \leq i < n$.

For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$, set $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.
The q-Kostant partition function of type A_{n-1} is defined by

$$\prod_{1 \leq i < j \leq n} \frac{1}{1 - q\frac{x_i}{x_j}} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q(\beta) x^\beta.$$

- $\mathcal{P}_q(\beta) \in \mathbb{Z}_{\geq 0}[q]$
- $\mathcal{P}_1(\beta)$ gives the number of nonnegative decompositions of β as a sum of $\varepsilon_i - \varepsilon_j, 1 \leq i < j \leq n$.
- $\mathcal{P}_q(\beta) = 0$ when $\beta \notin Q$.

A partition is a sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0) \in \mathbb{Z}^n$.
Each partition is encoded by its Young diagram. For example

$\lambda = (4, 3, 2, 1) \leftrightarrow$  and $|\lambda| = 10$.

Let $\rho = (n-1, n-2, \ldots, 1, 0)$.

Let $\mathfrak{S}_n$ be the symmetric group of rank n.

The group $\mathfrak{S}_n$ acts on $\mathbb{Z}^n$ by permutation
$\sigma \cdot (\beta_1, \dots, \beta_n) = (\beta_{\sigma(1)}, \dots, \beta_{\sigma(n)})$.

Consider λ and μ two partitions with at most n parts.

They can be identified with dominant weights of $\mathfrak{sl}_n$

$$\lambda = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\omega_i \text{ and } \mu = \sum_{i=1}^{n-1} (\mu_i - \mu_{i+1})\omega_i$$

Definition

The Kostka polynomial $K_{\lambda,\mu}(q)$ is the polynomial of $\mathbb{Z}[q]$ s.t.

$$K_{\lambda,\mu}(q) = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\mathcal{P}_q(\sigma(\lambda+\rho)-\rho-\mu)$$

where $\rho = (n-1,\dots,1,0) \in \mathbb{Z} = \mathbb{N}^n$.

By the Weyl character formula

$$K_{\lambda,\mu}(1) = \dim V(\lambda)_\mu$$

where $V(\lambda)_\mu$ is the space of weight μ in the f.d. representation $V(\lambda)$.

A semistandard tableau T of shape λ is a filling of λ by letters in $\{1, \ldots, n\}$ with

- strictly increasing columns from top to bottom
- weakly increasing rows from left to right.

Its weight is $\mathrm{wt}(T) = (\mu_1, \ldots, \mu_n)$ with $\mu_i = \#$ letters i in T

Example

$T =$

1	1	2	4
2	3	5	
3	4		
5			

with weight $\mathrm{wt}(T) = (2, 2, 2, 2, 2)$ and row reading

$$\mathrm{w}(T) = 4211532435$$

Theorem

1. $K_{\lambda,\mu}(1)$ *is equal to the number of SST of shape* λ *and weight* μ*.*
2. $K_{\lambda,\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$ *and* $K_{\lambda,\mu}(q) \neq 0$ *only if* $\lambda - \mu \in Q_+ = \oplus_{i=1}^{n-1} \mathbb{N}\alpha_i$*.*

For Assertion 2 :

- sophisticated geometric or algebraic proofs (affine Kazhdan-Lusztig polynomials, Brilinsky-Kostant filtration 1989).

Theorem

1. $K_{\lambda,\mu}(1)$ *is equal to the number of SST of shape* λ *and weight* μ.
2. $K_{\lambda,\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$ *and* $K_{\lambda,\mu}(q) \neq 0$ *only if* $\lambda - \mu \in Q_+ = \oplus_{i=1}^{n-1} \mathbb{N}\alpha_i$.

For Assertion 2 :

- sophisticated geometric or algebraic proofs (affine Kazhdan-Lusztig polynomials, Brilinsky-Kostant filtration 1989).
- a combinatorial proof and description by Lascoux & Schützenberger based on the charge statistics on SST.

Hall-Littlewood polynomials

For any $P \in \mathbb{Z}[x_1, \ldots, x_n]$ set

$$J(P) = \sum_{w \in W} \varepsilon(w) w \cdot P$$

The HL-polynomial associated to μ is defined by

$$P_\mu = \frac{1}{W_\mu(q)} \frac{J\left(\prod_{\alpha \in R_+}(1 - qe^{-\alpha}) e^{\mu+\rho}\right)}{J(e^\rho)}$$

where $W_\mu(q) = \sum_{\sigma \in \mathfrak{S}_n | \sigma(\mu)=\mu} q^{\ell(\sigma)}$.

Theorem

The HL polynomials are symmetric polynomials and for any partition λ

$$s_\lambda = \sum_\mu K_{\lambda,\mu}(q) P_\mu.$$

This permits to prove that $K_{\lambda,\mu}(q)$ is an affine KL-polynomial (Kato 1982).

Row insertions of letters in a SST.

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \cline{1-3} 3 & 4 \\ \cline{1-2}\end{array} \longleftarrow 5 = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & \mathbf{5} \\ \hline 2 & 3 & 5 \\ \cline{1-3} 3 & 4 \\ \cline{1-2}\end{array}$$

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 5 \\ \hline 2 & 5 & 5 \\ \cline{1-3} 3 & 6 \\ \cline{1-2}\end{array} \longleftarrow 3 = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & \mathbf{3} & 5 \\ \hline 2 & \mathbf{4} & 5 \\ \cline{1-3} 3 & \mathbf{5} \\ \cline{1-2} \mathbf{6} \\ \cline{1-1}\end{array}$$

Cyclage from T of weight μ:

Remove the southwest entry and insert it in the remaining tableau.

1	1	2	4
2	3	5	
3	4		
6			

$\longrightarrow$

1	1	2	4	6
2	3	5		
3	4			

$\longrightarrow$

1	1	2	3	6
2	3	4		
4	5			

$\longrightarrow$

1	1	2	3	4
2	3	4	6	
5				

$\longrightarrow \cdots \longrightarrow$

1	1	2	2	3	3	4	4	5	6

Theorem

Cocyclage operations eventually ends at the unique row R_μ of weight μ

Definition

For T of weight μ, set

$$\mathrm{ch}_n(T) = \sum_{i=1}^{n} (i-1)\mu_i - I = \mathrm{ch}_n(R_\mu) - I$$

where I the number of cyclage operations needed to get R_μ.

Theorem (LS 1980)

We have

$$K_{\lambda,\mu}(q) = \sum_{T \in SST_\mu(\lambda)} q^{\mathrm{ch}_n(T)}$$

where $SST_\mu(\lambda)$ is the set of SST of shape λ and weight μ.

Example

For $\lambda = (2,1,0)$ we have

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$$

Thus

$$\begin{aligned} \mathrm{ch}_n\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\right) &= 0 \times 1 + 1 \times 1 + 2 \times 1 = 3, \\ \mathrm{ch}_n\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) &= 1 \text{ and } \mathrm{ch}_n\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = 2. \end{aligned}$$

We get

$$K_{(2,1,0)}(q) = q + q^2.$$

Charge and crystals

Problem

Interpret the charge in crystal theory.

Let $w = x_1 \cdots x_\ell$ be a word on $\{1 < \cdots < n\}$.
For each $i = 1, \ldots, n-1$, form w_i the subword of w contained only the letters i and $i+1$.

Remind the definition of ε_i and φ_i (Part I).

Example: $w = \mathbf{2}\mathbf{1}1\mathbf{1}5324\mathbf{3}\mathbf{1}3\mathbf{1}$ with $n = 5$ $w_1 = 21(12)11$ and $w_1^{\text{red}} = 2111$. Thus $\varepsilon_1(w) = 1$ and $\varphi_1(w) = 3$.

Remind also the action of the Weyl group: s_i acts by symmetrizing each i-chain :

$$s_i(w) = \mathbf{2}4\mathbf{2}153243\mathbf{2}3\mathbf{1}.$$

Let $O(T)$ be the orbit of the tableau T under the $\mathfrak{S}_n$-crystal action.

Theorem (LLT 1995)

For any tableau T of dominant weight

$$\mathrm{ch}_n(T) = \frac{1}{|O(T)|} \sum_{T' \in O(T)} \chi(T')$$

where

$$\chi(T') = \sum_{i=1}^{n-1} (n-i)\varepsilon_i(\mathrm{w}(T)).$$

Fact

This gives an interpretation of ch_n independent of the tableau model.

III Affine type A quantum groups and 1-d sums

$U_q(\widehat{\mathfrak{sl}}_n)$ is the quantum group associated to the affine Lie algebra

$$\widehat{\mathfrak{sl}}_n = \mathfrak{sl}_n \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}d \oplus \mathbb{C}K.$$

It admits

1. highest weight representations but infinite dimensional (quite simple)

III Affine type A quantum groups and 1-d sums

$U_q(\widehat{\mathfrak{sl}}_n)$ is the quantum group associated to the affine Lie algebra

$$\widehat{\mathfrak{sl}}_n = \mathfrak{sl}_n \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}d \oplus \mathbb{C}K.$$

It admits

1. highest weight representations but infinite dimensional (quite simple)
2. finite dimensional representations but without h.w. vectors (more complicated).

K-R modules and crystals

For $\mu = (\mu_1, ..., \mu_m) \in \mathbb{Z}_{\geq 0}^m$, consider

$$B_\mu = B(\mu_1 \omega_1) \otimes \cdots \otimes B(\mu_m \omega_1)$$

the crystal of the $U_q(\mathfrak{sl}_n)$-module $V_{q,\mu}$

$$V_{q,\mu} = V_q(\mu_1 \omega_1) \otimes \cdots \otimes V_q(\mu_m \omega_1).$$

Theorem

1. *$V_{q,\mu}$ has also the structure of an irreducible affine $U_q(\widehat{\mathfrak{sl}}_n)$-module $\widehat{V}_{q,\mu}$.*
2. *$\widehat{V}_{q,\mu}$ has a crystal $\widehat{B}_\mu$ obtained by adding arrows of color 0 in B_μ.*

An example

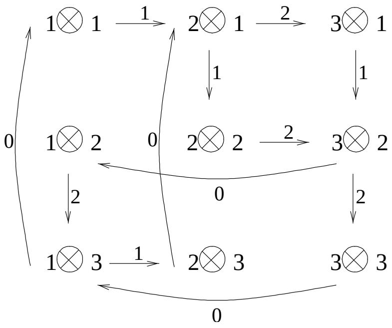


Figure: The affine $U_q(\widehat{\mathfrak{sl}}_3)$-crystal $\widehat{B}_{(1^2)}$

We have

$$\tilde{f}_0 = \mathrm{pr}^{-1} \circ \tilde{f}_1 \circ \mathrm{pr}$$

where the promotion operator pr changes each $i = 1, \ldots, n$ in $i + 1 \bmod n$.

$\widehat{B}_\mu$ is graded by the "energy" D defined from the graph structure.
D is constant on the classical connected components.

D is given on $\widehat{B}_{1^\ell}$ by

$$D(b) = \sum_{k=1}^{\ell-1} (\ell - k) H(x_k \otimes x_{k+1}) \text{ where } H(x_k \otimes x_{k+1}) = \begin{cases} 1 \text{ if } x_k \leq x_{k+1} \\ 0 \text{ if } x_k > x_{k+1} \end{cases}$$

for any $b = x_1 \otimes \cdots \otimes x_\ell$.

Example

On $\widehat{B}_{(1^2)}$, we have $D = 1$ on $\widehat{B}(1 \otimes 1)$ and $D = 0$ on $\widehat{B}(1 \otimes 2)$.

The definition is more complicated for a general μ.

One-dimensional sums and Kostka polynomials

For any partition λ, set

$$E_{\lambda,\mu} = \{b \in \widehat{B}_\mu \mid b \text{ is of highest weight } \lambda\}.$$

Definition

The one-dimensional sum $X_{\lambda,\mu}$ is defined by

$$X_{\lambda,\mu}(q) = \sum_{b \in E_{\lambda,\mu}} q^{D(b)}.$$

It is related to particle theory in mathematical physics.

Theorem (Nakayashiki-Yamada 1996)

The 1-d sums coincide with the Kostka polynomials.

Idea of the proof

Key ingredient: "Schur duality":

$$K_{\lambda,\mu} = \dim V_q(\lambda)_\mu = [V_{q,\mu} : V_q(\lambda)]$$

$$\text{SST } T \text{ of weight } \mu \overset{1:1}{\leftrightarrow} b \in \widehat{B}_\mu \text{ of h.w. } \lambda$$

Example

with $\lambda = (5,4,1)$ and $\mu = (2,2,3,3)$

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} 4 \\ \cline{1-1}\end{array} \overset{1:1}{\leftrightarrow}$$

$$\boxed{1}\boxed{1} \otimes \boxed{1}\boxed{2} \otimes \boxed{1}\boxed{2}\boxed{2} \otimes \boxed{1}\boxed{2}\boxed{3} = b$$

where b is indeed of highest weight λ in $\widehat{B}_\mu$ for

$$\mathrm{w}(b) = 1121221321.$$

It remains to prove that

$$\mathrm{ch}_n(T) = D(b).$$

This can be done

- either by using another expression of ch_n in terms of indices of T (NY 1995)

It remains to prove that

$$\mathrm{ch}_n(T) = D(b).$$

This can be done

- either by using another expression of ch_n in terms of indices of T (NY 1995)
- or directly by proving that cyclage on T corresponds to promotion on b (Gerber-L 2021).

Beyond type A

The Kostka polynomial $K_{\lambda,\mu}(q)$ with $\lambda, \mu \in P_+$ is defined similarly as a q-analogue of $\dim V(\lambda)_\mu$.

Problem

Find a description of $K_{\lambda,\mu}(q)$.

This is a elementary but difficult question and only partial answers are known.

Theorem (L-Okado-Shimozono 2011)

One-dimensional sums defined from classical types KR crystals of row or column shapes are Kostka polynomials.

Unfortunately, the converse is false, for example

$$K_{\lambda,0}(q) \text{ i.e. for } \mu = 0$$

is not a 1-d sum in general.

Some results

1. A conjectural description of the $K_{\lambda,\mu}(q)$ in type C based on cyclage on Kashiwara-Nakashima tableaux (L-2005).

All these results seems to indicate that the crystal structure does not suffice to capture the combinatorial complexity of the KF-polynomials.

Some results

1. A conjectural description of the $K_{\lambda,\mu}(q)$ in type C based on cyclage on Kashiwara-Nakashima tableaux (L-2005).
2. A description of $K_{\lambda,0}(q)$ in type C based on King tableaux and a proof of 1 for columns tableaux (L-Lenart 2018).

All these results seems to indicate that the crystal structure does not suffice to capture the combinatorial complexity of the KF-polynomials.

Some results

1. A conjectural description of the $K_{\lambda,\mu}(q)$ in type C based on cyclage on Kashiwara-Nakashima tableaux (L-2005).
2. A description of $K_{\lambda,0}(q)$ in type C based on King tableaux and a proof of 1 for columns tableaux (L-Lenart 2018).
3. A description of $K_{\lambda,0}(q)$ in type B, D (Jang-Known 2019)

All these results seems to indicate that the crystal structure does not suffice to capture the combinatorial complexity of the KF-polynomials.

Some results

1. A conjectural description of the $K_{\lambda,\mu}(q)$ in type C based on cyclage on Kashiwara-Nakashima tableaux (L-2005).
2. A description of $K_{\lambda,0}(q)$ in type C based on King tableaux and a proof of 1 for columns tableaux (L-Lenart 2018).
3. A description of $K_{\lambda,0}(q)$ in type B, D (Jang-Known 2019)
4. A proof of 1 for λ row partition (Gerber-Dolega-Torres 2020).

All these results seems to indicate that the crystal structure does not suffice to capture the combinatorial complexity of the KF-polynomials.

Some results

1. A conjectural description of the $K_{\lambda,\mu}(q)$ in type C based on cyclage on Kashiwara-Nakashima tableaux (L-2005).
2. A description of $K_{\lambda,0}(q)$ in type C based on King tableaux and a proof of 1 for columns tableaux (L-Lenart 2018).
3. A description of $K_{\lambda,0}(q)$ in type B, D (Jang-Known 2019)
4. A proof of 1 for λ row partition (Gerber-Dolega-Torres 2020).
5. A new approach based on atomic decomposition of characters (L-Lenart 2020).

All these results seems to indicate that the crystal structure does not suffice to capture the combinatorial complexity of the KF-polynomials.