2-core Littlewood identities

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Classical Littlewood identities

The first “Littlewood identity” is actually due to Schur (1918)

$$\sum_{\lambda} s_\lambda(x_1,\dots,x_n) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j}.$$

where for $\lambda=(\lambda_1,\dots,\lambda_n)$ a partition of length at most n the Schur function is given by

$$s_\lambda(x_1,\dots,x_n) = s_\lambda(x) := \frac{\det_{1\leqslant i,j\leqslant n}\left(x_i^{\lambda_j+n-i}\right)}{\det_{1\leqslant i,j\leqslant n}\left(x_i^{n-j}\right)}.$$

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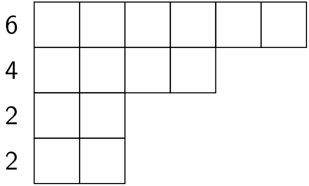
This can be proved by induction, combinatorially through the Robinson–Schensted–Knuth correspondence, or by using vanishing integrals.

In his 1940 text *The Theory of Group Characters and Matrix Representations of Groups*, Littlewood wrote down two more identities

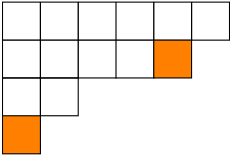
$$\sum_{\substack{\lambda \\ \lambda \text{ even}}} s_\lambda(x_1, \dots, x_n) = \prod_{i=1}^{n} \frac{1}{1 - x_i^2} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1 - x_i x_j}$$

$$\sum_{\substack{\lambda \\ \lambda' \text{ even}}} s_\lambda(x_1, \dots, x_n) = \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1 - x_i x_j},$$

where λ even means that the Young diagram of λ has only even rows:



Any partition λ can be decomposed into an even partition and a vertical strip by subtracting 1 from all the odd parts:



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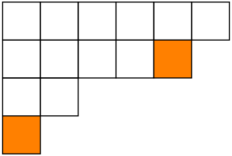

By the Pieri rule it then follows that

$$\sum_{\lambda} s_\lambda(x) = \Bigg(\sum_{r \geqslant 0} s_{(1^r)}(x)\Bigg)\Bigg(\sum_{\substack{\lambda \\ \lambda \text{ even}}} s_\lambda(x)\Bigg)$$

$$= \prod_{i=1}^{n}(1 + x_i) \sum_{\substack{\lambda \\ \lambda \text{ even}}} s_\lambda(x).$$

By Schur's identity

$$\sum_{\lambda} s_\lambda(x) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j},$$

it follows that

$$\begin{aligned}\sum_{\substack{\lambda \\ \lambda \text{ even}}} s_\lambda(x) &= \prod_{i=1}^{n} \frac{1}{1+x_i} \sum_{\lambda} s_\lambda(x) \\ &= \prod_{i=1}^{n} \frac{1}{1-x_i^2} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j}.\end{aligned}$$

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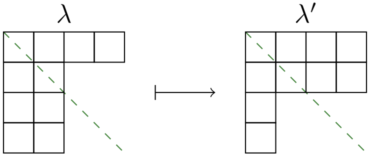
$$\sum_{\substack{\lambda \\ \lambda \text{ even}}} s_\lambda(x) = \prod_{i=1}^{n} \frac{1}{1+x_i} \sum_{\lambda} s_\lambda(x)$$

$$= \prod_{i=1}^{n} \frac{1}{1-x_i^2} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j}.$$

To obtain Littlewood's even column identity one only needs to take conjugates.



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In particular Macdonald's bounded analogue of the first identity is

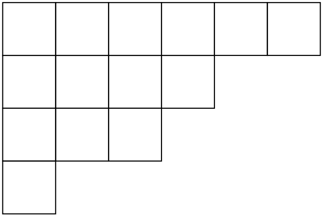
$$\sum_{\substack{\lambda \\ \lambda\subseteq(m^n)}} s_\lambda(x_1,\dots,x_n) = \frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{m+2n-j} - x_i^{j-1}\big)}{\prod_{i=1}^n (x_i-1)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(x_ix_j-1)}.$$

He used this to deduce MacMahon's famous conjecture for the number of symmetric plane partitions in a box in the form

$$\sum_{\substack{\lambda \\ \lambda\subseteq(m^n)}} s_\lambda(q,q^3,\dots,q^{2n-1}) = \prod_{i=1}^n \frac{1-q^{m+2i-1}}{1-q^{2i-1}} \prod_{1\leqslant i<j\leqslant n} \frac{1-q^{2(m+i+j-1)}}{1-q^{2(i+j-1)}}.$$

Hooks and 2-cores

We identify the Young diagram

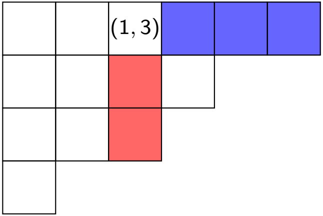


with a set of points (i, j) such that

$$1 \leqslant i \leqslant \ell(\lambda)$$
$$1 \leqslant j \leqslant \lambda_i,$$

and write s or (i, j) for a square.

For a square $s = (i, j) \in \lambda$ we have the arm and leg lengths



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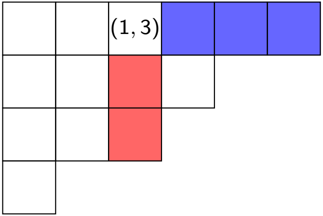

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The hook length for $s = (i, j)$ is then

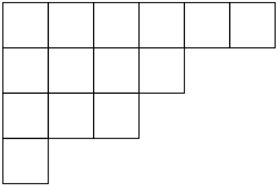
$$h(s) = a(s) + l(s) + 1 = \lambda_i + \lambda'_j - i - j + 1.$$

For a partition λ, set

$$\mathcal{H}^{\mathrm{e}}_\lambda = \{h(s) \text{ even} \mid s \in \lambda\}$$
$$\mathcal{H}^{\mathrm{o}}_\lambda = \{h(s) \text{ odd} \mid s \in \lambda\},$$

and $\mathcal{H}_\lambda = \mathcal{H}^{\mathrm{e}}_\lambda \cup \mathcal{H}^{\mathrm{o}}_\lambda$.

A partition has empty 2-core, written 2-core$(\lambda) = 0$, if it can be tiled by dominoes. For example, $\lambda = (6, 4, 3, 1)$



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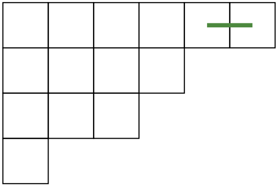

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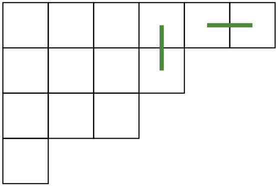

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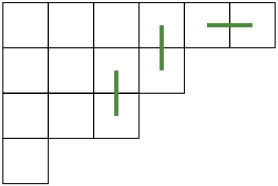

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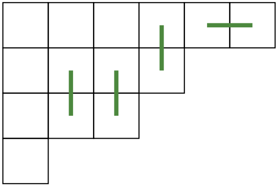

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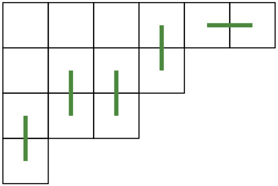

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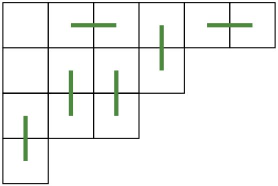

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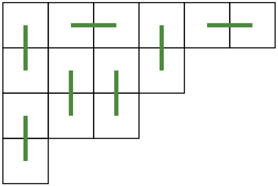

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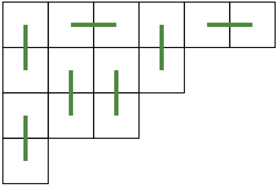
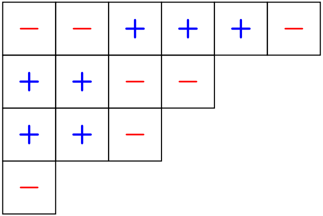


For us it's important to note that

$$\text{2-core}(\lambda) = 0 \quad \Longleftrightarrow \quad |\mathcal{H}_\lambda^{\mathrm{o}}| = |\mathcal{H}_\lambda^{\mathrm{e}}|.$$



Finally, we need a statistic

$$b(\lambda) := \sum_{(i,j)\in\lambda} (-1)^{\lambda_i+\lambda'_j-i-j+1}(\lambda_i - i).$$

For our running example

5	$-$	$-$	$+$	$+$	$+$	$-$
2	$+$	$+$	$-$	$-$		
0	$+$	$+$	$-$			
-3	$-$					

we compute

$$b\big((6,4,3,1)\big) = 3.$$

In fact for 2-core$(\lambda) = 0$ we have $b(\lambda) \geqslant 0$ with equality if and only if λ is even.

2-core condition

In their work on the branching problem, Lee, Rains and Warnaar were led to conjecture a swathe of curious formulae including integral evaluations, Littlewood identities, branching formulae, and hypergeometric summations.

The link between all of their conjectures is the "2-core condition". For example, an integral vanishes unless 2-core$(\lambda) = 0$.

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All of their conjectures are at the Macdonald, or (q, t), level. In the Schur case $q = t$, things simplify dramatically, and some of their conjectures can be resolved.

Recall the usual infinite q-shifted factorial

$$(a;q)_\infty := \prod_{i\geqslant 0}(1-aq^i).$$

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Then the following conjecture of Lee, Rains and Warnaar is true.

Theorem

There holds

$$\sum_{\substack{\lambda \\ \text{2-core}(\lambda)=0}} q^{b(\lambda)} \frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}}(1-q^h)}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}}(1-q^h)} s_\lambda(x) = \prod_{i=1}^n \frac{(qx_i^2;q^2)_\infty}{(x_i^2;q^2)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_ix_j},$$

and

$$\sum_{\substack{\lambda \\ \text{2-core}(\lambda)=0}} q^{b(\lambda')} \frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}}(1-q^h)}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}}(1-q^h)} s_\lambda(x) = \prod_{i=1}^n \frac{(q^2x_i^2;q^2)_\infty}{(qx_i^2;q^2)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_ix_j}.$$

For $q=0$ these are Littlewood's even row/even column identities i
respectively.

The previous identities are in the spirit of Kawanaka's 1999 formula

$$\sum_{\lambda}\Big(\prod_{h\in\mathcal{H}}\frac{1+q^h}{1-q^h}\Big)s_\lambda(x)=\prod_{i=1}^{n}\frac{(-qx_i;q)_\infty}{(x_i;q)_\infty}\prod_{1\leqslant i<j\leqslant n}\frac{1}{1-x_ix_j}$$

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Unlike Kawanaka's identity, the 2-core identities make sense for $q\to 1$ and produce the following corollary.

$$\sum_{\substack{\lambda\\ \text{2-core}(\lambda)=0}}\frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}}(h)}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}}(h)}s_\lambda(x)=\prod_{i\geqslant 1}\frac{1}{(1-x_i^2)^{1/2}}\prod_{i<j}\frac{1}{1-x_ix_j}$$

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The proof of the previous theorem relies on some basic Koornwinder polynomial theory together with vanishing integrals.

Vanishing integrals

Fix the measure

$$\mathrm{d}T(x) := \frac{1}{2^n n!(2\pi \mathrm{i})^n} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}.$$

For a function $f(x)$ of a single variable we define

$$\begin{aligned} f\left(x^{\pm}\right) &= f(x)f(1/x) \\ f\left(x^{\pm}y^{\pm}\right) &= f(xy)f(x/y)f(y/x)f(1/xy). \end{aligned}$$

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The prototypical example of a vanishing integral is Weyl's formula

$$\int_{\mathbb{T}^n} s_\lambda(x_1^{\pm},\dots,x_n^{\pm}) \prod_{i=1}^n \big(1-x_i^{\pm 2}\big) \prod_{1\leqslant i<j\leqslant n} \big(1-x_i^{\pm}x_j^{\pm}\big)\,\mathrm{d}T(x) = \begin{cases} 1 & \text{if } \lambda' \text{ even}, \\ 0 & \text{otherwise}. \end{cases}$$

Lee, Rains and Warnaar prove the following generalisation of the vanishing part of Weyl's integral.

For $a, b, q \in \mathbb{C}$ such that $|a|, |b|, |q| < 1$, the integral

$$\mathrm{I}_{\lambda}^{(n)}(a, b; q) := \int_{\mathbb{T}^n} s_{\lambda}\big(x^{\pm}\big) \prod_{i=1}^{n} \frac{(x_i^{\pm 2}; q)_{\infty}}{(a x_i^{\pm 2}; q^2)_{\infty} (b x_i^{\pm 2}; q^2)_{\infty}} \prod_{1 \leqslant i < j \leqslant n} \big(1 - x_i^{\pm} x_j^{\pm}\big) \mathrm{d}T(x)$$

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vanishes unless 2-core$(\lambda) = 0$.

For $a = b = q = 0$ the integral reduces to the vanishing part of Weyl's integral. Also define the scaled version

$$\hat{\mathrm{I}}_\lambda^{(n)}(a, b; q) := \frac{\mathrm{I}_\lambda^{(n)}(a, b; q)}{\mathrm{I}_0^{(n)}(a, b; q)},$$

where the denominator has an explicit product formula (Gustafson's generalised Selberg/Askey–Wilson integral).

For 2-core$(\lambda) = 0$, they also give integral evaluations in terms of Pfaffians in two important special cases. These Pfaffians may be evaluated, and yield the following pair of evaluations.

For 2-core$(\lambda) = 0$,

$$\hat{\mathrm{I}}_\lambda^{(n)}(q, q; q) = q^{b(\lambda')} \frac{C_\lambda^{\mathrm{e}}(q^{2n}; q) H_\lambda^{\mathrm{o}}(q)}{C_\lambda^{\mathrm{o}}(q^{2n}; q) H_\lambda^{\mathrm{e}}(q)}$$

and

$$\hat{\mathrm{I}}_\lambda^{(n)}(1, q^2; q) = q^{b(\lambda)} \frac{1 + q^{n+2(b(\lambda')-b(\lambda))}}{1 + q^n} \, \frac{C_\lambda^{\mathrm{e}}(q^{2n}; q) H_\lambda^{\mathrm{o}}(q)}{C_\lambda^{\mathrm{o}}(q^{2n}; q) H_\lambda^{\mathrm{e}}(q)}.$$

Here

$$H_\lambda^{\mathrm{e/o}}(q) := \prod_{\substack{s \in \lambda \\ h(s) \text{ even/odd}}} \big(1 - q^{h(s)}\big),$$

$$C_\lambda^{\mathrm{e/o}}(z; q) := \prod_{\substack{(i,j) \in \lambda \\ i+j \text{ even/odd}}} \big(1 - zq^{j-i}\big).$$

The key identity is the following due to Rains and Warnaar (stated in a special case).

For nonnegative integers n, m,

$$(x_1 \cdots x_n)^m K_{(m^n)}(x; q, q; \pm a, \pm b) = \sum_{\lambda} (-1)^{|\lambda|} \hat{\mathrm{I}}_{\lambda'}^{(m)}(a, b; q) s_\lambda(x).$$

Any closed form evaluation of the integral $\hat{\mathrm{I}}_{\lambda'}^{(m)}(a, b; q)$ thus gives a bounded Littlewood-type identity.

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For example with $(a, b) = (q, q)$ we obtain

$$(x_1 \cdots x_n)^m K_{(m^n)}(x; q, q; \pm q, \pm q) = \sum_{\substack{\lambda \\ \text{2-core}(\lambda)=0}} q^{b(\lambda')} \frac{C_\lambda^{\mathrm{e}}(q^{-2m}; q) H_\lambda^{\mathrm{o}}(q)}{C_\lambda^{\mathrm{o}}(q^{-2m}; q) H_\lambda^{\mathrm{e}}(q)} s_\lambda(x),$$

and a similar result for $(a, b) = (1, q^2)$. Sending $m \to \infty$ gives the unbounded identity from before.

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The q, t-analogues of the vanishing integrals and (bounded) Littlewood identities are still open. However, in the Hall–Littlewood case, the 2-core condition drops out and the two identities are known. For example

$$\sum_{\lambda} t^{o(\lambda)/2}\Bigg(\prod_{\substack{s\in\lambda\\ a(s)=0\\ l(s)\text{ even}}}\big(1-t^{l(s)+1}\big)\Bigg)P_\lambda(x;t)=\prod_{i\geqslant 1}\frac{1-tx_i^2}{1-x_i^2}\prod_{i<j}\frac{1-tx_ix_j}{1-x_ix_j},$$

where the sum is over all partitions such that odd parts have even multiplicity and $o(\lambda)$ is the sum of these multiplicities. This is due to Kawanaka. The other is an identity of Macdonald in this case.

One final curious conjecture of Lee, Rains and Warnaar is a C_n analogue of Andrews' q-analogue of Watson's ${}_3F_2$ summation

$$ {}_4\phi_3\left[\begin{matrix} a^{1/2}, -a^{1/2}, bq^{N-1}, q^{-N} \\ a, b^{1/2}, -b^{-1/2} \end{matrix}; q, q\right] = \begin{cases} \dfrac{a^{N/2}(q, b/a; q^2)_{N/2}}{(aq, b; q^2)_{N/2}} & \text{if } N \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} $$

Suppressing the details, it may be stated as

$$ \sum_{\mu\subseteq\lambda} f_{\lambda,\mu}(a; q, t) = \begin{cases} F_\lambda(a; q, t) & \text{if 2-core}(\lambda) = 0, \\ 0 & \text{otherwise.} \end{cases} $$

For $\lambda = (N)$ this is Andrews' formula.

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$${}_4\phi_3\left[\begin{matrix} a^{1/2}, -a^{1/2}, bq^{N-1}, q^{-N} \\ a, b^{1/2}, -b^{-1/2} \end{matrix}; q, q\right] = \begin{cases} \dfrac{a^{N/2}(q, b/a; q^2)_{N/2}}{(aq, b; q^2)_{N/2}} & \text{if } N \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Suppressing the details, it may be stated as

$$\sum_{\mu\subseteq\lambda} f_{\lambda,\mu}(a; q, t) = \begin{cases} F_\lambda(a; q, t) & \text{if 2-core}(\lambda) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $\lambda = (N)$ this is Andrews' formula.

the end