Bailey pairs and strange identities

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April 2022

Recap

q-hypergeometric series are everywhere!

q-series identities are fundamental and beautiful.

The structure of Bailey pairs lies behind many important identities.

Once the iterative framework is set up, results like the Andrews-Gordon identities come easily.

One identity gives infinitely many identities.

Bailey pairs can be used to express Ramanujan's mock theta functions in terms of Appell-type series or indefinite theta functions, whose modular transformation properties were determined by Zwegers.

Recap

Ramanujan's mock theta functions can then be completed to non-holomorphic modular forms via an Eichler integral.

Standard applications of the Bailey chain do not produce families of q-hypergeometric mock theta functions, but a more nuanced approach does.

And the corresponding Bailey pairs have nice applications, including to colored Jones polynomials of certain torus knots and WRT invariants of 3-manifolds.

In fact, the colored Jones polynomials and their cyclotomic coefficients essentially form a Bailey pair!

Remark

If you thought that the two "mini-courses" at SLC 87 were completely unrelated, see:

S.O. Warnaar, The Bailey lemma and Kostka polynomials, *J. Algebraic Combin.* **20** (2004), no. 2, 131–171.

(This paper concers a "higher level" Bailey lemma...)

Outline

1. Zagier's strange identity and applications
2. General strange identities
3. New families of strange identities
4. Proofs using Bailey pairs

Zagier's "identity"

Zagier (2001) recorded the "strange identity"

$$\sum_{n\geq 0}(q)_n \text{ “} = \text{” } -\frac{1}{2}\sum_{n\geq 1} n\left(\frac{12}{n}\right)q^{(n^2-1)/24}.$$

Here

$$\left(\frac{12}{n}\right)=\begin{cases}1, & \text{if } n\equiv \pm 1 \pmod{12},\\ -1, & \text{if } n\equiv \pm 5 \pmod{12},\\ 0, & \text{otherwise.}\end{cases}$$

Note that

$$(q)_\infty = \sum_{n\geq 0}\left(\frac{12}{n}\right)q^{(n^2-1)/24}.$$

Zagier's "identity"

The right-hand side of Zagier's "identity" is

$$-\frac{1}{2}\left(1 - 5q - 7q^2 + 11q^5 + 13q^7 - 17q^{12} - \cdots\right).$$

This is an analytic function for $|q| < 1$.

The left-hand side is not an analytic function on any open subset of $\mathbb{C}$.

So, what does Zagier mean by " $=$ " ?

Zagier's "identity"

Note that the left-hand side is a finite sum (and hence well-defined) when q is a root of unity.

In particular, if $q^N = 1$ then

$$\sum_{n=0}^{\infty} (q)_n = \sum_{n=0}^{N-1} (q)_n.$$

Similarly, for any root of unity ζ, the left-hand side is a well-defined power series in t when

$$q = \zeta e^{-t} = \zeta \left(\sum_{k \geq 0} \frac{(-1)^k t^k}{k!} \right).$$

Zagier's "identity"

If we replace q by ζe^{-t} on the right-hand side, classical results in asymptotic analysis give that as $t \to 0^+$,

$$-\frac{1}{2}\sum_{n\geq 1} n\left(\frac{12}{n}\right)\zeta^{(n^2-1)/24}e^{-t(n^2-1)/24} \sim \text{a power series in } t.$$

The " $=$ " in Zagier's "identity" means that at any root of unity these two power series are equal.

Zagier's "identity"

For example, at $q = 1$ we have

$$\sum_{n\geq 0}(e^{-t};e^{-t})_n = 3 + 11t + \frac{133}{2}t^2 + \frac{3389}{6}t^3 + \frac{148177}{24}t^4 + \cdots$$

$$\sim -\frac{1}{2}\sum_{n\geq 1} n \left(\frac{12}{n}\right) e^{-t(n^2-1)/24}.$$

At $q = -1$ we have

$$\sum_{n\geq 0}(-e^{-t};-e^{-t})_n = 1 + t + \frac{3}{2}t^2 + \frac{19}{6}t^3 + \frac{207}{24}t^4 + \cdots$$

$$\sim -\frac{1}{2}\sum_{n\geq 1} n \left(\frac{12}{n}\right) (-1)^{(n^2-1)/24} e^{-t(n^2-1)/24}.$$

Applications

It turns out that Zagier's strange identity has many interesting applications.

Let

$$F(q) = \sum_{n \geq 0} (q)_n.$$

This is called the "Kontsevich-Zagier function".

Applications - values of $F(q)$

Habiro (2000) and Lê (2003) showed that the colored Jones polynomial of the trefoil knot can be written

$$J_N(q) = q^{1-N} \sum_{n \geq 0} q^{-nN} (q^{1-N})_n.$$

Therefore

$$F(e^{2\pi i/N}) = J_N(e^{2\pi i/N}) \text{ (The Kashaev invariant).}$$

Using the strange identity, one can show that if L is such that $\zeta^{\frac{L}{12}} = 1$, then

$$F(\zeta) = \frac{1}{4L} \sum_{m=1}^{L} m^2 \left(\frac{12}{m} \right) \zeta^{(m^2-1)/24}.$$

Applications - quantum modularity

A quantum modular form of weight k is a function $f : \mathbb{Q} \to \mathbb{C}$ such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in (a subgroup of) $\mathrm{SL}_2(\mathbb{Z})$, one has that

$$g_\gamma(x) = f\left(\frac{ax+b}{cx+d}\right) - \bullet(cx+d)^k f(x)$$

is a a continuous (or smooth) function on $\mathbb{R}\backslash\{\frac{-d}{c}\}$.

Using the strange identity one can show that the function $\phi : \mathbb{Q} \to \mathbb{C}$ defined by

$$\phi(x) = e^{\pi i x/12} F(e^{2\pi i x})$$

is a weight $3/2$ quantum modular form.

Applications - congruences for Fishburn numbers

The Kontsevich-Zagier series at $1 - q$,

$$F(1-q) = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \cdots ,$$

is a well-known combinatorial generating function.

It counts Fishburn matrices, $(2+2)$-free posets, ascent sequences, linearized chord diagrams,... (Bousquet-Mélou, Claesson, Dukes, Kitaev, 2010)

For example, a Fishburn matrix is an upper-triangular matrix with non-negative integer entries such that no row or column consists only of zeros.

Applications - congruences for Fishburn numbers

The five Fishburn matrices for $n = 3$ are

$$(3),\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix},\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix},\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Using the strange identity, one can show that the Fishburn numbers satisfy many Ramanujan-type congruences, like

$$\begin{aligned}a(5n+r) &\equiv 0 \pmod 5, r \in \{3,4\}\\a(7n+6) &\equiv 0 \pmod 7,\\a(11n+s) &\equiv 0 \pmod{11}, s \in \{8,9,10\},\\&\vdots\end{aligned}$$

Strange identity - definition

Generalizing Zagier's example, a strange identity

$$\sum_{n\geq 0}(q)_n f_n(q) \text{ “}=\text{” } \sum_{n\geq 0} n^v \chi(n) q^{(n^2-a)/b}$$

means that for $q=\zeta e^{-t}$, the right-hand side has an asymptotic expansion as a power series as $t\to 0^+$ and this power series is given by the left-hand side at ζe^{-t}.

Here the f_n are polynomials, $v\in\{0,1\}$, $a\geq 0$ and $b>0$ are integers, and $\chi:\mathbb{Z}\to\mathbb{C}$ is a periodic function such that:

(i) $\chi(n)\neq 0$ if and only if $\frac{n^2-a}{b}\in\mathbb{Z}$,

(ii) The function $n\to\zeta^{(n^2-a)/b}\chi(n)$ is a periodic function with mean value zero for any root of unity ζ.

Strange identity - definition

This definition can be relaxed in several ways, notably by replacing $(q)_n$ on the left-hand side by other q-factorials, such as $(q;q^2)_n$ or $\frac{(q)_n}{(-q)_n}$, and appropriately restricting to a subset of the roots of unity.

In these two cases, for example, we only consider the odd roots of unity.

One can also let v be any positive integer.

General results

Following Zagier's work, many applications of Zagier's strange identity were extended to general strange identities:

1) Bringmann-Rolen and Goswami-Osburn showed that q-hypergeometric series satisfying a strange identity are quantum modular forms.

2) Guerzhoy-Kent-Rolen and Ahlgren-Kim-L. showed that if $G(q)$ satisfies a strange identity, then the coefficients of $G(1-q)$ will have many Ramanujan-type congruences.

3) Goswami-Jha-Kim-Osburn found an asymptotic formula for the coefficients of $G(1-q)$ when $G(q)$ satisfies a strange identity.

Hikami's strange identity

Many nice results and very few examples.

Aside from a few isolated examples, until recently the only other strange identities are due to Hikami (2006).

We need the q-binomial coefficient (or "Gaussian polynomial") defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_{n-k}(q)_k}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Hikami's strange identity

For $0 \leq a \leq k-1$ we have

$$\sum_{n_1,\dots,n_k \geq 0} (q)_{n_k} q^{n_1^2+\cdots+n_{k-1}^2+n_{a+1}+\cdots+n_{k-1}} \prod_{i=1}^{k-1} \begin{bmatrix} n_{i+1}+\delta_{i,a} \\ n_i \end{bmatrix}$$

$$\text{“}=\text{”} -\frac{1}{2}\sum_{n\geq 0} n\chi_{8k+4}^{(a)}(n) q^{\frac{n^2-(2k-2a-1)^2}{8(2k+1)}},$$

where $\chi_{8k+4}^{(a)}(n)$ is the even periodic function modulo $8k+4$ defined by

$$\chi_{8k+4}^{(a)}(n) = \begin{cases} 1, & \text{if } n \equiv 2k-2a-1 \text{ or } 6k+2a+5 \pmod{8k+4}, \\ -1, & \text{if } n \equiv 2k+2a+3 \text{ or } 6k-2a+1 \pmod{8k+4}, \\ 0, & \text{otherwise.} \end{cases}$$

Hikami's strange identity

The case $k = 1$ is Zagier's strange identity.

For $k = 2$ we have

$$\sum_{n_1,n_2 \geq 0} (q)_{n_2} q^{n_1^2+n_1} \begin{bmatrix} n_2 \\ n_1 \end{bmatrix}$$
$$\text{``}=\text{''} -\frac{1}{2} \sum_{n \geq 0} n \chi_{20}^{(0)}(n) q^{\frac{n^2-9}{40}}$$

and

$$\sum_{n_1,n_2 \geq 0} (q)_{n_2} q^{n_1^2} \begin{bmatrix} n_2+1 \\ n_1 \end{bmatrix}$$
$$\text{``}=\text{''} -\frac{1}{2} \sum_{n \geq 0} n \chi_{20}^{(1)}(n) q^{\frac{n^2-1}{40}}.$$

Hikami's strange identity - Remarks

One can deduce formulas, quantum modularity, asymptotics, congruences, etc.

The case $(k, 0)$ corresponds to the Kashaev invariant for the torus knot $(2, 2k + 1)$.

Very recently another family of strange identities was found, corresponding to the torus knots $(3, 2^t)$.

Can any of this be understood in the context of Bailey pairs?

Yes!

Results (L., 2022)

Here is a sample of what one can prove using the Bailey machinery.

Let $\chi_{4k}(n)$ be the even periodic function modulo $4k$ defined by

$$\chi_{4k}(n) = \begin{cases} 1, & \text{if } n \equiv k-1 \text{ or } 3k+1 \pmod{4k}, \\ -1, & \text{if } n \equiv k+1 \text{ or } 3k-1 \pmod{4k}, \\ 0, & \text{otherwise.} \end{cases}$$

Results (L., 2022)

Then

$$1)\ \sum_{n_1,\dots,n_k\geq 0} (q)_{n_k} \frac{q^{n_1^2+\cdots+n_{k-1}^2+n_1+\cdots+n_{k-1}}}{(-q)_{n_1}} \prod_{i=1}^{k-1} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}$$

$$\text{``}=\text{''} -(1+\delta_{k,1}) \sum_{n\geq 0} n\chi_{4k}(n) q^{\frac{n^2-(k-1)^2}{4k}}$$

and

$$2)\ \sum_{n_1,\dots,n_k\geq 0} (q^2;q^2)_{n_k} \frac{q^{2n_1^2+2n_1+\cdots+2n_{k-1}^2+2n_{k-1}}(q;q^2)_{n_1}}{(-q)_{2n_1+1}} \prod_{i=1}^{k-1} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}_{q^2}$$

$$\text{``}=\text{''} -(1+\delta_{k,1}) \frac{1}{2} \sum_{n\geq 0} n\chi_{8k-4}(n) q^{\frac{n^2-(2k-2)^2}{8k-4}}.$$

Results (L., 2022)

Let $\chi_{8k}^{(a)}(n)$ be the even periodic function modulo $8k$ defined by

$$\chi_{8k}^{(a)}(n) = \begin{cases} 1, & \text{if } n \equiv 2k-2a-1 \text{ or } 6k+2a+1 \pmod{8k}, \\ -1, & \text{if } n \equiv 2k+2a+1 \text{ or } 6k-2a-1 \pmod{8k}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$3) \sum_{n_1,\dots,n_k \geq 0} (q^2;q^2)_{n_k} \frac{q^{2n_1^2+\cdots+2n_{k-1}^2+2n_{a+1}+\cdots+2n_{k-1}}}{(-q;q^2)_{n_1+\delta_{a,0}}} \prod_{i=1}^{k-1} \begin{bmatrix} n_{i+1}+\delta_{i,a} \\ n_i \end{bmatrix}_{q^2}$$
$$\text{“}=\text{”} -\frac{1}{2}\sum_{n\geq 0} n\chi_{8k}^{(a)}(n) q^{\frac{n^2-(2k-2a-1)^2}{8k}}.$$

Results (L., 2022)

There are many more.

One has all of the usual applications.

In addition, one obtains interesting q-series identities at roots of unity by comparing strange identities.

Results (L., 2022)

Comparing the first two strange identities we have that

$$\sum_{n_1,\dots,n_{2k-1}\geq 0} (q)_{n_{2k-1}} \frac{q^{n_1^2+\cdots+n_{2k-2}^2+n_1+\cdots+n_{2k-2}}}{(-q)_{n_1}} \prod_{i=1}^{2k-2} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}$$

$$= 2 \sum_{n_1,\dots,n_k\geq 0} (q^2;q^2)_{n_k} \frac{q^{2n_1^2+2n_1+\cdots+2n_{k-1}^2+2n_{k-1}}(q;q^2)_{n_1}}{(-q)_{2n_1+1}} \prod_{i=1}^{k-1} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}_{q^2}$$

at any odd root of unity (in which case the sums become finite), but of course not as functions inside the unit disk (where neither series converges, anyway).

The case $k=1$ reads

$$\sum_{n\geq 0} \frac{(q)_n}{(-q)_n} = 2\sum_{n\geq 0} \frac{(q)_{2n}}{(-q)_{2n+1}}.$$

Proof of Zagier's identity

Using q-difference equations, Zagier showed that for $|x| < 1$,

$$(1-x)\sum_{n\geq 0}(xq)_n x^n = \sum_{n\geq 0}(-1)^n x^{3n} q^{n(3n+1)/2}(1-x^2q^{2n+1}).$$

We want to take $\frac{d}{dx}|_{x=1}$.

Zagier added and subtracted $(x)_\infty$ on the left-hand side to obtain

$$(1-x)\sum_{n\geq 0}\left((xq)_n - (xq)_\infty\right)x^n + (xq)_\infty$$
$$= \sum_{n\geq 0}(-1)^n x^{3n} q^{n(3n+1)/2}(1-x^2q^{2n+1}).$$

Proof of Zagier's identity

Replacing x by x^2 on both sides, multiplying by x, and taking $\frac{d}{dx}\big|_{x=1}$ gives the "sum of tails" identity,

$$2\sum_{n\geq 0}((q)_n - (q)_\infty) + (q)_\infty\left(-1 + 2\sum_{n\geq 1}\frac{q^n}{1-q^n}\right)$$
$$= -\sum_{n\geq 1} n\left(\frac{12}{n}\right) q^{(n^2-1)/24}.$$

Letting q approach a root of unity, we obtain Zagier's strange identity, since $(q)_\infty$ vanishes to infinite order.

Proof of Hikami's identity

Using q-difference equations and some impressive calculations, Hikami showed that

$$
\begin{aligned}
(1-x) \sum_{n_1,\ldots,n_k\geq 0} & (xq)_{n_k} q^{n_1^2+\cdots+n_{k-1}^2+n_{a+1}+\cdots+n_{k-1}} \\
& \times x^{2n_1+\cdots+2n_{k-1}+n_k} \prod_{i=1}^{k-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix} \\
&= \sum_{n\geq 0} (-1)^n x^{(2k+1)n} q^{\binom{n+1}{2}+(a+1)n^2+(k-a-1)(n^2+n)} \\
& \quad \times \left(1-x^{2(a+1)} q^{(a+1)(2n+1)}\right) \\
&= \sum_{n\geq 0} \chi_{8k+4}^{(a)}(n) q^{\frac{n^2-(2k-2a-1)^2}{8(2k+1)}} x^{\frac{n-(2k-2a-1)}{2}}.
\end{aligned}
$$

Proof of Hikami's identity

Arguing along the lines of Zagier (with considerably more complications), Hikami arrived at his strange identity.

One new element he required was the "Andrews-Gordon variant"

$$\sum_{n_1,n_2,\dots,n_{k-1}\geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+n_{a+1}+\cdots+n_{k-1}}}{(q)_{n_{k-1}}} \prod_{i=1}^{k-2} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}$$

$$= \frac{1}{(q)_\infty} \sum_{n\geq 0} \chi_{8m+4}^{(a)}(n) q^{\frac{n^2-(2k-2a-1)^2}{8(2k+1)}}$$

$$= \prod_{n\not\equiv 0,\pm(a+1) \pmod{2k+1}} \frac{1}{1-q^n}.$$

Bailey pairs

How does this all fit into the framework of Bailey pairs?

Recall from classical work of Bailey that if (α_n, β_n) is a Bailey pair relative to a, then

$$\sum_{n\geq 0}(b)_n(c)_n(aq/bc)^n\beta_n = \frac{(aq/b)_\infty(aq/c)_\infty}{(aq)_\infty(aq/bc)_\infty}\sum_{n\geq 0}\frac{(b)_n(c)_n(aq/bc)^n}{(aq/b)_n(aq/c)_n}\alpha_n.$$

Using $(a,b,c) = (x^2q, xq, q)$ we have the key identity

$$(1-x)\sum_{n\geq 0}(xq)_n(q)_n x^n\beta_n = (1-x^2q)\sum_{n\geq 0}\frac{(q)_n}{(x^2q)_n}x^n\alpha_n.$$

Bailey pairs

Still using work of Bailey and Slater, we have the Bailey pair relative to x^2q,

$$\alpha_n = \frac{(x^2q)_n(1 - x^2q^{2n+1})(-1)^n x^{2n} q^{n(3n+1)/2}}{(q)_n(1 - x^2q)}$$

and

$$\beta_n = \frac{1}{(q)_n}.$$

This gives

$$(1 - x)\sum_{n\geq 0}(xq)_n x^n = \sum_{n\geq 0}(-1)^n x^{3n} q^{n(3n+1)/2}(1 - x^2q^{2n+1}),$$

the starting point for Zagier's proof.

Bailey pairs

To get to the starting point for Hikami's strange identity requires much more work.

The result we need is that for $k \geq 1$ and $0 \leq a \leq k-1$, the following is a Bailey pair relative to x^2q:

$$\alpha_n = \frac{(x^2q)_n}{(q)_n(1-x^2q)}(-1)^n x^{2kn} q^{\binom{n+1}{2}+(a+1)n^2+(k-a-1)(n^2+n)} \times (1 - x^{2(a+1)} q^{(a+1)(2n+1)})$$

and

$$\beta_n = \beta_{n_k} = \sum_{n_1,n_2,\ldots,n_{k-1}\geq 0} \frac{q^{n_1^2+\cdots+n_{k-1}^2+n_{a+1}+\cdots+n_{k-1}} x^{2n_1+\cdots+2n_{k-1}}}{(q)_{n_k}} \times \prod_{i=1}^{k-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}.$$

Bailey pairs

Using this in the key identity one obtains the starting point for Hikami's proof.

It also gives Hikami's Andrews-Gordon variant using a different application of the Bailey lemma.

Its proof requires a sequence of new Bailey-type lemmas, culminating in a key lemma, which says that if (α_n, β_n) is a Bailey pair relative to a, then (α''_n, β''_n) is a Bailey pair relative to aq, where

$$\alpha''_n = \frac{1}{1-aq}\left(\frac{1-q^{n+1}}{1-aq^{2n+2}}\alpha_{n+1} + \frac{q^n(1-aq^n)}{1-aq^{2n}}\alpha_n\right)$$

and

$$\beta''_n = (1-q^{n+1})\beta_{n+1}.$$

The proof of the desired Bailey pair then goes as follows:

Start with the unit Bailey pair relative to x^2, iterate $a+1$ times along the Bailey chain with $b, c \to \infty$, apply the key lemma to obtain a Bailey pair relative to x^2q, iterate $k-1-a$ times along the Bailey chain with $b, c \to \infty$, and simplify to obtain the result.

Conclusion

Many other Bailey pairs can be used in the key identity to (ultimately) obtain strange identities.

The presence of the term $(q)_n/(x^2q)_n$ does restrict the possibilities.

Behind the scenes, we've always iterated along the Bailey chain using $b, c \to \infty$. Other choices give, for example,

$$\sum_{n_1,\dots,n_k\geq 0} \frac{(q)_{n_k}(-q)_{n_{k-1}} q^{n_1^2+\cdots+n_{k-2}^2+\binom{n_{k-1}+1}{2}+n_{a+1}+\cdots+n_{k-2}}}{(-q)_{n_k}} \times \prod_{i=1}^{k-1} \begin{bmatrix} n_{i+1}+\delta_{i,a} \\ n_i \end{bmatrix} \text{“}=\text{''} -\sum_{n\geq 0} n\chi_{4k}^{(a)}(n) q^{\frac{n^2-(k-a-1)^2}{4k}}.$$

Conclusion

Here $0 \leq a < k-1$, and $\chi_{4k}^{(a)}(n)$ is the even periodic function modulo $4k$ defined by

$$\chi_{4k}^{(a)}(n) = \begin{cases} 1, & \text{if } n \equiv k-a-1 \text{ or } 3k+a+1 \pmod{4k}, \\ -1, & \text{if } n \equiv k+a+1 \text{ or } 3k-a-1 \pmod{4k}, \\ 0, & \text{otherwise.} \end{cases}$$

It does not appear possible to obtain a strange identity for every periodic function.

Conclusion

One important class we seem to miss is

$$\sum_{n\geq 0} n\chi_{2st}(n)q^{\frac{n^2-(st-s-t)^2}{4st}},$$

where $\chi_{2st}(n)$ is the even periodic function modulo $2st$ defined by

$$\chi_{2st}(n) = \begin{cases} 1, & \text{if } n \equiv st-s-t \text{ or } st+s+t \pmod{2st}, \\ -1, & \text{if } n \equiv st-s+t \text{ or } st+s-t \pmod{2st}, \\ 0, & \text{otherwise.} \end{cases}$$

These are related to the torus knots $T_{s,t}$.

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