Symplectic cactus action on crystals of Kashiwara-Nakashima tableaux

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Plan

- Basics and cacti
- Normal crystals, Levi branching, cacti action and partial Schützenberger-Lusztig involutions
- Symplectic cactus action on crystals of KN tableaux
  - Direct algorithms for partial Schützenberger-Lusztig involutions
  - Baker virtualization
- A symplectic Berenstein-Kirillov group

The cactus group $J_{\mathfrak{g}}$

- Let $\mathfrak{g}$ be a finite dimensional, complex, semisimple Lie algebra and
  - I its Dynkin diagram, $\Delta = \{\alpha_i\}_{i \in I}$ the simple roots.
  - W the Weyl group, $w_0 \in W$ the longest element.
  - $\theta : I \to I$ the Dynkin diagram automorphism of I defined by

$$\alpha_{\theta(i)} = -w_0.\alpha_i, \ i \in I.$$

  - $\theta_J : J \to J$ the Dynkin diagram automorphism of a connected subdiagram $J \subseteq I$, defined by

$$\alpha_{\theta_J(j)} = -w_0^J.\alpha_j, \ j \in J,$$

    w_0^J the long element of the parabolic subgroup $W^J \subseteq W$.

- [Halacheva 2016]. The *cactus group* $J_{\mathfrak{g}}$ corresponding to $\mathfrak{g}$ is the group defined by:
  - **Generators**: s_J, $J \subseteq I$ running over all connected subdiagrams of the Dynkin diagram I of $\mathfrak{g}$, and
  - **Relations**:

    1 $\mathfrak{g}$. $s_J^2 = 1$, for all $J \subseteq I$,

    2 $\mathfrak{g}$. $s_J s_{J'} = s_{J'} s_J$, for all $J, J' \subseteq I$ such that $J \cup J'$ is not connected,

    3 $\mathfrak{g}$. $s_J s_{J'} = s_{\theta_J(J')} s_J$, for all $J' \subseteq J \subseteq I$.

The cacti $J_{\mathfrak{gl}_n}$ and $J_{\mathfrak{sp}_{2n}}$

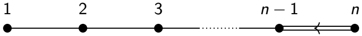
- The cactus group $J_{\mathfrak{sp}_{2n}}$ is the group defined by
  - Generators: s_J, J connected subdiagrams of the C_n Dynkin diagram,
  - Relations:

    1C. $s_J^2 = 1$, $J \subseteq [n]$,

    2C. $s_J s_{J'} = s_{J'} s_J$, $J, J' \subseteq [n]$ such that $J \cup J'$ is not connected,

    3C. (1) $s_{[p,q]} s_{[k,l]} = s_{[p+q-l,p+q-k]} s_{[p,q]}$, $[k,l] \subset [p,q] \subseteq [n-1]$.

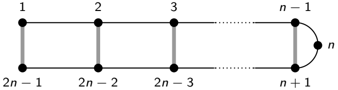
    (2) $s_{[p,n]} s_{[q,l]} = s_{[q,l]} s_{[p,n]}$, $[q,l] \subset [p,n] \subseteq [n]$,



- $J_n = J_{\mathfrak{gl}_n} \subseteq J_{\mathfrak{sp}_{2n}}$.
- Alternative $n-1$ generators for $J_n = J_{\mathfrak{gl}_n}$, and $2n-1$ generators for $J_{\mathfrak{sp}_{2n}}$

$$s_{[1,p]}, \ 1 \leq p \leq n-1, \quad s_{[p,n]}, \ 1 \leq p \leq n.$$

- $J_n \subseteq J_{\mathfrak{sp}_{2n}} \hookrightarrow J_{2n}$ [A–Tarighat–Torres 2022].



$$\begin{array}{rcll} \iota : J_{\mathfrak{sp}_{2n}} & \hookrightarrow & J_{2n} & \\ s_{[1,p]} & \mapsto & s_{[1,p]} s_{[2n-p,2n-1]}, & [1,p] \subseteq [n-1], \\ s_{[p,n]} & \mapsto & s_{[p,2n-p]}, & [p,n] \subseteq [n]. \end{array}$$

Normal crystals, Levi branching and cacti action

- Let B be a normal crystal
  - For $J \subseteq I$, the Levi branched crystal $\mathrm{B}_J =$ the restriction of B to the subdiagram J of I.
  - The crystal graph of B_J has the same vertices as B but the arrows are only those labelled in J, that is, we forget the crystal maps e_i, f_i, φ_i, and ε_i, for $i \notin J$.
  - [Halacheva 2016] The cactus group $J_{\mathfrak{g}} \curvearrowright \mathrm{B} = \sqcup \mathrm{B}(\lambda)$ via partial Schützenberger-Lusztig involutions.
  - For $J \subseteq I$, the partial Schützenberger-Lusztig involution $\xi_J =$ restriction Schützenberger-Lusztig involution to the normal Levi branched crystal $\mathrm{B}_J(\lambda)$.
- $\mathfrak{g} = \mathrm{gl}_n$: $\mathrm{B}(\lambda) = \mathrm{SSYT}(\lambda, n)$ crystal of A_{n-1} semistandard Young tableaux of straight shape λ in the alphabet $[n]$.
- $\mathfrak{g} = \mathfrak{sp}_{2n}$: $\mathrm{B}(\lambda) = \mathrm{KN}(\lambda, n)$ crystal of C_n Kashiwara-Nakashima (De Concini) tableaux of straight shape λ in the alphabet $[\pm n] = \{1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}\}$

$$n = 4 \qquad Q = \begin{array}{c} 1 \\ 2 \\ 4 \\ \bar{2} \end{array} \qquad \begin{array}{c} 1 \\ \emptyset \end{array} \quad \begin{array}{c} 2 \\ \bar{2} \end{array} \quad \begin{array}{c} \emptyset \\ \emptyset \end{array} \quad \begin{array}{c} 4 \\ \emptyset \end{array} \quad Q \text{ not KN column}, \qquad T = \begin{array}{c} 2 \\ 4 \\ \bar{2} \end{array} \qquad \begin{array}{c} \emptyset \\ \emptyset \end{array} \quad \begin{array}{c} 2 \\ \bar{2} \end{array} \quad \begin{array}{c} \emptyset \\ \emptyset \end{array} \quad \begin{array}{c} 4 \\ \emptyset \end{array} \quad OK$$

$$P = \begin{array}{ccc} 2 & 2 & \bar{1} \\ 4 & \bar{3} & \\ \bar{2} & \bar{1} & \end{array} \qquad (\ell P, rP) = \begin{array}{cccccc} 1 & 2| & 2 & 2| & \bar{1} & \bar{1} \\ 4 & 4| & \bar{3} & \bar{3}| & & \\ \bar{2} & \bar{1}| & \bar{1} & \bar{1}| & & \end{array} \qquad \text{KN tableau}$$

Virtualization via the Baker embedding (2000)

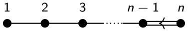
- Embedding of the C_n crystal $\mathrm{KN}(\lambda, n)$ into the A_{2n-1} crystal $\mathrm{SSYT}(\lambda^A, n, \bar{n})$ [Baker 2000]

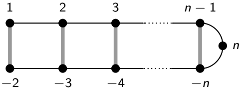
$$b = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \bar{5} \\ \hline \bar{4} & \bar{3} \\ \hline \bar{3} & \\ \cline{1-1}\end{array} \mapsto \Psi_{Baker}(C_1) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 5 & \bar{4} \\ \hline \bar{5} & \bar{2} \\ \hline \bar{4} & \\ \cline{1-1} \bar{3} & \\ \cline{1-1}\end{array}, \quad \Psi_{Baker}(C_2) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \bar{5} \\ \hline 4 & \bar{3} \\ \hline \bar{5} & \\ \cline{1-1} \bar{4} & \\ \cline{1-1} \bar{3} & \\ \cline{1-1} \bar{2} & \\ \cline{1-1}\end{array}$$

$$\Psi_{Baker}(b) = w_{col}(\Psi_{Baker}(C_2)).w_{col}(\Psi_{Baker}(C_1))$$

$$\begin{array}{rcl} E : \mathrm{KN}(\lambda, n) & \hookrightarrow & \mathrm{SSYT}(\lambda^A, n, \bar{n}) \\ b & \mapsto & E(b) = [\emptyset \leftarrow_{column} \Psi_{Baker}(b)] \\ u_\lambda & \mapsto & u_{\lambda^A} \end{array}$$

where $E(f_i^C(b)) = f_i^E(E(b))$, $f_i^E = f_i^A \circ f_{\overline{i+1}}^A$, $i < n$, and $f_n^E = (f_n^A)^2$.





- [A–Tarighat–Torres 2022] The Baker recording tableau $Q_{Baker}(u_\lambda) := Q(\Psi_{Baker}(u_\lambda)) = Q(\Psi_{Baker}(b))$ only depends on λ.
- Unbumping: $RSK^{-1}_{|E(\mathrm{KN}(\lambda,n))\times\{Q_{Baker}(u_\lambda)\}}$.

  $E^{-1} = \Psi^{-1} \circ RSK^{-1}_{|E(\mathrm{KN}(\lambda,n))\times\{Q_{Baker}(u_\lambda)\}}$.

Virtualization and Levi branching

Embedding of the Levi branched crystal $\mathrm{KN}_J(\lambda, n)$ into the Levi branched crystal $SSYT_{J\cup\bar{J}}(\lambda^A, n, \bar{n})$

- U connected component of $KN_J(\lambda, n)$ with u^{high}, u^{low} $\Rightarrow$ $E(U)$ is contained in a connected component of $SSYT_{J\cup\bar{J}}(\lambda^A, n, \bar{n})$, with high. and low. weight elements $E(u^{\text{high}})$, $E(u^{\text{low}})$ respectively.

- The virtualization map E behaves very nicely with respect to Levi restriction!

$$\begin{array}{ccc} \mathrm{KN}_J(\lambda, n) & \underset{E}{\hookrightarrow} & \mathrm{SSYT}_{J\cup\bar{J}}(\lambda^A, n, \bar{n}) \\ \mathrm{KN}_{[1,p]}(\lambda, n) & \underset{E}{\hookrightarrow} & \mathrm{SSYT}_{[1,p]\cup[\overline{p+1},\bar{2}]}(\lambda, n, \bar{n}), \quad p < n, \\ \mathrm{KN}_{[p,n]}(\lambda, n) & \underset{E}{\hookrightarrow} & \mathrm{SSYT}_{[p,\overline{p+1}]}(\lambda, n, \bar{n}), \quad p \leq n \end{array}$$

Figure: I is the Dynkin diagram of type C_6 and $J = \{1, 2\}$ or $J = \{4, 5, 6\}$

Schützenberger-Lusztig (SL) involution and direct algorithms

- Let $\mathrm{B}(\lambda)$ be a normal crystal with highest weight λ, and
  - u_λ^{high}, u_λ^{low} highest and lowest weight elements.
- The Schützenberger-Lusztig involution ξ is the unique set involution $\xi : \mathrm{B}(\lambda) \to \mathrm{B}(\lambda)$ such that, for all $b \in \mathrm{B}(\lambda)$, and $i \in I$,
  - $e_i \xi(b) = \xi f_{\theta(i)}(b)$
  - $f_i \xi(b) = \xi e_{\theta(i)}(b)$
  - $\mathrm{wt}(\xi(b)) = w_0.\mathrm{wt(b)}$, w_0 the long element of the Weyl group W.

- Let $b \in \mathrm{B}(\lambda)$ and $b = f_{j_r} \cdots f_{j_1}(u_\lambda^{\text{high}})$.
  - in type A_{n-1}, $\xi(b) = e_{n-j_r} \cdots e_{n-j_1}(u_\lambda^{\text{low}}) = \mathrm{evac}^A(b)$, [Schützenberger 1976] $\mathrm{wt}(\xi(b)) = [n \cdots 2\,1]\,\mathrm{wt(b)}$
  - in type C_n, $\xi(b) = e_{j_r} \cdots e_{j_1}(u_\lambda^{\text{low}}) = \mathrm{evac}^C(b)$, [Santos 2021], $\mathrm{wt}(\xi(b)) = -\mathrm{wt(b)}$

Partial SL-involutions and direct algorithms

- For $J \subseteq I$, let b_J^{high}, b_J^{low} be the highest and lowest weight vertices of the connected component of $B_J(\lambda)$ containing b, and $b = f_{j_r} \cdots f_{j_1}(b_J^{\text{high}})$, for $j_r, \ldots, j_1 \in J$.

1. $SSYT_J(\lambda, n)$, $J = [p, q]$, $q < n$, type A_{q-p} crystal,

$$\xi_J(b) = e_{q-p-j_r+1} \cdots e_{q-p-j_1+1}(b_J^{\text{low}}) = \text{reversal}^A{}_J(b),$$ [Benkart–Sottile–Stroomer, 1999].

partial reversal : $\text{reversal}_J^A(b) = \text{reversal}(b_{[p,q+1]}) = \text{rectification}^{-1}.\text{evacuation}.\text{rectification}(b_{[p,q+1]})$.

2. $\text{KN}_J(\lambda, n)$ and $\text{KN}_J(\lambda, n) \underset{E}{\hookrightarrow} \text{SSYT}_{J \cup \bar{J}}(\lambda^A, n, \bar{n})$

   - $J = [p, n]$, $\text{KN}_J(\lambda, n)$ type C_{n-p+1} crystal [A–Tarighat–Torres, 2022]

$$\xi_J(b) = e_{j_r} \cdots e_{j_1}(b_J^{\text{low}}) = \text{reversal}^{C_n}{}_{[p,n]}(b) = E_{left}^{-1}\text{reversal}^A_{\overline{[p,p+1]}}E(b)$$

partial symplectic reversal:

$$\text{reversal}^C_{[p,n]}(b) = \text{reversal}^C(b_{[\pm p,n]})$$
$$= (\text{rectification}^C)^{-1}.\text{evacuation}^C.\text{rectification}^C(b_{[\pm p,n]}).$$

   - $J = [1, p]$, $p < n$, $\text{KN}_J(\lambda, n)$ type A_p crystal [A– Tarighat–Torres, 2022]

$$\xi_J(b) = e_{p-j_r+1} \cdots e_{p-j_1+1}(b_J^{\text{low}}) = E_{left}^{-1}\text{evac}^A_{[1,p]}\text{reversal}^A_{[\overline{p+1},\bar{2}]}E(b)$$

Example: Virtualization of the partial SL involution $\xi^{C_5}_{[1,4]}$

- $n = 5$, Baker splitting Ψ

$$b = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \overline{5} \\ \hline \overline{4} & \overline{3} \\ \hline \overline{3} & \\ \cline{1-1} \end{array} \quad \Psi_{Baker}(C_1) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 5 & \overline{4} \\ \hline \overline{5} & \overline{2} \\ \hline \overline{4} & \\ \cline{1-1} \overline{3} & \\ \cline{1-1} \end{array}, \quad \Psi_{Baker}(C_2) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \overline{5} \\ \hline 4 & \overline{3} \\ \hline \overline{5} & \\ \cline{1-1} \overline{4} & \\ \cline{1-1} \overline{3} & \\ \cline{1-1} \overline{2} & \\ \cline{1-1} \end{array}$$

- Baker embedding E and recording tableau Q_{Baker}

$$E(b) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \overline{5} \\ \hline 4 & \overline{3} \\ \hline \overline{5} & \\ \cline{1-1} \overline{4} & \\ \cline{1-1} \overline{3} & \\ \cline{1-1} \overline{2} & \\ \cline{1-1} \end{array} \underset{\textit{column insertion}}{\longleftarrow} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 5 & \overline{4} \\ \hline \overline{5} & \overline{2} \\ \hline \overline{4} & \\ \cline{1-1} \overline{3} & \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 4 & \overline{5} \\ \hline 3 & \overline{5} & \overline{4} & \overline{3} \\ \hline 5 & \overline{4} & \overline{2} & \\ \cline{1-3} \overline{5} & \overline{3} & & \\ \cline{1-2} \overline{4} & \overline{2} & & \\ \cline{1-2} \overline{3} & & & \\ \cline{1-1} \end{array} \quad Q_{Baker} = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 11 & 15 \\ \hline 2 & 5 & 12 & 16 \\ \hline 3 & 6 & 13 & 17 \\ \hline 7 & 14 & 18 & \\ \cline{1-3} 8 & 19 & & \\ \cline{1-2} 9 & 20 & & \\ \cline{1-2} 10 & & & \\ \cline{1-1} \end{array}.$$

Virtualization of partial LS involution $\xi^{C_5}_{[1,4]}$

- $\xi^{A_9}_{[1,4]}\xi^{A_9}_{[\bar{5},\bar{2}]}E(b)$:

$$\xi^{A_9}_{[1,4]}E(b) = evac \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & \\ 3 & & & \\ 5 & & & \end{array} = \begin{array}{cccc} 1 & 3 & 4 & 5 \\ 2 & 5 & 5 & \\ 4 & & & \\ 5 & & & \end{array},$$

$$\xi^{A_9}_{[\bar{5},\bar{2}]}E(b) = reversal \begin{array}{cccc} * & * & * & * \\ * & * & * & \bar{5} \\ * & \bar{5} & \bar{4} & \bar{3} \\ * & \bar{4} & \bar{2} & \\ \bar{5} & \bar{3} & & \\ \bar{4} & \bar{2} & & \\ \bar{3} & & & \end{array} = \begin{array}{cccc} * & * & * & * \\ * & * & * & \bar{3} \\ * & \bar{4} & \bar{2} & \bar{2} \\ * & \bar{3} & \bar{1} & \\ \bar{4} & \bar{2} & & \\ \bar{3} & \bar{1} & & \\ \bar{1} & & & \end{array}.$$

$$\xi^{A_9}_{[1,4]}\cdot\xi^{A_9}_{[\bar{5},\bar{2}]}E(b) = \begin{array}{cccc} 1 & 3 & 4 & 5 \\ 2 & 5 & 5 & \bar{3} \\ 4 & \bar{4} & \bar{2} & \bar{2} \\ 5 & \bar{3} & \bar{1} & \\ \bar{4} & \bar{2} & & \\ \bar{3} & \bar{1} & & \\ \bar{1} & & & \end{array}.$$

Virtualization of partial LS involution $\xi^{C_5}_{[1,4]}$

- $\underbrace{\Psi^{-1}.\text{unbumping}}_{E^{-1}}(\xi^{A_9}_{[1,4]}\xi^{A_9}_{[\bar{5},\bar{2}]}E(b))$ gives $\xi^{C_5}_{[1,4]}(b)$

$$b = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \overline{5} \\ \hline \overline{4} & \overline{3} \\ \hline \overline{3} & \\ \cline{1-1} \end{array} \mapsto \xi^{C_5}_{[1,4]}(b) = E^{-1}\xi^{A_9}_{[1,4]}\xi^{A_9}_{[\bar{5},\bar{2}]}E(b) = \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 5 & \overline{3} \\ \hline \overline{3} & \overline{2} \\ \hline \overline{1} & \\ \cline{1-1} \end{array}$$

Example: partial LS involution $\xi^{C_4}_{[2,4]}$ - Colourful Letters

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & \bar{1} \\ \hline 4 & 4 & \bar{3} \\ \cline{1-3} \bar{4} & \bar{2} & \bar{1} \\ \cline{1-3} \bar{3} \\ \cline{1-1}\end{array} \qquad P_{\pm 2,4} = \begin{array}{|c|c|c|c|}\hline & 2 & 2 & \\ \hline 4 & 4 & \bar{3} \\ \cline{1-3} \bar{4} & \bar{2} & \\ \cline{1-3} \bar{3} \\ \cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline a & 2 & 2 & \\ \hline 4 & 4 & \bar{3} \\ \cline{1-3} \bar{4} & \bar{2} & \\ \cline{1-3} \bar{3} \\ \cline{1-1}\end{array} \underset{\text{contraction}}{\overset{\text{symplectic Knuth}}{\longrightarrow}}$$

$$\text{symplectic Knuth contraction} : 24\bar{2} \equiv 4 \rightarrow \begin{array}{|c|c|c|c|}\hline a & b & 2 & \\ \hline 4 & 4 & \bar{3} \\ \cline{1-3} \bar{4} & b' & \\ \cline{1-3} \bar{3} \\ \cline{1-1}\end{array} \overset{\text{SJDT}}{\rightarrow} \ldots \overset{\text{SJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline 2 & 4 & \bar{3} & \\ \hline 4 & a & b \\ \cline{1-3} \bar{4} & b' & \\ \cline{1-3} \bar{3} \\ \cline{1-1}\end{array}$$

$a < b < b'$

symplectic Knuth contraction: $24\bar{4}\bar{3} \equiv 2\bar{3}$

$$\overset{\text{SJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline c & 4 & \bar{3} & \\ \hline 2 & a & b \\ \cline{1-3} \bar{3} & b' & \\ \cline{1-3} c' \\ \cline{1-1}\end{array} \overset{\text{SJDT}}{\rightarrow} \begin{array}{|c|c|c|c|}\hline 2 & 4 & \bar{3} & \\ \hline \bar{3} & a & b \\ \cline{1-3} c & b' & \\ \cline{1-3} c' \\ \cline{1-1}\end{array}. \qquad \text{rect} P_{[\pm 2,4]} = \begin{array}{|c|c|c|c|}\hline 2 & 4 & \bar{3} & \\ \hline \bar{3} & & \\ \cline{1-3} & & \\ \cline{1-3} \\ \cline{1-1}\end{array}$$

$c < c' < a < b < b'$

Example: Colourful Letters

$$evac^C \begin{array}{cccc} 2 & 4 & \bar{3} & \\ \bar{3} & & & \\ & & & \\ & & & \end{array} = \begin{array}{cccc} 3 & 3 & \bar{2} & \\ \bar{4} & & & \\ & & & \\ & & & \end{array}$$

$c < c' < a < b < b'$

$$\begin{array}{cccc} 3 & 3 & \bar{2} & \\ \bar{4} & a & b & \\ c & b' & & \\ c' & & & \end{array} \overset{\mathrm{SJDT}^{-1}}{\longrightarrow} \begin{array}{cccc} c & 3 & \bar{2} & \\ 3 & a & b & \\ \bar{4} & b' & & \\ c' & & & \end{array} \overset{\mathrm{SJDT}^{-1}}{\longrightarrow} \begin{array}{cccc} 2 & 3 & \bar{2} & \\ 3 & a & b & \\ \bar{4} & b' & & \\ \bar{2} & & & \end{array} \overset{\mathrm{SJDT}^{-1}}{\longrightarrow} \begin{array}{cccc} a & 2 & \bar{2} & \\ 3 & 3 & b & \\ \bar{4} & b' & & \\ \bar{2} & & & \end{array}$$

$$\overset{\mathrm{SJDT}^{-1}}{\longrightarrow} \begin{array}{cccc} a & b & 3 & \\ 3 & 3 & \bar{3} & \\ \bar{4} & b' & & \\ \bar{2} & & & \end{array} \overset{\mathrm{SJDT}^{-1}}{\longrightarrow} \begin{array}{cccc} a & 2 & 3 & \\ 3 & 3 & \bar{3} & \\ \bar{4} & \bar{2} & & \\ \bar{2} & & & \end{array} \qquad \xi^C(P_{[\pm 2,4]}) = \begin{array}{cccc} & 2 & 3 & \\ 3 & 3 & \bar{3} & \\ \bar{4} & \bar{2} & & \\ \bar{2} & & & \end{array}$$

$$\xi^{C_4}_{[2,4]}(P) = \begin{array}{cccc} 1 & 2 & 3 & \bar{1} \\ 3 & 3 & \bar{3} & \\ \bar{4} & \bar{2} & \bar{1} & \\ \bar{2} & & & \end{array}$$

The Berenstein–Kirillov group

The *Berenstein–Kirillov group* $\mathcal{BK}$ (*Gelfand-Tsetlin group*) [Berenstein, Kirillov, 1995], is the free group generated by the Bender-Knuth involutions t_i, for $i > 0$, modulo the relations they satisfy on `straight shaped semistandard Young tableaux`.

$$t_1 \begin{array}{ccccccc} \cdot & \cdot & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 2 \\ 3 \end{array} = \begin{array}{ccccccc} \cdot & \cdot & 1 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 \\ 3 \end{array} \neq \xi_1 \begin{array}{ccccccc} \cdot & \cdot & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 2 \\ 3 \end{array} = \begin{array}{ccccccc} \cdot & \cdot & 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 2 \\ 3 \end{array}$$

Proposition

[Berenstein–Kirillov, 1995] *Let $\mathcal{BK}_n$ be the subgroup of $\mathcal{BK}$ generated by $t_1, \ldots, t_{n-1}$.*

- *The elements $q_{[1,1]}, \ldots, q_{[1,n-1]}$ are generators of $\mathcal{BK}_n$, $q_{[1,i]} = \xi_{[1,i]}$, $i \geq 1$.*
- $t_1 = q_{[1,1]}, \qquad t_i = q_{[1,i-1]} q_{[1,i]} q_{[1,i-1]} q_{[1,i-2]}$, *for* $i \geq 2$, $\quad q_{[1,0]} := 1$.

- The following are group epimorphisms from J_n to $\mathcal{BK}_n$.
  1. $s_{[i,j]} \mapsto q_{[i,j]}$ [Chmutov–Glick–Pylyavskii 2016].
  2. $s_{[1,j]} \mapsto q_{[1,j]}$ [Halacheva 2016, 2020].

  The group $\mathcal{BK}_n$ is isomorphic to a quotient of J_n.

The type C Berenstein–Kirillov group $\mathcal{BK}^C$

Definition (A–Tarighat–Torres 2022)

The *symplectic Berenstein–Kirillov group* $\mathcal{BK}_n^C$, $n \geq 1$, is the free group generated by the $2n-1$ symplectic partial Schützenberger-Lusztig involutions

$$q_{[1,i]}^C =: \xi_{[1,i]}^{C_n},\ 1 \leq i < n, \quad \text{and} \quad q_{[i,n]}^C =: \xi_{[i,n]}^{C_n},\ 1 \leq i \leq n,$$

on straight shaped KN tableaux on the alphabet $[\pm n]$ modulo the relations they satisfy on those tableaux.

- [A–Tarighat–Torres 2022] The following is a group epimorphism from $J_{\mathfrak{sp}_{2n}}$ to $\mathcal{BK}_n^C$:

$$s_{[1,j]} \mapsto q_{[1,j]}^{C_n},\ 1 \leq j < n, \qquad s_{[j,n]} \mapsto q_{[j,n]}^C,\ 1 \leq j \leq n.$$

  $\mathcal{BK}_n^C$ is isomorphic to a quotient of $J_{\mathfrak{sp}_{2n}}$.

- [A–Tarighat–Torres 2022] For $n \geq 1$, the *symplectic Bender–Knuth involutions* $t_i^{C_n}$, $1 \leq i \leq 2n-1$, on straight shaped KN tableaux on the alphabet $[\pm n]$, are defined as

$$\begin{aligned} t_i^{C_n} &:= q_{[1,i-1]}^{C_n} q_{[1,i]}^{C_n} q_{[1,i-1]}^{C_n} q_{[1,i-2]}^{C_n} = E^{-1} t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} E,\ \ 1 \leq i \leq n-1, \\ \tilde{t}_{2n-i}^{A_{2n-1}} &:= q_{[1,2n-1]}^{A_{2n-1}} t_i^{A_{2n-1}} q_{[1,2n-1]}^{A_{2n-1}}\ \ 1 \leq i \leq n-1, \\ t_{n-1+i}^{C_n} &:= q_{[n-i+1,n]}^{C_n} q_{[n-i+2,n]}^{C_n} = E^{-1} q_{[n-(i-1),n+(i-1)]}^{A_{2n-1}} q_{[n-(i-2),n+(i-2)]}^{A_{2n-1}} E,\ \ 1 \leq i \leq n. \end{aligned}$$

  The symplectic Bender-Knuth involutions $t_i^{C_n}$, $1 \leq i \leq 2n-1$ also generate $\mathcal{BK}_n^C$.

- $q_{[1,n-1]}^{C_n} = t_1^{C_n}(t_2^{C_n} t_1^{C_n}) \cdots (t_{n-1}^{C_n} t_{n-2}^{C_n} \cdots t_1^{C_n}), \qquad q_{[1,n]}^{C_n} = t_{2n-1}^{C_n} t_{2n-2}^{C_n} \cdots t_n^{C_n}.$

List of relations for $\mathcal{BK}_n^C$

$$(t_i^{C_n})^2 = 1, \;\; i = 1, \dots, n, \dots, 2n-1,$$
$$(t_i^{C_n} t_j^{C_n})^2 = 1, \;\; |i-j| > 1, 1 \leq i, j < n,$$
$$(t_{n+i-1}^{C_n} t_{n+j-1}^{C_n})^2 = 1, \;\; 1 \leq i, j \leq n,$$
$$(t_i^{C_n} t_{n+j-1}^{C_n})^2 = 1, \;\; i < n-j,$$
$$(t_1^{C_n} t_2^{C_n})^6 = 1,$$
$$(t_i^{C_n} q_{[j,k-1]}^{C_n})^2 = 1, \;\; i+1 < j < k \leq n,$$
$$(t_i^{C_n} q_{[j,n]}^{C_n})^2 = 1, \;\; i+1 < j \leq n,$$
$$(t_{n+i-1}^{C_n} q_{[j,n]}^{C_n})^2 = 1, \;\; 1 \leq i, j \leq n,$$
$$(t_{n+i-1}^{C_n} q_{[j,k-1]}^{C_n})^2 = 1, \;\; n-i+1 < j < k \leq n.$$