Crystal graph theory and some of its generalizations III: random walks

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I. Simple random walk

Let $B = \{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and let $\overline{C}$ be the cone

$$\overline{C} = \{x \in \mathbb{R}^n \mid x_1 \geq \cdots \geq x_n \geq 0\} \subset \mathbb{R}^n.$$

The elements of $\overline{C} \cap \mathbb{Z}^n$ are partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$.
Set

$$|\lambda| = \lambda_1 + \cdots + \lambda_n.$$

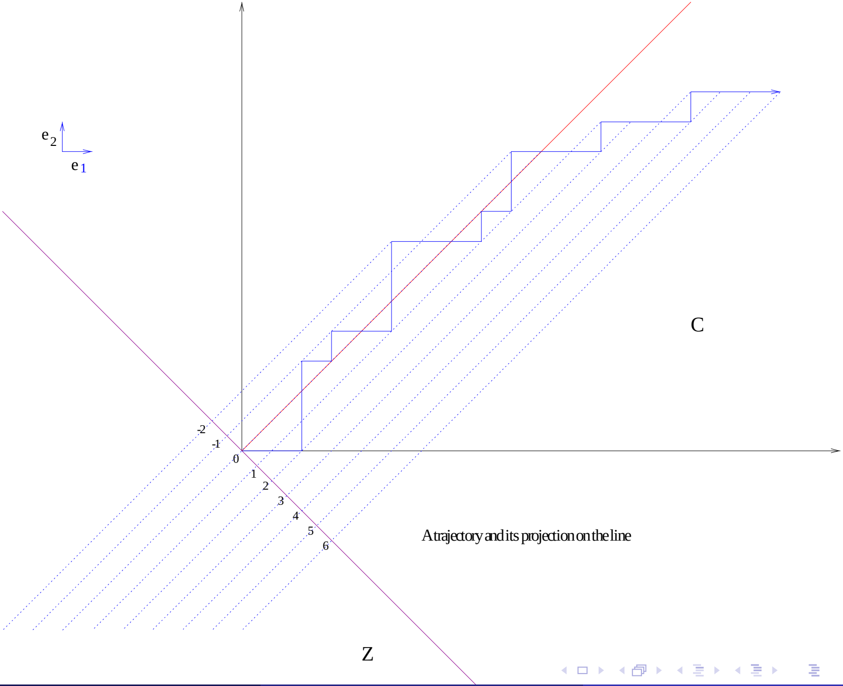
Let $(X_\ell)_{\ell \geq 1}$ be a sequence of random variables in B (i.i.d.)

$$\mathbb{P}(X_\ell = e_i) = p_{e_i} \in]0,1[\text{ for } i = 1, \ldots, n$$
$$p_{e_1} + \cdots + p_{e_n} = 1$$

$$m := E(X_\ell) = \sum_{i=1}^{n} p_{e_i} e_i.$$

$S_\ell = X_1 + \cdots + X_\ell$ defines a random walk on $\mathbb{Z}^n$ with steps in B.
It is a Markov chain with transition matrix

$$\Pi(\alpha, \beta) = \begin{cases} p_{e_i} \text{ if } \beta - \alpha = e_i \in B, \\ 0 \text{ otherwise} \end{cases}$$


e2
e1
-2
-1
0
1
2
3
4
5
6
C
A trajectory and its projection on the line
Z

Assume $E(X_\ell) = m = (p_{e_1}, \dots, p_{e_n}) \in C$.

For any partition μ, set $\psi(\mu) = \mathbb{P}_\mu(S_\ell \in \overline{C}, \forall \ell \geq 1)$ and $\hat{\psi}(\mu) = p^\mu \psi(\mu)$

Lemma

1. *The function ψ is positive on $\overline{C} \cap \mathbb{Z}^n$*

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Lemma

1. *The function ψ is positive on $\overline{C} \cap \mathbb{Z}^n$*
2. *The function $\hat{\psi}$ is harmonic on $\overline{C} \cap \mathbb{Z}^n$*

$$\hat{\psi}(\mu) = \sum_{\mu \rightsquigarrow \lambda} \hat{\psi}(\lambda)$$

where the sum is over the partitions $\lambda \supset \mu$ such that $|\lambda| - |\mu| = 1$.

Since $\psi > 0$, the conditioning of $(S_\ell)_{\ell \geq 0}$ to stay in $\overline{C}$ is well-defined.
For partitions $\lambda \supset \mu$ such that $|\lambda| - |\mu| = 1$, set

$$\Pi_{\overline{C}}(\mu, \lambda) = \mathbb{P}(S_{\ell+1} = \lambda \mid S_\ell = \mu, S_k \in \overline{C}, \forall k \geq 1).$$

Theorem (O'Connell 2004)

The conditioning of $(S_\ell)_{\ell \geq 0}$ to stay in $\overline{C}$ is a Markov chain with transitions

$$\Pi_{\overline{C}}(\mu, \lambda) = \Pi(\mu, \lambda) \frac{\hat{\psi}(\lambda)}{\hat{\psi}(\mu)} = \frac{\hat{\psi}(\lambda)}{\hat{\psi}(\mu)} 1_B(\lambda - \mu).$$

Problem

Compute the function $\hat{\psi}$.

Theorem (O'Connell (2004))

Assume $m = (p_{e_1} > \cdots > p_{e_n})$. *For any partition* λ,

$$\hat{\psi}(\lambda) = \prod_{1 \leqslant i < j \leqslant n} \left(1 - \frac{p_{e_j}}{p_{e_i}}\right) s_\lambda(p_{e_1}, \ldots, p_{e_n})$$

where s_λ *is the Schur polynomial associated to* λ.

Corollary

We have

$$\Pi_{\overline{C}}(\mu, \lambda) = \frac{s_\lambda(p_{e_1}, \ldots, p_{e_n})}{s_\mu(p_{e_1}, \ldots, p_{e_n})} 1_B(\lambda - \mu).$$

Idea of the proof

Based on 3 ingredients

1. The insertion procedure on SST (RSK)

Remark: there is a simpler proof based on the reflection principle of Gessel and Zeilberger.

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3. The limit

$$\lim_{\ell \to +\infty} \frac{f_{\lambda^{(\ell)}/\mu}}{f_{\lambda^{(\ell)}}} = s_\mu(p_{e_1}, \ldots, p_{e_n})$$

   when $\lambda^{(\ell)}$ is a sequence of partitions such that

$$\lim_{\ell \to +\infty} \frac{1}{\ell} \lambda^{(\ell)} = m = (p_{e_1} > \cdots > p_{e_n}).$$

   Here $f_{\lambda^{(\ell)}/\mu}$ is the number of standard skew tableaux of shape $\lambda^{(\ell)}/\mu$.

Remark: there is a simpler proof based on the reflection principle of Gessel and Zeilberger.

II Generalizations

Problem

- *Study conditioned random walks with steps the weights of any f.d. irreducible representation V of any simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, for example $\pm e_i$ in $\mathbb{Z}^n$.*

Ideas

II Generalizations

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- *Study conditioned random walks with steps the weights of any f.d. irreducible representation V of any simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, for example $\pm e_i$ in $\mathbb{Z}^n$.*
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Ideas

1. Replace random walks by random (continuous) trajectories obtained as the concatenation of Littelmann paths in the crystal of V.
2. Replace the reflection principle by the Weyl character formula (Littelmann proof of the WCF generalizes the reflection principle).
3. Use a transformation on trajectories inspired by the Pitman transform on the line instead of RSK.

Littelmann path model 1

Tableaux give particular Littelmann path models. For example

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \text{ corresponds to}$$

$$\mathrm{w}_R(T) = 2 * 1 * 1 * 3 * 3 * 2 * 3 \text{ or}$$
$$\text{the path } \mathrm{w}_R(T) = 2 * 3 * 1 * 3 * 1 * 2 * 3$$

in $\mathbb{R}^3 = \mathbb{R}\varepsilon_1 \oplus \mathbb{R}\varepsilon_2 \oplus \mathbb{R}\varepsilon_3$.
The operator $\tilde{f}_i$ changes a precise i (parenthezing process) in a $i+1$ thus applies $s_{\varepsilon_i - \varepsilon_{i+1}}$ to this i.
Vertices of $B(\lambda)$ can be realized as piecewise continous paths $\eta : [0,1] \to P_{\mathbb{R}}$

Littelmann path model 2

Let $\mathfrak{g}$ be a simple Lie algebra with root system R, simple roots $\alpha_1, \ldots, \alpha_n$ and weight lattice P.

- A Littelmann path is a piecewise linear map $\eta : [0,1] \rightarrow P_{\mathbb{R}}$ such that $\eta(0) = 0$ and $\eta(1) \in P$.
- The crystal operators $\tilde{e}_i, \tilde{f}_i$, $i = 1, \ldots, n$ act on η by reflecting some parts of η by s_{α_i}.
- A highest weight path η is such that $\mathrm{Im}\, \eta \subset \overline{C}$ (equivalent to $\tilde{e}_i(\eta) = 0$ for any i).
- Given $\kappa \in P_+$ and η_κ a h.w.p such that $\eta(1) = \kappa$. The set

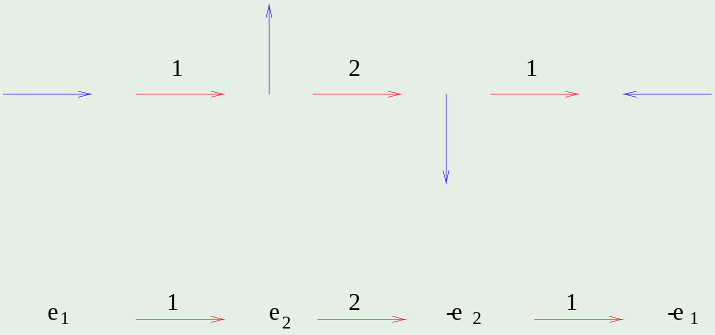
$$B(\kappa) \simeq B(\eta_\kappa) = \{\tilde{F} \cdot \eta_\kappa \mid \tilde{F} \text{ product of } \tilde{f}_i\}$$

is the crystal associated to η_κ.

Example

In type C_2, $P = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset \mathbb{R}^2$ and $\overline{C} = \{x = (x_1, x_2) \mid x_1 \geq x_2 \geq 0\}$.
For $\kappa = \omega_1 = e_1$,

Crystal of the vector representation in type C2

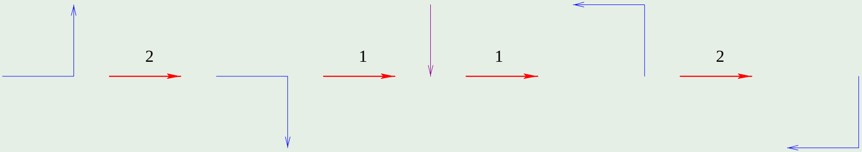


The Littelmann paths are lines (as in any minuscule representation)

Example

For $\kappa = \omega_2 = e_1 + e_2$,

In type C2, the crystal of the fundamental representation with dimension 5 with its 5 elementary Littelman paths



 is the up-down path of weight 0

Random trajectory

Assume $B(\eta_\kappa)$ has probability distribution $p = (p_\eta)_{\eta \in B(\eta_\kappa)}$

Let X be a random variable with values in $B(\eta_\kappa)$ s.t.

$$\mathbb{P}(X = \eta) = p_\eta \text{ for any } \eta \in B(\eta_\kappa).$$

Set

$$\mathbf{m} := E(X) = \sum_{\eta \in B(\eta_\kappa)} p_\eta \eta$$

and $\mathbf{m}(1) = m$.

Let $(X_\ell)_{\ell \geq 1}$ be a i.i.d. sequence of random variables with the same law as X.

The random trajectory $\mathcal{W}$ is defined by

$$\mathcal{W}(t) := X_1(1) + X_2(1) + \cdots + X_{\ell-1}(1) + X_\ell(t - \ell)$$

for any $\ell \in \mathbb{Z}_{>0}$ and $t \in [\ell, \ell + 1]$.

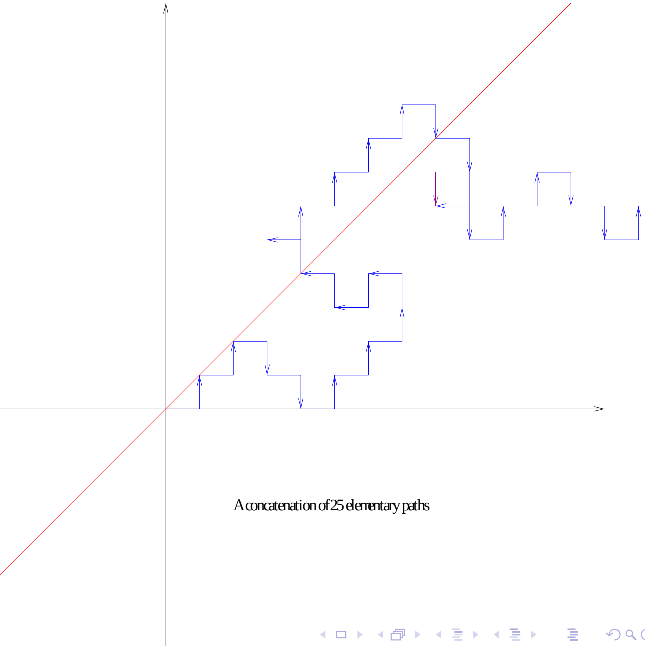
Set $W_\ell = \mathcal{W}(\ell)$.

The sequence $W = (W_\ell)_{\ell \geq 1}$ is a random walk with steps the weights of $V(\kappa)$.

In type C2 the 5 elementary Littelman paths

are the vertices of the crystal of the fun. rep. of dim 5

 is the up-down path



Central measures on trajectories

A trajectory η of length ℓ is the concatenation

$$\eta = \pi_1 * \cdots * \pi_\ell \in B(\eta_\kappa)^{*\ell}$$

of ℓ paths in $B(\eta_\kappa)$.

It has probability

$$p_\eta = p_{\pi_1} \times \cdots \times p_{\pi_\ell}.$$

Definition

The distribution p on $B(\eta_\kappa)$ is central when for any $\ell \geq 1$ and η, η' in $B(\eta_\kappa)^{*\ell}$ such that $\eta(\ell) = \eta'(\ell)$, we have $p_\eta = p_{\eta'}$.

Theorem (L., Lesigne, Peigné)

The distribution p is central i.f.f. there exists $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_{>0}$ such that

$$p_{\eta'} = p_\eta \times \tau_i$$

as soon as $\eta \xrightarrow{i} \eta'$ in $B(\eta_\kappa)$

Example

In type C_2 with $\kappa = \omega_1$, choose $\tau = (\tau_1, \tau_2) \in \mathbb{R}^2_{>0}$

$$e_1 \underset{\times\tau_1}{\overset{1}{\longrightarrow}} e_2 \underset{\times\tau_2}{\overset{2}{\longrightarrow}} -e_2 \underset{\times\tau_1}{\overset{1}{\longrightarrow}} -e_1$$

$$p_{e_1} = \frac{1}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2},\ p_{e_2} = \frac{\tau_1}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}$$

$$p_{-e_2} = \frac{\tau_1\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2},\ p_{-e_1} = \frac{\tau_1^2\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}$$

and

$$m(\tau) = \frac{1-\tau_1^2\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}e_1 + \frac{\tau_1-\tau_1\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}e_2$$

Observe that $m(\tau) \in C$ i.f.f. $(\tau_1, \tau_2) \in]0,1[^2$.

Generalization of O'Connell's results

Assume $\tau \in]0,1[^n$ (this is equivalent to $m(\tau) \in C$).

For any $\beta = a_1\alpha_1 + \cdots + a_n\alpha_n \in Q_+$, set $\tau^\beta = \tau_1^{a_1} \cdots \tau_n^{a_n}$

Consider $\lambda \in P_+$. Let $V(\lambda)$ be the f.d. representation of $\mathfrak{g}$ of h.w. λ.

Define the harmonic function ψ on P_+ by

$$\psi(\lambda) = \mathbb{P}_\lambda(\mathcal{W}(t) \in \overline{C} \text{ for any } t \geq 0).$$

Theorem (L., Lesigne, Peigné)

1. *We have*

$$\psi(\lambda) = \prod_{\alpha \in R_+} (1 - \tau^{\alpha}) S_{\lambda}(\tau)$$

where $S_\lambda \in \mathbb{Z}_{\geq 0}[e^{\alpha_1}, \dots, e^{\alpha_n}]$ *is the (renormalized) Weyl character of* $V(\lambda)$.

Proof.

Based on WCF, LLN and the path model. □

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2. *The law of the random walk* W *conditioned to stay in* $\overline{C}$ *is given by*

$$\Pi_{\overline{C}}(\mu, \lambda) = \frac{S_{\lambda}(\tau)}{S_{\kappa}(\tau) S_{\mu}(\tau)} \tau^{\kappa + \mu - \lambda} m^{\lambda}_{\mu,\kappa}$$

where $m^{\lambda}_{\mu,\kappa}$ *is the multiplicity of* $V(\lambda)$ *in* $V(\mu) \otimes V(\kappa)$.

Proof.

Based on WCF, LLN and the path model. □

III Generalized Pitman transform

$B(\pi_\kappa)^{*\ell}$ has the structure of a crystal graph.
Each trajectory $\eta \in B(\pi_\kappa)^{*\ell}$ of length ℓ belongs to a connected component $B(\eta) \subset B(\pi_\kappa)^{*\ell}$.
$B(\eta)$ contains a unique trajectory $\mathcal{P}(\eta)$ such that $\tilde{e}_i(\mathcal{P}(\eta)) = 0$ for any $i = 1, \dots, n$. Thus

$$\mathrm{Im}\,\mathcal{P}(\eta) \subset \overline{C}.$$

Definition (Biane, Bougerol, O'Connell (2005))

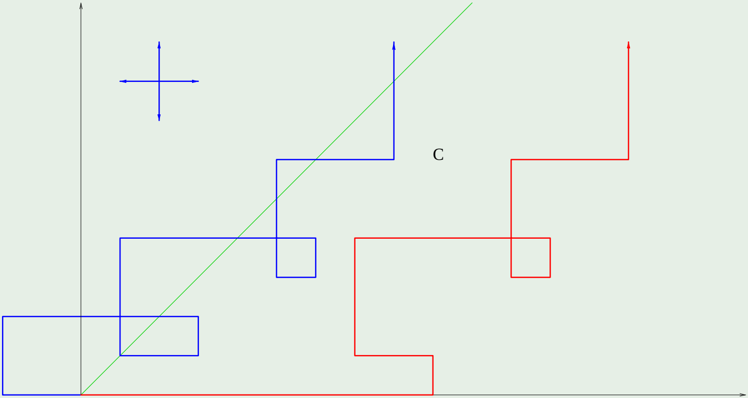
The map

$$\mathcal{P} : \eta \to \mathcal{P}(\eta) \in \overline{C}$$

is the generalized Pitman transform on trajectories.

Example

Type C_2 and $\kappa = \omega_1$



Un chemin (en bleu) et son image par P (en rouge) pour la representation vectorielle de sp(4,C)

A path (in blue) and its image by $\mathcal{P}$ (in red).

Set $\mathcal{H} = \mathcal{P}(\mathcal{W})$.

Theorem (L., Lesigne, Peigné, Tarrago)

- *$\mathcal{H}$ is a Markov chain and its law coincides with the law of $\mathcal{W}$ conditioned to stay in $\overline{C}$.*

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- *$\mathcal{H}$ is a Markov chain and its law coincides with the law of $\mathcal{W}$ conditioned to stay in $\overline{C}$.*
- *$\mathcal{P}$ is almost surely invertible on infinite trajectories and $\mathcal{P}^{-1}$ can be made explicit using Lusztig involution on crystals.*

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- *$\mathcal{P}$ is almost surely invertible on infinite trajectories and $\mathcal{P}^{-1}$ can be made explicit using Lusztig involution on crystals.*
- *$\mathcal{W}$ and $\mathcal{H}$ satisfy a law of large numbers and a central limit theorem.*

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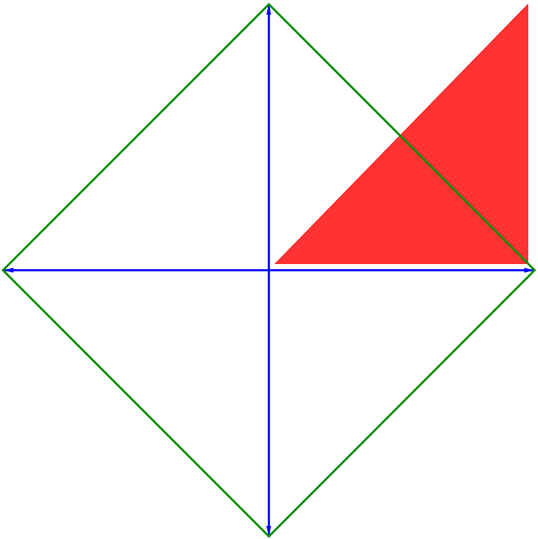
- *$\mathcal{P}$ is almost surely invertible on infinite trajectories and $\mathcal{P}^{-1}$ can be made explicit using Lusztig involution on crystals.*
- *$\mathcal{W}$ and $\mathcal{H}$ satisfy a law of large numbers and a central limit theorem.*
- *When τ runs over $]0,1[^n$, the drifts $m(\tau)$ parametrize $C \cap \Pi_\kappa$ where Π_κ is the convex hull of the weights for $V(\kappa)$.*



The set $C \cap \Pi_\kappa$ for $\kappa = \omega_1$ in type C_2

Some perpectives

Interesting random processes are controled by positive harmonic functions on rooted graded graphs e.g.

vertices	harmonic functions	markov chain
partitions $\lambda \in \mathcal{P}_n$	$\lambda \to s_\lambda(p)$	on $\mathcal{P}_n$
dominant weights $\lambda \in P_+$	$\lambda \to \mathrm{char}\, V(\lambda)(\boldsymbol{\tau})$	on P_+
$(n+1)$-core partitions	k-Schur polynomials	on type A alcoves
parabolic cosets W/W_I	hom. aff. grassm.	on alcoves
partition of $\mathcal{P}_{n,\ell}$	fusion ring	on $\mathcal{P}_{n,\ell}$

For the 3 last examples, no combinatorial description of the structure constants is known.