Alternating sign matrices and totally symmetric plane partitions

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joint work with I. Fischer, M. Konvalinka, P. Nadeau and V. Tewari.

87th SLC

Overview

- Definitions
- A multivariate generating function for monotone triangles
- Restricting to alternating sign matrices
- Connection to cyclically symmetric lozenge tilings

Alternating sign matrices

Definition (Robbins-Rumsey)

An alternating sign matrix (ASM) of size n is an $n \times n$ matrix with entries $1, 0, -1$, such that

- all row- and column-sums are equal to 1,
- in each row and column, the non-zero entries alternate.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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Theorem (Zeilberger, 1996)

The number of ASMs of size n is given by

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The number of ASMs of size n is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 3^{-\binom{n}{2}} s_{(n-1,n-1,n-2,n-2,\ldots,1,1)}(\mathbf{1}_{2n}).$$

Monotone Triangles

A monotone triangle (= strict GT pattern) is an array of integers of the form

$$\begin{array}{ccccccc} & & & m_{1,1} & & & \\ & & m_{2,1} & & m_{2,2} & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ m_{n,1} & & \cdots & \cdots & \cdots & & m_{n,n} \end{array} \qquad \begin{array}{ccc} & m_{i,j} & \\ \leq & & \leq \\ m_{i+1,j} & < & m_{i+1,j+1} \end{array}$$

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 & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\
m_{n,1} & & \cdots & \cdots & \cdots & & m_{n,n}
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & m_{i,i} & \\
\rotatebox{45}{$\le$} & & \rotatebox{-45}{$\le$} \\
m_{i+1,j} & < & m_{i+1,j+1}
\end{array}
$$

ASMs are in bijection to MTs with bottom row $(1, 2, \dots, n)$.

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
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$$4$$

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Weights for monotone triangles

Let $M = (m_{i,j})$ be a monotone triangle. We define

$$
\begin{aligned}
s_i(M) &= \#j : \; m_{i+1,j} < m_{i,j} < m_{i+1,j+1}, && \text{(special entries)} \\
l_i(M) &= \#j : m_{i,j} = m_{i+1,j}, && \text{(left-leaning entries)} \\
r_i(M) &= \#j : m_{i,j} = m_{i+1,j+1}, && \text{(right-leaning entries)} \\
\widetilde{d}_i(M) &= \sum_j (m_{i,j}) - \sum_j (m_{i-1,j}) - i.
\end{aligned}
$$

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The weight $\omega(M)$ is defined as

$$\omega(M) = \prod_{i=1}^{n} u^{r_i(M)} v^{l_i(M)} x_i^{\widetilde{d}_i(M)+2r_{i-1}(M)} (v + wx_i + ux_i^2)^{s_{i-1}}.$$

Example - MTs with bottom row $(1, 2, 3)$

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$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 2 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3 \end{array}$$

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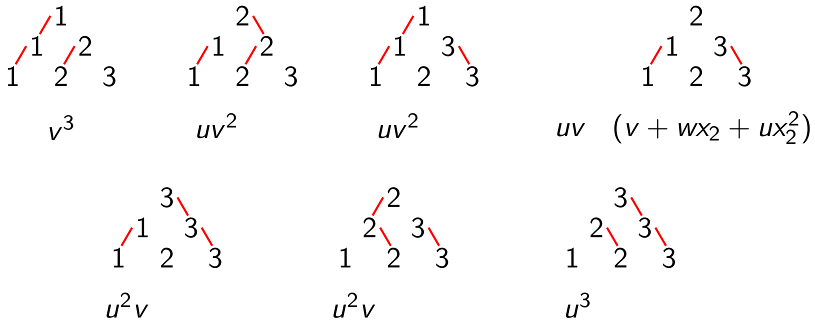

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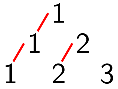
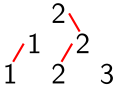
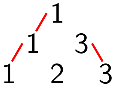


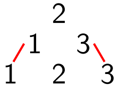
v^3



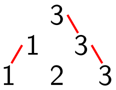
$uv^2 x_1 x_2$



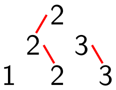
$uv^2 x_2 x_3$



$uvx_1(v + wx_2 + ux_2^2)x_3$



$u^2 v x_1^2 x_2 x_3$



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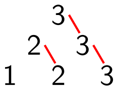


$u^3 x_1^2 x_2^2 x_3^2$

Schur polynomials

Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $L = (L_1, \ldots, L_n)$ be a sequence of non-negative integers, then we define the Schur polynomial indexed by L as

$$s_{(L_1,\ldots,L_n)}(\mathbf{x}) := \frac{\det\limits_{1 \leq i,j \leq n} \left(x_i^{L_j+n-j} \right)}{\prod\limits_{1 \leq i < j \leq n} (x_i - x_j)}.$$

A multivariate generating function for MTs

Denote by E_x denote the *shift operator* $E_x f(x) = f(x+1)$.

Theorem (A.-Fischer)

The multivariate generating function for monotone triangles with bottom row $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *w.r.t. the weight* ω *is*

$$\sum_M \omega_M(\mathbf{x})$$

$$= \prod_{1 \le i < j \le n} \left(v E_{k_j}^{-1} + w E_{k_i} E_{k_j}^{-1} + u E_{k_i} \right) s_{(k_n, \ldots, k_1)}(\mathbf{x}) \Bigg|_{k_i = \lambda_i - 1},$$

where the sum is over all monotone triangles with bottom row $(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

In the ASM case

$$n = 1: \qquad 1,$$

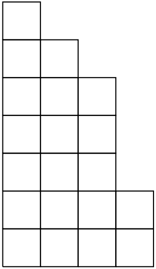
$$n = 2: \qquad v + u\, s_{(1,1)}(\mathbf{x}),$$

$$n = 3: \qquad v^3 + uv^2\, s_{(1,1)}(\mathbf{x}) + uvw\, s_{(1,1,1)}(\mathbf{x}) + u^2 v\, s_{(2,1,1)}(\mathbf{x}) + u^3\, s_{(2,2,2)}(\mathbf{x}),$$

$$n = 4: \qquad v^6 + uv^5\, s_{(1,1)}(\mathbf{x}) + uv^4 w\, s_{(1,1,1)}(\mathbf{x}) + u^2 v^4\, s_{(2,1,1)}(\mathbf{x}) + u^3 v^3\, s_{(2,2,2)}(\mathbf{x}) + uv^3 w^2\, s_{(1,1,1,1)}(\mathbf{x}) + 2u^2 v^3 w s_{(2,1,1,1)}(\mathbf{x}) + 2u^3 v^2 w s_{(2,2,2,1)}(\mathbf{x}) + u^3 v w^2\, s_{(2,2,2,2)}(\mathbf{x}) + u^3 v^3\, s_{(3,1,1,1)}(\mathbf{x}) + u^4 v^2 s_{(3,2,2,1)}(\mathbf{x}) + u^4 vw\, s_{(3,2,2,2)}(\mathbf{x}) + u^5 v\, s_{(3,3,2,2)}(\mathbf{x}) + u^6\, s_{(3,3,3,3)}(\mathbf{x}).$$

Frobenius notation for partitions

Let λ be a partition and l the length of the Durfee square. The Frobenius notation of λ is $(\lambda_1 - 1, \ldots, \lambda_l - l | \lambda'_1 - 1, \ldots, \lambda'_l - l)$.



$$\lambda = (4, 4, 3, 3, 3, 2, 1)$$

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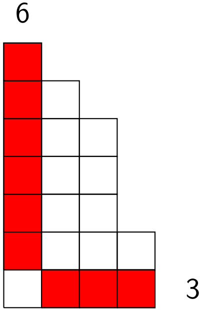

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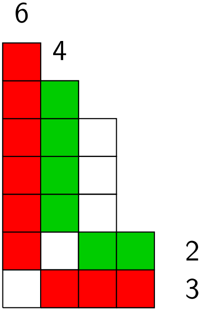

$$\lambda = (4, 4, 3, 3, 3, 2, 1)$$
$$= (3, 2, \quad | 6, 4, \quad)$$

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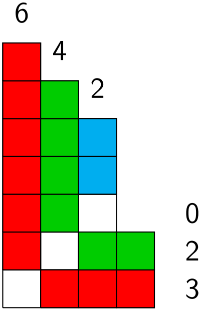


$$\lambda = (4, 4, 3, 3, 3, 2, 1)$$
$$= (3, 2, 0 | 6, 4, 2)$$

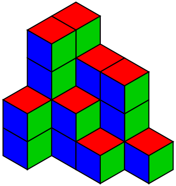
Plane partitions

A *plane partition* $\pi = (\pi_{i,j})$ inside an (a, b, c)-box is an array of non-negative integers

$$\begin{matrix} \pi_{1,1} & \pi_{1,2} & \cdots & \pi_{1,b} \\ \pi_{2,1} & \pi_{2,2} & \cdots & \pi_{2,b} \\ \vdots & \vdots & & \vdots \\ \pi_{a,1} & \pi_{a,2} & \cdots & \pi_{a,b} \end{matrix}$$

such that $\pi_{i,j} \leq c$ and all rows and columns are weakly decreasing.

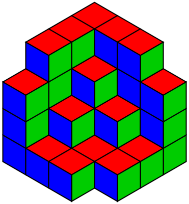
$$\begin{matrix} 4 & 3 & 3 & 1 \\ 4 & 2 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{matrix}$$



Totally symmetric plane partitions

Denote by TSPP_n the set of totally symmetric plane partitions inside an (n, n, n)-box. For $T \in \mathrm{TSPP}_n$ define

$$\begin{matrix} 4 & 4 & 4 & 3 \\ 4 & 3 & 2 & 1 \\ 4 & 2 & 1 & 1 \\ 3 & 1 & 1 & 0 \end{matrix}$$



T

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Denote by TSPP_n the set of totally symmetric plane partitions inside an (n, n, n)-box. For $T \in \mathrm{TSPP}_n$ define

$$\mathrm{diag}(T) = (T_{i,i})' = (a_1, \ldots, a_l | b_1, \ldots, b_l),$$

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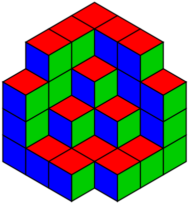

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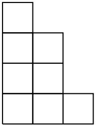

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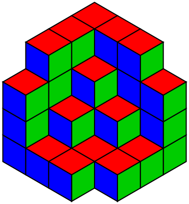

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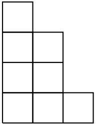
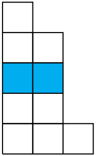


$\operatorname{diag}(T)$



$\pi(T)$

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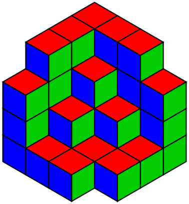
$$\mathrm{diag}(T) = (T_{i,i})' = (a_1, \ldots, a_l | b_1, \ldots, b_l),$$

$$\pi(T) = (a_1, \ldots, a_l | b_1 + 1, \ldots, b_l + 1),$$

$$\omega_T(r,u,v,w) = r^l u^{\sum_{i=1}^{l}(a_i+1)} v^{\binom{n}{2} - \sum_{i=1}^{l}(b_i+1)} w^{\sum_{i=1}^{l}(b_i - a_i)}.$$

4	4	4	3
4	3	2	1
4	2	1	1
3	1	1	0



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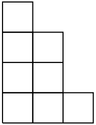

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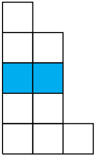


$\pi(T)$

A multivariate generating function for ASMs

Theorem (A.-Fischer-Konvalinka-Nadeau-Tewari)

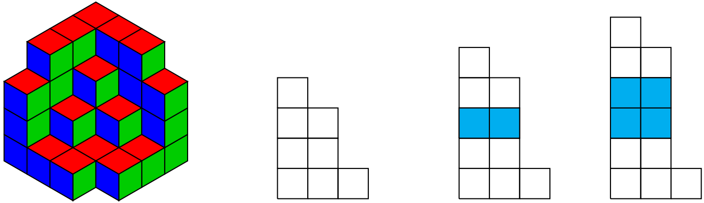
The multivariate generating function for ASMs w.r.t. ω is

$$\sum_{M} \omega(M) = \sum_{T \in \mathrm{TSPP}_{n-1}} \omega_T(1, u, v, w) s_{\pi(T)}(\mathbf{x}),$$

where the sum is over all monotone triangles with bottom row $(1, 2, \ldots, n)$.

Extending the family of symmetric polynomials

For $T \in \mathrm{TSPP}_n$ with diagonal $\mathrm{diag}(T) = (a_1, \dots, a_l | b_1, \dots, b_l)$ define $\pi_k(T) = (a_1, \dots, a_l | b_1 + k, \dots, b_l + k)$.

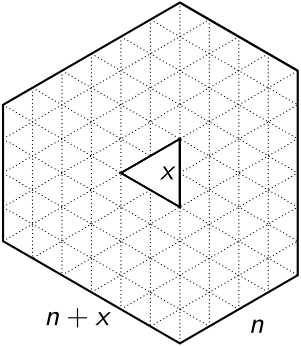


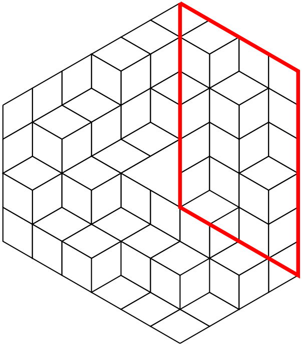
T $\quad$ $\mathrm{diag}(T) = \pi_0(T)$ $\quad$ $\pi_1(T)$ $\quad$ $\pi_2(T)$

We define the symmetric polynomial in $\mathbf{x} = (x_1, \dots, x_{n+k-1})$

$$\mathcal{A}_{n,k}(r, u, v, w; \mathbf{x}) = \sum_{T \in \mathrm{TSPP}_{n-1}} \omega_T(r, u, v, w) s_{\pi_k(T)}(\mathbf{x}).$$

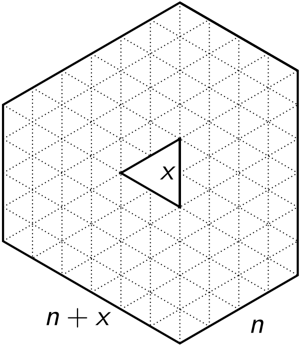
Cyclically symmetric lozenge tilings

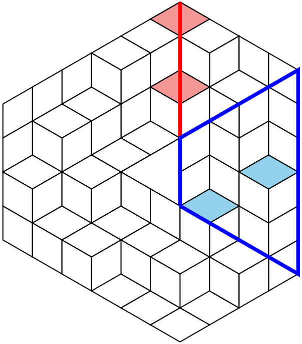




Denote by $\mathrm{CS}_{n,x}(r,t)$ the generating function of cyclically symmetric lozenge tilings in a cored hexagon with side lengths $(n, n+x, n, n+x, n, n+x)$ with respect to the weight

Cyclically symmetric lozenge tilings





Denote by $\mathrm{CS}_{n,x}(r,t)$ the generating function of cyclically symmetric lozenge tilings in a cored hexagon with side lengths $(n, n+x, n, n+x, n, n+x)$ with respect to the weight

$$r^{\#\lozenge \text{ on the red line}}\, t^{\#\lozenge \text{ in the blue region}}$$

Three enumeration formulas

Remember, the symmetric polynomials $\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x})$ were defined as

$$\mathcal{A}_{n,k}(r,u,v,w;\mathbf{x}) = \sum_{T \in \mathrm{TSPP}_{n-1}} \omega_T(r,u,v,w) s_{\pi_k(T)}(\mathbf{x}).$$

Theorem (A.-Fischer)

Let n be a positive integer and let $\mathbf{1}=(1,\ldots,1)$. Then,

$$\mathcal{A}_{n,0}(r,1,t,1;\mathbf{1}) = \mathrm{CS}_{n-1,0}(r,t+2),$$
$$\mathcal{A}_{n,k}(r,1,-1,1;\mathbf{1}) = \mathrm{CS}_{n-1,2k}(r,1),$$
$$\mathcal{A}_{n,k}(r,1,0,1;\mathbf{1}) = \mathrm{CS}_{n-1,k}(r,2).$$