Bailey pairs in combinatorics, number theory, and knot theory

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Three talks

1. Introduction to q-series and Bailey pairs
2. Bailey pairs, mock theta functions, and indefinite quadratic forms
3. Bailey pairs and strange identities

Outline

1. Introduction to q-series and Bailey pairs

   (*i*) Background on q-series

   (*ii*) Work of Bailey and Slater

   (*iii*) Work of Andrews

   (*iv*) Work of Warnaar on partial theta functions

q-series

q-series (Eulerian series, basic hypergeometric series, q-hypergeometric series) are constructed from the q-rising factorials (q-Pochhammer symbols),

$$(a)_n := (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$
$$(a)_\infty := (a;q)_\infty := (1-a)(1-aq)\cdots .$$

Generically, q-series take the form

$$\sum_{n\geq 0} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n z^n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}.$$

Convergence conditions

q-series

Historically, q-series were studied as combinatorial or analytic objects.

Euler, Heine, Sylvester, Rogers, Ramanujan, Bailey, Slater, Andrews, . . .

Today, q-series are everywhere!

Combinatorics, number theory, knot theory, mathematical physics, algebra, . . .

They are interesting enough in their own right.

Rogers-Ramanujan identities

That q-series are everywhere is best exemplified by the Rogers-Ramanujan identities,

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

In q-series, one typically studies identities.

Here are some examples of q-series identities:

Identities - Examples

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

$$\sum_{n\geq 0} \frac{q^{n^2}}{(-q)_n} = \frac{1}{(q)_\infty} \sum_{\substack{n\geq 0 \\ |j|\leq n}} (-1)^j q^{n(5n+1)/2-j^2}(1-q^{4n+2}),$$

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n^2} = \frac{1}{(q)_\infty},$$

$$\sum_{n\geq 0} \frac{q^{n}}{(q)_n^2} = \frac{1}{(q)_\infty^2} \sum_{n\geq 0} (-1)^n q^{\binom{n+1}{2}}$$

$$\sum_{n\geq 0} \frac{q^{n^2}}{(-q)_n^2} = \frac{2}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n(3n+1)/2}}{(1+q^n)}.$$

Identities - Examples

The right-hand sides are, respectively, a weight 0 modular form, a fifth order mock theta function, a weight $-1/2$ modular form,a false theta function, and a third order mock theta function.

The left-hand sides are generating functions for partitions.

q-series identities are the basis for the connection between partitions and number theory.

Note that q-series identities are not always "robust".

Generally speaking, the number-theoretic properties can never be established using the q-series expression.

Identities - Examples

Here are more examples of identities.

$$\sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n} = (q)_\infty,$$

$$\sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(-q;q)_n} = \sum_{\substack{n\geq 0 \\ |j|\leq n}} (-1)^{n+j} q^{n(3n+1)/2-j^2}(1-q^{2n+1}),$$

$$\sum_{n\geq 0} \frac{q^{n(n-1)/2}}{(-q;q)_n} = 2.$$

The left hand sides are, respectively, a weight $1/2$ modular form, a Hecke theta function, and a constant.

Proving identities

How does one prove q-series identities?

A "bag of tricks"?

Arthur C. Clarke's 3rd law: "Any sufficiently advanced technology is indistinguishable from magic."

In fact, there is a great deal of structure behind many q-series identities!

Perhaps the most important structural elements in q-series are Bailey pairs.

Work of Bailey

Building on L.J. Rogers' work on the Rogers-Ramanujan identities, W.N. Bailey (1949) showed that the identity

$$\sum_{n\geq 0}(b)_n(c)_n(aq/bc)^n\beta_n = \frac{(aq/b)_\infty(aq/c)_\infty}{(aq)_\infty(aq/bc)_\infty}\sum_{n\geq 0}\frac{(b)_n(c)_n(aq/bc)^n}{(aq/b)_n(aq/c)_n}\alpha_n$$

holds whenever the pair of sequences (α_n, β_n) satisfies the key relation

$$\beta_n = \sum_{k=0}^{n}\frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}.$$

Work of Bailey

It can be shown that the key relation is satisfied for $a = 1$ by

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{n(3n-1)/2}(1+q^n), & \text{if } n > 0, \end{cases}$$

and

$$\beta_n = \frac{1}{(q)_n}.$$

Inserting this into Bailey's general identity, taking $b, c \to \infty$, and using the fact that

$$\lim_{x \to \infty} (x)_n (1/x)^n = (-1)^n q^{\binom{n}{2}},$$

we have the identity

Work of Bailey

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q)_n}=\frac{1}{(q)_\infty}\left(1+\sum_{n\geq 1}(-1)^n q^{n(5n-1)/2}(1+q^n)\right)$$
$$=\sum_{n\in\mathbb{Z}}(-1)^n q^{n(5n-1)/2}$$
$$=\frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q)_\infty},$$

by Jacobi's triple product identity,

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2}=(-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty.$$

Work of Bailey

This is the first Rogers-Ramanujan identity.

The second Rogers-Ramanujan identity follows from a similar pair with $a = q$,

$$\alpha_n = \frac{(1 - q^{2n+1})(-1)^n q^{n(3n+1)/2}}{1 - q}$$

and

$$\beta_n = \frac{1}{(q)_n}.$$

There is some freedom here. Choosing other values of b and c leads to other identities, like

$$\sum_{n \geq 0} \frac{(-1)_n q^{\binom{n+1}{2}}}{(q)_n} = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty}.$$

Work of Slater

A student of Bailey, L.J. Slater, wrote two fundamental papers (1951–52).

In the first paper, she compiled over 70 pairs (α_n, β_n) satisfying Bailey's key relation.

In the second paper, she used these pairs in Bailey's identity for various choices of b and c to prove 130 Rogers-Ramanujan type identities.

This included most of the known Rogers-Ramanujan type identities at the time and much more.

These papers are often called "Slater's lists".

Work of Slater

An excerpt from Slater's first paper:

a	c	d	β_n	α_r	x
1	$-q^{\frac12}$	$q^{\frac12}$	$2(-q)_{n-1}/(q^{\frac12})_n(-q^{\frac12})_n(q)_n$	$q^{r^2}(q^{\frac12 r}+q^{-\frac12 r})$	q
1	$-q^{\frac12}$	∞	$1/(-q^{\frac12})_n(q)_n$	$(-1)^r q^{r^2}(q^{\frac12 r}+q^{-\frac12 r})$	q
q	0	0	$(-1)^n/q^{\frac12(n^2+3n)}(q)_n$	$(-1)^r q^{-\frac12 r^2}(q^{-\frac32 r}+q^{\frac32 r})$	q
			$(-1)^n/q^{\frac12(n^2+n)}(q)_n$	$(-1)^r q^{-\frac12 r^2}(q^{-\frac12 r}+q^{\frac12 r})$	q
q	q	0	$1/q^n(q)_n(q)_n$	$q^{-r}-q^r$	q
			$1/(q)_n(q)_n$	0	q
q	$-q$	0	$(-1)^n/(q)_n(-q)_n$	$2(-1)^r$	q
			$(-1)^n/q^n(q)_n(-q)_n$	$(-1)^r(q^{-r}+q^r)$	q
q	$-q$	∞	$q^n/(q)_n(q)_n$	$q^{r^2}(q^r-q^{-r})$	q
q	q	$-q$	$2/(q)_n(q)_n(1+q^n)$	$q^{\frac12 r^2}(q^{-\frac12 r}-q^{\frac12 r})$	q
			$2q^n/(q)_n(q)_n(1+q^n)$	$q^{\frac12 r^2}(q^{\frac12 r}-q^{-\frac12 r})$	q
q^2	q	0	$1/q^n(q^2)_n(q)_n$	$q^{-r}(1-q^{2r+1})$	q^2
			$1/(q^2)_n(q)_n$	0	q^2
q^2	q	∞	$q^n/(q^2)_n(q)_n$	$-q^{r^2-1}(1-q^{2r+1})$	q^2
q^2	q	$q^{\frac32}$	$(q^{\frac12})_n/(q^2)_n(q)_n(q^{\frac32})_n$	$(-1)^r(q^{\frac12 r^2}+q^{\frac12(r+1)^2})$	q^2
			$q^n(q^{\frac12})_n/(q^2)_n(q)_n(q^{\frac32})_n$	$(-1)^r(q^{\frac12 r^2+r}+q^{\frac12(r^2-1)})$	q^2

Work of Slater

And an excerpt from her second paper:

$$\Pi(1-q^{5n-2})(1-q^{5n-3})(1-q^{5n}) = \Pi(1-q^n)\,\Sigma\,\frac{q^{n^2}}{(q;\,n)} \qquad \mathrm{B}(1)\ y,\,z\to\infty. \qquad (18)$$

$$= \Pi(1-q^{2n})\,\Sigma\,\frac{(-1)^n q^{3n^2}}{(q^4;\,q^4,\,n)(-q;\,q^2,\,n)}$$

$$\mathrm{G}(4)\ y,\,z\to\infty. \qquad (19)$$

$$= \Pi\,\frac{(1-q^{2n})}{(1+q^{2n-1})}\,\Sigma\,\frac{q^{n^2}}{(q^4;\,q^4,\,n)}$$

$$\mathrm{G}(1)\quad y=-q^{\frac{1}{2}},\ z\to\infty. \qquad (20)$$

$$\Pi(1+q^{5n-2})(1+q^{5n-3})(1-q^{5n}) = \Pi\,\frac{(1-q^{2n})}{(1-q^{2n-1})}\,\Sigma\,\frac{(-1)^n (q;\,q^2,\,n)\,q^{n^2}}{(q^4;\,q^4,\,n)(-q;\,q^2,\,n)}$$

$$\mathrm{G}(1)\ y=q^{\frac{1}{2}},\ z\to\infty. \qquad (21)$$

$$\Pi(1-q^{6n-1})(1-q^{6n-5})(1-q^{6n}) = \Pi\,\frac{(1-q^n)}{(1+q^n)}\,\Sigma\,\frac{q^{n(n+1)}(-q;\,n)}{(q;\,q^2,\,n+1)(q;\,n)}$$

$$\mathrm{C}(7)\ y=-q,\ z\to\infty. \qquad (22)$$

Work of Andrews

Some 30 years after the papers of Bailey and Slater, Andrews revisited their work. Three of his important contributions were:

He defined a Bailey pair relative to a to be a pair of sequences (α_n, β_n) satisfying Bailey's key relation, namely

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}.$$

He inverted this relation, showing that it is equivalent to

$$\alpha_n = \frac{1-aq^{2n}}{1-a} \sum_{j=0}^{n} \frac{(a)_{n+j}(-1)^{n-j} q^{\binom{n-j}{2}}}{(q)_{n-j}} \beta_j.$$

Work of Andrews

Most importantly, he proved what he called the Bailey lemma:

If (α_n, β_n) is a Bailey pair relative to a, then so is (α'_n, β'_n), where

$$\alpha'_n = \frac{(b)_n(c)_n(aq/bc)^n}{(aq/b)_n(aq/c)_n}\alpha_n$$

and

$$\beta'_n = \sum_{k=0}^{n} \frac{(b)_k(c)_k(aq/bc)_{n-k}(aq/bc)^k}{(aq/b)_n(aq/c)_n(q)_{n-k}}\beta_k.$$

Work of Andrews

This may be iterated, giving rise to what is called the Bailey chain.

Now, one Bailey pair gives infnitely many.

So one identity gives infinitely many.

And such families of identities are far from artificial – they often play the same natural role as the base identity.

Moreover, the number-theoretic properties of the α side are often preserved.

Work of Andrews

We illustrate this in the simplest possible case, the so-called unit Bailey pair.

Take $\beta_n = \delta_{n,0}$ and $a = 1$ in Andrews' Bailey pair inversion.

Then

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ (1+q^n)(-1)^n q^{\binom{n}{2}}, & \text{if } n > 0. \end{cases}$$

Applying the Bailey lemma once with $b, c \to \infty$ gives

$$\alpha'_n = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{n(3n-1)/2}(1+q^n), & \text{if } n > 0, \end{cases}$$

and

$$\beta'_n = \frac{1}{(q)_n}.$$

Work of Andrews

Applying it again with $b, c \to \infty$ gives

$$\alpha_n'' = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{n(5n-1)/2}(1+q^n), & \text{if } n > 0, \end{cases}$$

and

$$\beta_n'' = \sum_{n \geq n_1 \geq 0} \frac{q^{n_1^2}}{(q)_{n-n_1}(q)_{n_1}}.$$

One more time and we obtain

$$\alpha_n''' = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{n(7n-1)/2}(1+q^n), & \text{if } n > 0, \end{cases}$$

and

$$\beta_n''' = \sum_{n \geq n_2 \geq n_1 \geq 0} \frac{q^{n_1^2+n_2^2}}{(q)_{n-n_2}(q)_{n_2-n_1}(q)_{n_1}}.$$

Work of Andrews

Iterating k times along the Bailey chain in the same way we have

$$\alpha_n^{(k)} = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{n((2k+1)n-1)/2}(1+q^n), & \text{if } n > 0, \end{cases}$$

and

$$\beta_n^{(k)} = \sum_{n \geq n_{k-1} \geq \cdots n_1 \geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2}}{(q)_{n-n_{k-1}} \cdots (q)_{n_2-n_1}(q)_{n_1}}.$$

Work of Andrews

Inserting this Bailey pair into the original definition and letting $n \to \infty$, we obtain

$$\sum_{n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2}}{(q)_{n_{k-1}-n_{k-2}} \cdots (q)_{n_2 - n_1} (q)_{n_1}} = \frac{1}{(q)_\infty} \sum_{n \in \mathbb{Z}} (-1)^n q^{n((2k+1)n-1)/2} = \frac{(q^k; q^{2k+1})_\infty (q^{k+1}; q^{2k+1})_\infty (q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty}.$$

These are a subfamily of the Andrews-Gordon identities.

Once the Bailey framework is set up, these come almost for free.

The Bailey machinery

If we iterate the same unit Bailey pair relative to a and leave b, c unspecialized at each step, we obtain "Andrews' generalization of the Watson-Whipple transformation," one of the most useful identities in q-series.

Since Andrews' work on the Bailey lemma and the Bailey chain, the theory of Bailey pairs has undergone considerable development.

There are Bailey lattices, Bailey trees, dual Bailey pairs, conjugate Bailey pairs, changes-of-base in Bailey pairs, well-poised Bailey pairs, . . .

These are statements of the form "If (α_n, β_n) is a Bailey pair relative to a, then . . ."

The Bailey machinery

For example, the Bailey lattice combines the Bailey chain with the fact that if (α_n, β_n) is a Bailey pair relative to a, then (α'_n, β'_n) is a Bailey pair relative to a/q, where $\alpha'_0 = \alpha_0$ and for $n \geq 1$,

$$\alpha'_n = (1-a)\left(\frac{a}{\rho\sigma}\right)^n \frac{(\rho)_n(\sigma)_n}{(a/\rho)_n(a/\sigma)_n} \times \left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right)$$

and

$$\beta'_n = \sum_{k=0}^{n} \frac{(\rho)_k(\sigma)_k(a/\rho\sigma)_{n-k}(a/\rho\sigma)^k}{(a/\rho)_n(a/\sigma)_n(q)_{n-k}} \beta_k$$

The Bailey machinery

The Bailey lattice is used to obtain the full family of Andrews-Gordon identities,

$$\sum_{n_{k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+n_1+\cdots+n_{k-i}}}{(q)_{n_{k-1}-n_{k-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}} = \prod_{n\not\equiv 0,\pm i \pmod{2k+1}} \frac{1}{1-q^n}.$$

Here $k \geq 2$ and $1 \leq i \leq k$.

The Bailey machinery

Another Bailey lattice combines the Bailey chain with the fact that if (α_n, β_n) is a Bailey pair relative to a, then (α'_n, β'_n) is a Bailey pair relative to aq, where

$$\alpha'_n = \frac{(1-aq^{2n+1})(aq/b;q)_n(-b)^n q^{n(n-1)/2}}{(1-aq)(bq;q)_n} \times \sum_{r=0}^{n} \frac{(b;q)_r}{(aq/b;q)_r}(-b)^{-r}q^{-r(r-1)/2}\alpha_r$$

and

$$\beta'_n = \frac{(b;q)_n}{(bq;q)_n}\beta_n.$$

The Bailey machinery

This can be used to obtain Bailey pairs with indefinite quadratic forms, leading to identities like

$$\sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(-q;q)_n} = \sum_{\substack{n\geq 0\\ |j|\leq n}} (-1)^{n+j} q^{n(3n+1)/2-j^2}(1-q^{2n+1})$$

and

$$\sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n} = \frac{1}{(q;q)_\infty} \sum_{\substack{n\geq 0\\ |j|\leq n}} (-1)^{j} q^{n(5n+1)/2-j^2}(1-q^{4n+2}).$$

The Bailey machinery

Another example:

If (α_n, β_n) is a Bailey pair relative to 1 with $\alpha_0 = \beta_0 = 0$, then (α'_n, β'_n) is a Bailey pair relative to q, where

$$\alpha'_n = \frac{1}{1-q}\left(-\frac{\alpha_{n+1}}{1-q^{2n+2}} + \frac{q^{2n}\alpha_n}{1-q^{2n}}\right)$$

and

$$\beta'_n = -\beta_{n+1}.$$

The Bailey machinery

Some other examples:

If (α_n, β_n) is a Bailey pair relative to a, then

$$\sum_{n\geq 0} q^n \beta_n = \frac{1}{(aq)_\infty (q)_\infty} \sum_{r,n\geq 0} (-a)^n q^{\binom{n+1}{2}+(2n+1)r} \alpha_r.$$

and

$$\sum_{n\geq 0} (aq)_{2n} q^n \beta_n = \frac{1}{(q)_\infty} \sum_{r,n\geq 0} (-a)^n q^{3n(n+1)/2+(2n+1)r} \alpha_r.$$

The Bailey machinery

In practice, q-series identities are sometimes seen as annoying obstacles and/or they are given (seemingly) ad hoc proofs.

As much as possible, the goal should be to understand identities systematically using the framework of Bailey pairs.

Often there is hidden structure, and this opens up new worlds.

Let's briefly look at a perfect illustration of this – work of Warnaar on partial theta functions.

Work of Warnaar

In his lost notebook, Ramanujan recorded a number of partial theta identities, like

$$\sum_{n\geq 0}\frac{q^n}{(aq)_n(q/a)_n} = (1-a)\sum_{n\geq 0}(-1)^n a^{3n}q^{n(3n+1)/2}(1-a^2q^{2n+1}) + \frac{1}{(aq)_\infty(q/a)_\infty}\sum_{n\geq 0}(-1)^n a^{2n+1}q^{\binom{n+1}{2}}$$

and

$$\sum_{n\geq 0}\frac{(q^{n+1})_n q^n}{(aq)_n(q/a)_n} = (1-a)\sum_{n\geq 0}a^n q^{n^2+n} + \frac{1}{(aq)_\infty(q/a)_\infty}\sum_{n\geq 0}a^{3n+1}q^{n(3n+2)}(1-aq^{2n+1}).$$

Work of Warnaar

Ramanujan gave no proofs.

“Unsolved problems tremble with fear as he approaches” - H.S. Wilf

Who is “he”?

Ramanujan’s partial theta identities were first proved by Andrews (1981).

Work of Warnaar

Quoting Warnaar, “Andrews’ proofs are at times quite intricate and rely heavily on standard and some not-so-standard identities for basic hypergeometric series. This perhaps partially explains why Ramanujan’s partial theta function identities, though beautiful and deep, have remained rather isolated and have not become as widely appreciated and studied as, for example, Ramanujan’s mock theta functions.”

Work of Warnaar

Warnaar went on to prove that if (α_n, β_n) is a Bailey pair relative to q, then

$$\sum_{n\geq 0} \frac{(q)_{2n}}{(a)_{n+1}(q/a)_n}\beta_n - (1-q)\sum_{n\geq 0}\frac{(-a)^n q^{-\binom{n}{2}}}{1-q^{2n+1}}\alpha_n$$
$$= \frac{-1}{(q^2)_\infty (a)_\infty (q/a)_\infty}\sum_{r\geq 1}(-a)^r q^{\binom{r}{2}} \sum_{n\geq 0} q^{(1-r)n}\frac{1-q^{r(2n+1)}}{1-q^{2n+1}}\alpha_n.$$

Work of Warnaar

Now, all of Ramanujan's partial theta identities follow from Bailey pairs on Slater's list (or similar).

For example, the second identity cited above follows from Slater's Bailey pair corresponding to the second Rogers-Ramanujan identity.

Other simple Bailey pairs lead to new partial theta identities.

Iterating along the Bailey chain gives infinite families of such identities.

In addition to being "beautiful and deep," these partial theta identities have connections to Rogers-Ramanujan type identities and even to WRT invariants of certain 3-manifolds.

References I

W.P. Johnson, *An introduction to q-analysis*, American Mathematical Society, Providence, RI, 2020.

G. Gasper and M. Rahman, *Basic hypergeometric series*, Second edition, Encyclopedia of Mathematics and its Applications, 96, Cambridge University Press, Cambridge, 2004.

J. McLaughlin, *Topics and methods in q-series*, Monographs in Number Theory, 8, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.

References II

W. N. Bailey, Identities of the Rogers-Ramanujan type, *Proc. London Math. Soc. (2)* **50** (1948), 1–10.

L.J. Slater, A new proof of Rogers's transformations of infinite series, *Proc. London Math. Soc. (2)* **53** (1951), 460–475.

L.J. Slater, Further identities of the Rogers-Ramanujan type, *Proc. London Math. Soc. (2)* 54 (1952), 147–167.

G.E. Andrews, Multiple series Rogers-Ramanujan type identities, *Pacific J. Math.* **114** (1984), no. 2, 267–283.

S.O. Warnaar, Partial theta functions. I. Beyond the lost notebook, *Proc. London Math. Soc. (3)* **87** (2003), no. 2, 363–395.