Bailey pairs, mock theta functions, and indefinite quadratic forms

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Outline

Mock theta functions

In his last letter to Hardy, dated January 12, 1920, Ramanujan wrote:

I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (partially studied by Prof. Rogers), they enter into mathematics as beautifully as the ordinary ϑ-functions. I am sending you with this letter some examples.

Ramanujan included 17 examples to which he assigned the "order" 3, 5, or 7.

Mock theta functions

For example, there is the “third order”

$$f(q) = \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

and the ”fifth order”

$$f_0(q) = \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n}.$$

Mock theta functions

Ramanujan claimed that these functions had asymptotic properties resembling those for modular forms.

More precisely, he said that $g(q)$ is a mock theta function if:

(1) there are infinitely many roots of unity which are exponential singularities;

(2) for every root of unity ζ, there is a theta function ϑ_ζ, such that the difference $G(q) - \vartheta_\zeta(q)$ is bounded as $q \to \zeta$ radially;

(3) g is not the sum of two functions, one of which is a theta function and the other a function that is bounded radially toward all roots of unity.

Mock theta functions

Ramanujan also claimed that the mock theta functions satisfied identities involving modular forms.

More mock theta functions and more identities were found in the lost notebook - sixth and tenth “order”.

Ramanujan’s claimed identities were settled over time, notably by Watson, Andrews, Hickerson, and Choi.

Despite the hints at connections with modular forms, the precise modularity properties of the mock theta functions remained elusive until work of Zwegers (2002) on mock modular forms.

What is a mock modular form?

A function $f : \mathbb{H} \to \mathbb{C}$ is a modular form on Γ of weight k if

$$f\left(\frac{az+b}{cz+d}\right) = \bullet(cz+d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$.

A function $f : \mathbb{H} \to \mathbb{C}$ is a mock modular form on Γ of weight k if there is a modular form g of weight $2-k$ such that

$$f(\tau) + \int_{-\bar{\tau}}^{i\infty} \frac{g(z)}{(-i(z+\tau))^k} dz$$

transforms like a weight k modular form on Γ.

Remarks

The weight k can be an integer or a half-integer.

The function g is called the shadow.

The completed function is called a harmonic Maass form.

If the shadow is not identically 0, then the completed function is not holomorphic.

If the shadow is identically 0, we have a modular form.

Remarks

It turns out that all of Ramanujan's mock theta functions are mock modular forms of weight $1/2$ whose shadows are weight $3/2$ unary theta functions.

For example, for the third order mock theta function

$$f := \sum_{n \geq 0} \frac{q^{n^2}}{(-q;q)_n^2},$$

we have

$$g = \sum_{n \geq 1} \left(\frac{-12}{n} \right) n q^{n^2/24}.$$

Remarks

This mock modular structure explains what had been observed and/or proven about mock theta functions since the time of Ramanujan.

For example, the identity

$$2\phi(-q) + f(q) = (q;q^2)_\infty (q;q)_\infty,$$

where $\phi(q)$ is another third order mock theta function, exists because the shadow of $f(q)$ is proportional to the shadow of $\phi(-q)$.

And we can now do much more, such as prove asymptotic formulas, establish congruences, find new identities,...

Identities

How do we know that the mock theta functions satisfy the definition of mock modular form?

q-hypergeometric representations are not very helpful. What can one do with

$$\sum_{n \geq 0} \frac{q^{n^2}}{(-q;q)_n^2}?$$

We need identities.

Classical mock theta functions have been written in terms of Appell-Lerch (generalized Lambert) series, indefinite theta (Hecke-type) series, and as constant terms of (meromorphic) Jacobi forms.

Identities

For example, the function $f(q)$ satisfies

$$f(q) = \frac{2}{(q;q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}.$$

This is an Appell-Lerch-type series.

The fifth order mock theta function $f_0(q)$ satisfies

$$f_0(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q;q)_n} = \frac{1}{(q;q)_\infty} \sum_{\substack{n \geq 0 \\ |j| \leq n}} (-1)^j q^{n(5n+1)/2 - j^2} (1 - q^{4n+2}).$$

This is an indefinite theta series.

Identities

Zwegers studied transformation properties of Appell-Lerch series and indefinite theta series (as well as constant terms of meromorphic Jacobi forms) and showed how to complete them to non-holomorphic modular forms (by adding the shadow integral).

The idea of such completions goes back to the Zagier-Eisenstein series, i.e. the generating function for the Hurwitz-Kronecker class numbers $H(n)$. In 1975 Zagier showed that this series is a weight $3/2$ mock modular form with shadow proportional to

$$\theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2}.$$

Bailey pairs

So, if we can express a q-series in terms of Appell-Lerch series or indefinite theta functions, we can now understand its modular behavior.

Our principal tool for proving q-series identities is the theory of Bailey pairs.

10 years after Zwegers' work, there were still only around 45 q-series that were known to be mock theta functions.

Let us see what happens if we try to use the Bailey chain to produce families of mock theta functions.

The Bailey chain

Take the Bailey pair relative to 1,

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ \frac{4(-1)^n q^{\binom{n+1}{2}}}{(1+q^n)}, & \text{otherwise,} \end{cases}$$

and

$$\beta_n = \frac{1}{(-q)_n^2}.$$

This can be deduced from work of Slater.

The Bailey chain

Iteration along the Bailey chain with $b, c \to \infty$ at each step gives

$$\sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_k^2+n_{k-1}^2+\cdots+n_1^2}}{(q)_{n_k-n_{k-1}} \cdots (q)_{n_2-n_1}(-q)_{n_1}^2} = \frac{2}{(q)_\infty} \sum_{n \in \mathbb{Z}} \frac{q^{kn^2+\binom{n+1}{2}}(-1)^n}{(1+q^n)}.$$

The case $k = 1$ is the Appell-Lerch type series for $f(q)$.

The case $k \geq 2$ is not a mock theta function.

The Bailey chain

For example, when $k = 2$ we have the identity

$$\sum_{n_2 \geq n_1 \geq 0} \frac{q^{n_2^2+n_1^2}}{(q)_{n_2-n_1}(-q)_{n_1}^2} = -\frac{2}{(q^2,q^3;q^5)_\infty}\chi(q) + \frac{2}{(q,q^4;q^5)_\infty}X(q) - \frac{(q)_\infty}{(-q)_\infty^2},$$

where χ and X are two tenth order mock theta functions,

$$\chi(q) = \sum_{n \geq 0} \frac{(-1)^n q^{(n+1)^2}}{(-q)_{2n+1}},$$

$$X(q) = \sum_{n \geq 0} \frac{(-1)^n q^{n^2}}{(-q)_{2n}}.$$

This is a mixed mock modular form.

Mixed mock modular forms

A mixed mock modular form is a finite sum $\sum f_i g_i$ where each f_i is a modular form and each g_i is a mock modular form.

Note that we have the inclusions

$$\begin{aligned}\{\text{modular forms}\} &\subsetneq \{\text{mock modular forms}\} \\ &\subsetneq \{\text{mixed mock modular forms}\}\end{aligned}$$

Some of the appearances of mock theta functions in combinatorics, physics, and algebra are of the mixed mock type.

But we lose much of the structure of "pure" mock theta functions.

What went wrong?

A level ℓ Appell sum

$$A_\ell(a,b,q) := a^{\ell/2} \sum_{n\in\mathbb{Z}} \frac{(-1)^{\ell n} q^{\ell n(n+1)/2} b^n}{1-aq^n}$$

or indefinite theta series

$$f_{a,b,c}(x,y,q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}$$

is in general a mixed mock modular form ($\sum f_i g_i$).

In very special cases $f_i = f$ for all i and one can divide by f.

Mock theta functions are very special.

What went wrong?

For example,

$$\frac{2}{(q;q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}$$

has just the right infinite product in front of the sum.

But,

$$\frac{2}{(q;q)_\infty} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n(5n+1)/2}}{1+q^n}$$

does not.

Moreover, there is no right product.

Something similar happens for $f_0(q)$ (and any other of Ramanujan's mock theta functions).

What went wrong

In the case of $f_0(q)$, Andrews established the indefinite theta series representation using the Bailey pair relative to 1,

$$\alpha_n = q^{n(3n+1)/2} \sum_{|j| \leq n} (-1)^j q^{-j^2} - \chi(n \neq 0) q^{n(3n-1)/2} \sum_{|j| \leq n-1} (-1)^j q^{-j^2},$$

$$\beta_n = \frac{1}{(-q)_n}.$$

Iterating along the Bailey chain in the standard way leads to the mixed mock modular series

$$\sum_{\substack{n \geq 0 \\ |j| \leq n}} (-1)^j q^{n((2k+1)n+1)/2} (1 - q^{2kn+k}).$$

The technical problem is that the quadratic term is "too far" from the linear term.

Questions

Is there any way at all to use the Bailey machinery to construct families of q-hypergeometric multisums which are mock theta functions?

If so, would they look good?

Would the Bailey pairs play a natural role in other areas?

Answers

The answer to the first question is yes.

The second question is rather subjective.

The answer to the third question is yes.

The construction

Bailey pairs with indefinite quadratic forms (in the exponent of q on the α-side) can be used to express all of Ramanujan's mock theta functions in terms of indefinite theta series.

(They also have a number of other important applications).

Often such pairs are quoted from work of Andrews or Andrews-Hickerson, or produced *ad hoc* as needed.

What about constructing classes of Bailey pairs with general indefinite quadratic forms?

The construction

We have done this for the indefinite quadratic forms

$$(K+1)n^2+(m+1)n-((2k+1)j^2+(2\ell+1)j)/2,$$
$$(K+1)n^2+(m+1)n-kj^2-\ell j,$$
$$((2K+1)n^2+(2m+1)n)/2-((2k+1)j^2+(2\ell+1)j)/2,$$

and

$$((2K+1)n^2+(2m+1)n)/2-kj^2-\ell j,$$

where k, K, m, and ℓ are intgers with $k, K \geq 1$, $0 \leq m < K$ and $0 \leq \ell < k$.

And the β-side is a nice multisum.

Results (L., 2014)

Suppose that $k, K \geq 1$, $0 \leq \ell < k$ and $0 \leq m < K$.
(1) The sequences $(\alpha_n^{(k,K,\ell)}, \beta_n^{(k,K,\ell)})$ form a Bailey pair relative to q, where

$$\alpha_n^{(k,K,\ell)} = \frac{q^{(K+1)n^2+Kn}(1-q^{2n+1})}{(1-q)} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$

and

$$\beta_n^{(k,K,\ell)} =$$

$$\sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}(n_{k+i}+1) + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i} (-1)^{n_k}}{(q)_{n-n_{k+K-1}} (q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}}.$$

Results

(2) The sequences $(\alpha_n^{(k,K,\ell,m)}, \beta_n^{(k,K,\ell,m)})$ form a Bailey pair relative to 1, where

$$\alpha_n^{(k,K,\ell,m)} = q^{(K+1)n^2+(m+1)n} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$
$$- \chi(n \neq 0) q^{(K+1)n^2-(m+1)n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$

and

$$\beta_n^{(k,K,\ell,m)} =$$
$$\sum_{n \geq n_{k+K-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{\sum_{i=1}^{K-1} n_{k+i}^2 + \sum_{i=1}^{m} n_{k+i} + \binom{n_k+1}{2} - \sum_{i=1}^{k-1} n_i n_{i+1} - \sum_{i=1}^{\ell} n_i} (-1)^{n_k}}{(q)_{n-n_{k+K-1}} (q)_{n_{k+K-1}-n_{k+K-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}}.$$

Results

There is such a result for each of the four indefinite quadratic forms mentioned above.

The proof uses the Bailey lemma, the Bailey lattice, another Bailey lattice, the Bailey lattice replacement, and dual Bailey pairs.

These pairs can be used to give multisum mock theta functions.

For example, suppose that $k \geq 1$, $0 \leq \ell < k$, and $0 \leq m \leq k$.

Results

Then

$$\sum_{n_{2k}\geq n_{2k-1}\geq\cdots\geq n_1\geq 0} \frac{q^{\sum_{i=1}^{k} n_{k+i}^2+\sum_{i=1}^{m} n_{k+i}+\binom{n_k+1}{2}-\sum_{i=1}^{k-1} n_i n_{i+1}-\sum_{i=1}^{\ell} n_i}(-1)^{n_k}}{(q)_{n_{2k}-n_{2k-1}}(q)_{n_{2k-1}-n_{2k-2}}\cdots(q)_{n_2-n_1}(q)_{n_1}}$$

$$= \frac{1}{(q)_\infty}\Bigg(\sum_{\substack{n\geq 0\\ |j|\leq n}} q^{(k+2)n^2+(m+1)n-((2k+1)j^2+(2\ell+1)j)/2}(-1)^j - \sum_{\substack{n\geq 1\\ |j|\leq n-1}} q^{(k+2)n^2-(m+1)n-((2k+1)j^2+(2\ell+1)j)/2}(-1)^j \Bigg)$$

$$= \frac{1}{(q)_\infty}\Big(f_{3,4k+5,3}(q^{m+2-\ell}, q^{m+3+\ell}, q) + q^{k+m+3} f_{3,4k+5,3}(q^{2k+m+6-\ell}, q^{2k+m+7+\ell}, q)\Big),$$

Outline of Proof

Begin with the “unit” Bailey pair relative to 1,

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{\binom{n}{2}}(1+q^n), & \text{if } n > 0, \end{cases}$$

and

$$\beta_n = \delta_{n,0}.$$

Outline of Proof

In words:

Alternately apply the Bailey lemma with $b, c \to \infty$ and the Bailey lattice replacement ℓ times, then apply the Bailey lemma with $b, c \to \infty$ $k - \ell$ times, then compute the dual Bailey pair, then apply the second Bailey lattice statement with $b = 0$, then apply the Bailey lemma with $b, c \to \infty$ $m + 1$ times, then (if $a = 1$ in the theorem) apply the first Bailey lattice statement with $\rho = \sigma = \sqrt{q}$, and finally apply the Bailey lemma with $b, c \to \infty$ $K - 1 - m$ times.

Outline of Proof

In math (I'll only show the α-side):

$$\begin{cases} 1, & \text{if } n = 0, \\ (-1)^n q^{\binom{n}{2}}(1+q^n), & \text{if } n > 0, \end{cases}$$

$$\to \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n \left(q^{((2\ell+1)n^2+(2\ell+1)n)/2} + q^{((2\ell+1)n^2-(2\ell+1)n)/2} \right), & \text{if } n > 0, \end{cases}$$

$$\to \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n \left(q^{((2k+1)n^2+(2\ell+1)n)/2} + q^{((2k+1)n^2-(2\ell+1)n)/2} \right), & \text{if } n > 0, \end{cases}$$

Outline of Proof

$$\rightarrow \begin{cases} 1, & \text{if } n = 0, \\ (-1)^n \left(q^{(-(2k-1)n^2-(2\ell+1)n)/2} + q^{(-(2k-1)n^2+(2\ell+1)n)/2} \right), & \text{if } n > 0, \end{cases}$$

$$\rightarrow \frac{q^{n^2}(1-q^{2n+1})}{1-q} \sum_{|j|\leq n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$

$$\rightarrow \frac{q^{(m+2)n^2+(m+1)n}(1-q^{2n+1})}{1-q} \sum_{|j|\leq n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$

Outline of Proof

$$\to q^{(m+2)n^2+(m+1)n} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$
$$- \chi(n \neq 0) q^{(m+1)n^2-(m+1)n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$

$$\to q^{(K+1)n^2+(m+1)n} \sum_{j=-n}^{n} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$
$$- \chi(n \neq 0) q^{(K+1)n^2-(m+1)n} \sum_{j=-n+1}^{n-1} (-1)^j q^{-((2k+1)j^2+(2\ell+1)j)/2}$$

The colored Jones polynomial

The colored Jones polynomial $J_N(K) = J_N(K, q)$ generalizes the classical Jones polynomial (the case $N = 2$).

It is an important knot invariant.

$J_N(e^{2\pi i/N})$ is called the Kashaev invariant, which appears in the "Volume Conjecture." (And there are other conjectures related to the colored Jones polynomial.)

There are many formulas in the literature for various knots.

The cyclotomic expansion

Habiro (2008) defined the cyclotomic expansion of the colored Jones polynomial for a knot K to be

$$J_N(K;q) = \sum_{n \geq 0} C_n(K;q)\,(q^{1+N})_n\,(q^{1-N})_n$$

and proved that

$$C_n(K;q) \in \mathbb{Z}[q, q^{-1}].$$

The $C_n(K;q)$ are called the cyclotomic coefficients.

They are important and elusive.

The cyclotomic expansion

Recall that (α_n, β_n) is a Bailey pair relative to a if and only if

$$\alpha_n = \frac{1-aq^{2n}}{1-a}\frac{(a)_n}{(q)_n}(-1)^n q^{n(n-1)/2}\sum_{j=0}^{n}(q^{-n})_j(aq^n)_j q^j \beta_j.$$

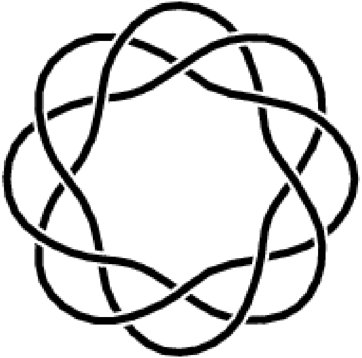
Compare this to Habiro's cyclotomic expansion,

$$J_N(K;q) = \sum_{n\geq 0} C_n(K;q)\,(q^{1+N})_n\,(q^{1-N})_n.$$

The colored Jones polynomial and its cyclotomic coefficients are essentially a Bailey pair relative to q^2!

Torus knots

The torus knots are knots which lie on a torus in $\mathbb{R}^3$.

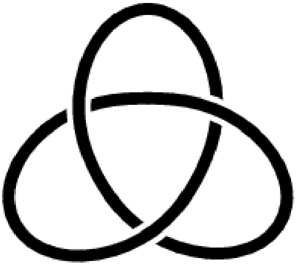


Torus knots

Torus knots are described by two positive coprime integers (s, t) and an "orientation" (left-handed or right-handed).

The previous example is the left-handed torus knot $(3, 8)$.

The case $(2, 3)$ is the trefoil knot:



Torus knots

Lê and Habiro showed that the cyclotomic expansion of the left-handed trefoil knot $T_{(2,3)}$ is

$$J_N(T_{(2,3)}; q) = \sum_{n=0}^{\infty} q^n (q^{1-N})_n (q^{1+N})_n.$$

This was the only torus knot for which this expansion was known.

In joint work with K. Hikami (2015), we used Bailey pairs to find the cyclotomic expansion for the torus knots $T_{(2,2t+1)}$.

Torsu knots

First, using the "Rosso-Jones formula" we deduced that

$$(1-q^N)\,J_N(T_{(2,2t+1)}) = (-1)^N q^{-t+\frac{1}{2}N+\frac{2t+1}{2}N^2} \times \sum_{k=-N}^{N-1} (-1)^k q^{-\frac{2t+1}{2}k(k+1)+k}.$$

This is the α side of a Bailey pair.

Note the indefinite quadratic forms!

Torus knots

Then we use results on Bailey pairs and indefinite quadratic forms to find the β side.

The (preliminary) result is

$$- q^{t-n} C_{n-1}(T_{(2,2t+1)}; q) = \sum \frac{q^{\sum_{i=1}^{t-1} n_{t+i}^2 + \binom{n_t}{2} - \sum_{i=1}^{t-1} n_i n_{i+1} - \sum_{i=1}^{t-2} n_i} (-1)^{n_t} (1 - q^{n_t - \chi(t \geq 2) n_{t-1}})}{(q)_{n-n_{2t-1}} (q)_{n_{2t-1}-n_{2t-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}},$$

where the sum is over $n \geq n_{2t-1} \geq \cdots \geq n_1 \geq 0$.

Why is this a (Laurent) polynomial?

Torus knots (cont.)

Using the q-binomial identity, we reduce the $2t$-fold sum to the t-fold sum

$$C_{n-1}(T_{(2,2t+1)}; q) =$$

$$q^{n+1-t} \sum_{n+1=k_t \geq k_{t-1} \geq \cdots \geq k_1 \geq 1} \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1} + k_i - i + 2\sum_{j=1}^{i-1} k_j \\ k_{i+1} - k_i \end{bmatrix}.$$

References (I)

More applications in:

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