The combinatorics of (k, l)-lecture hall partitions

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SLC 87

Finite sequences and integer partitions

Let λ be a finite sequence $(\lambda_1, \ldots, \lambda_t)$ of non-negative integers.

- The parts: $\lambda_1, \ldots, \lambda_t$.
- The weight: $|\lambda| = \lambda_1 + \cdots + \lambda_t$.
- The odd weight: $|\lambda|_o = \sum_{i \text{ odd}} \lambda_i$.
- The even weight: $|\lambda|_e = \sum_{i \text{ even}} \lambda_i$.

Partition of n: λ such that $\lambda_1 \geq \cdots \geq \lambda_t \geq 1$ and $|\lambda| = n$.

Theorem 1: Distinct-odd identity (Euler)

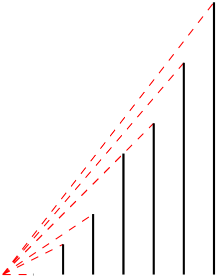
Let n be a non-negative integer. Then, the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts. The corresponding identity is

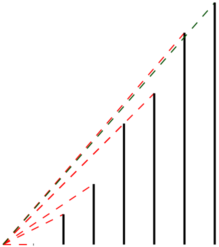
$$\prod_{n\geq 1}(1-q^n) = \prod_{n\geq 1} \frac{1}{1-q^{2n-1}}\,.$$

Partitions of 6 into distinct parts: $(6), (5,1), (4,2), (3,2,1)$.
Partitions of 6 into odd parts: $(5,1), (3,3), (3,1,1,1), (1,1,1,1,1,1)$.

Lecture-hall partitions

Let n be a positive integer.
Set of lecture-hall partitions $\mathcal{L}_n$: sequences $\lambda = (\lambda_1, \ldots, \lambda_n)$ of non-negative integers, such that $\left(\frac{\lambda_i}{i}\right)_{i=1}^{n}$ is non-decreasing. Example: $(0, 1, 2, 4, 5, 7, 9) \in \mathcal{L}_7$ but $(0, 1, 2, 4, 5, 7, 8) \notin \mathcal{L}_7$.





Bousquet-Mélou–Eriksson's refinement of Euler's theorem

Theorem 2: Lecture-hall theorem (Bousquet-Mélou and Eriksson 1997)

Let m be a non-negative integer. Then, the number of sequences in $\mathcal{L}_n$ with weight m is equal to the number of partitions of m into odd parts less than $2n$. The corresponding identity is

$$\sum_{\lambda \in \mathcal{L}_n} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1 - q^{2i-1}}.$$

We have

$$\{\lambda \text{ partitions into distinct parts}\} \equiv \lim_{n \to \infty} \mathcal{L}_n.$$

By tending n to ∞, the Lecture-hall theorem gives the distinct-odd theorem.

The (k, l)-sequence

Let k, l be positive integers such that $kl \geq 4$.
The (k, l)-sequence $\left(a_n^{(k,l)}\right)_{n\in\mathbb{Z}}$ is such that

$$\begin{cases} a_{2n}^{(k,l)} = l a_{2n-1}^{(k,l)} - a_{2n-2}^{(k,l)}, \\ a_{2n+1}^{(k,l)} = k a_{2n}^{(k,l)} - a_{2n-1}^{(k,l)}, \end{cases} \tag{1}$$

for $n \in \mathbb{Z}$, with $a_i^{(k,l)} = i$ for $i \in \{0, 1\}$.
Set $u_{kl} = \frac{\sqrt{kl}+\sqrt{kl-4}}{2}$, and for $n \in \mathbb{Z}$, set $s_{2n+1}^{(k,l)} = u_{kl}^{-2n}$ and $s_{2n}^{(k,l)} = \sqrt{l/k} \cdot u_{kl}^{-2n+1}$.
The sequence $\left(s_n^{(k,l)}\right)_{n\in\mathbb{Z}}$ satisfies (1).

The (k, l)-lecture-hall partitions

Let n be a positive integer.
Set of (k, l)-lecture hall partitions $\mathcal{L}_n^{(k,l)}$: $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \geq 0$ and $\left(\frac{\lambda_i}{a_i^{(k,l)}}\right)_{i=1}^{n}$ is non-decreasing.
Set $b_n^{(k,l)} = a_n^{(k,l)} + a_{n-1}^{(l,k)}$. The set $\mathcal{B}_n^{(k,l)}$: sequences $\lambda = \left(b_{i_1}^{(k,l)}, \ldots, b_{i_t}^{(k,l)}\right)$ such that $1 \leq i_1 \leq \cdots \leq i_t \leq n$.

$$\mathcal{B}^{(k,l)} = \lim_{n\to\infty} \mathcal{B}_n^{(k,l)} .$$

Write $\lambda = \prod_{i\geq 1} \left(b_i^{(k,l)}\right)^{m_i}$ where m_i is the number of parts $b_i^{(k,l)}$ in λ.

The (k, l)-lecture-hall theorem

Theorem 3: The (k, l)-lecture hall identity (Bousquet-Mélou and Eriksson 1997)

Let k, l, n be positive integers such that $kl \geq 4$. Then,

$$\sum_{\lambda \in \mathcal{L}_{2n}^{(k,l)}} x^{|\lambda|_o} y^{|\lambda|_e} = \prod_{i=1}^{2n} \frac{1}{1 - x^{a_{i-1}^{(l,k)}} y^{a_i^{(k,l)}}},$$

$$\sum_{\lambda \in \mathcal{L}_{2n-1}^{(k,l)}} x^{|\lambda|_o} y^{|\lambda|_e} = \prod_{i=1}^{2n-1} \frac{1}{1 - x^{a_i^{(l,k)}} y^{a_{i-1}^{(k,l)}}}.$$

This implies that, for a fixed weight $m \geq 0$, there are as many (k, l)-lecture hall partitions in $\mathcal{L}_{2n}^{(k,l)}$ as sequences in $\mathcal{B}_{2n}^{(k,l)}$, and there are as many (k, l)-lecture hall partitions in $\mathcal{L}_{2n-1}^{(k,l)}$ as sequences in $\mathcal{B}_{2n-1}^{(l,k)}$.

The (k, l)-Euler theorem

The set of (k, l)-Euler partitions $\mathcal{L}^{(k,l)}$: $\lambda = (\lambda_1, \ldots, \lambda_{2t})$ such that $0 = \lambda_{2t} \leq \lambda_{2t-1}$ and for $1 \leq i \leq t-1$,

$$s_0^{(l,k)} \cdot \lambda_{2i+1} < \lambda_{2i} < \left(s_0^{(k,l)}\right)^{-1} \cdot \lambda_{2i-1}.$$

Theorem 4: The (k, l)-Euler identity (Bousquet-Mélou and Eriksson)

Let k, l be positive integers such that $kl \geq 4$. Then,

$$\sum_{\lambda \in \mathcal{L}^{(k,l)}} x^{|\lambda|_o} y^{|\lambda|_e} = \prod_{i=1}^{\infty} \frac{1}{1 - x^{a_i^{(k,l)}} y^{a_{i-1}^{(l,k)}}}.$$

This implies that, for fixed weight $m \geq 0$, there are as many (k, l)-Euler partitions in $\mathcal{L}^{(k,l)}$ as sequences in $\mathcal{B}^{(k,l)}$.

We have

$$\mathcal{L}^{(k,l)} \equiv \lim_{n \to \infty} \mathcal{L}_{2n}^{(k,l)}.$$

Hence, by tending n to ∞, the (k, l)-Lecture-hall theorem gives the (k, l)-Euler theorem.

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- Recursive analytic proof of the (k, l)-lecture hall theorem (BME), that induces a recursive bijective proof.
- Proof of the (k, l)-Euler theorem from the limit of the (k, l)-lecture hall.
- In the case $k = l \geq 2$, bijective proof of l-lecture hall theorem and l-Euler theorem (Savage and Yee 2008), and a conjectured bijection for the case $k, l \geq 2$, and a conjecture that the BME recursive bijection and the SY bijection are the same

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What we bring to the table.

- Proof of the conjectured bijection for $k, l \geq 2$, and construction of the bijection for the case $k = 1$ and the case $l = 1$.
- Proof that the BME recursive bijection and our bijection are the same in all the cases for the (k, l)-lecture hall theorem.
- Construction of a recursive bijection for the (k, l)-Euler theorem.

The map $\Phi^{(k,l)}$ from $\mathcal{B}^{(k,l)}$ to $\mathcal{L}^{(k,l)}$

Let $\nu = (b_{i_1}^{(k,l)}, \ldots, b_{i_r}^{(k,l)}) \in \mathcal{B}^{(k,l)}$ and set $\lambda = (\lambda_i)_{i\geq 1}$ an infinite sequence of terms all equal to 0. Proceed by inserting the parts $b_i^{(k,l)}$ into the pairs $(\lambda_{2j-1}, \lambda_{2j})$, starting from the smallest j and the greatest i.

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- To insert $b_i^{(k,l)}$ with $i > 1$ into $(\lambda_{2j-1}, \lambda_{2j})$: if

$$\lambda_{2j-1} - s_0^{(k,l)} \cdot \lambda_{2j} > s_{i-1}^{(k,l)} - s_i^{(k,l)},$$

then do

$$(\lambda_{2j-1}, \lambda_{2j}) \mapsto (\lambda_{2j-1} + a_i^{(k,l)} - a_{i-1}^{(k,l)}, \lambda_{2j} + a_{i-1}^{(l,k)} - a_{i-2}^{(l,k)}) \quad (1)$$

and store $b_{i-1}^{(k,l)}$ for the insertion into the pair $(\lambda_{2j+1}, \lambda_{2j+2})$. Else, do

$$(\lambda_{2j-1}, \lambda_{2j}) \mapsto (\lambda_{2j-1} + a_i^{(k,l)}, \lambda_{2j} + a_{i-1}^{(l,k)}). \quad (2)$$

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$$\lambda_{2j-1} - s_0^{(k,l)} \cdot \lambda_{2j} > s_{i-1}^{(k,l)} - s_i^{(k,l)},$$

  then do

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  and store $b_{i-1}^{(k,l)}$ for the insertion into the pair $(\lambda_{2j+1}, \lambda_{2j+2})$. Else, do

$$(\lambda_{2j-1}, \lambda_{2j}) \mapsto (\lambda_{2j-1} + a_i^{(k,l)}, \lambda_{2j} + a_{i-1}^{(l,k)}). \tag{2}$$

- To insert $b_1^{(k,l)}$: do (2) for $i = 1$.

After all the insertions, we set $\Phi^{(k,l)}(\nu) = (\lambda_j)_{j=1}^{2t}$ where t is the smallest positive j such that $\lambda_{2j} = 0$.

The (k, l)-admissible words

Set $o_{2i-1}^{(k,l)} = l-2$ and $o_{2i}^{(k,l)} = k-2$ for $i \geq 1$. A (k,l)-admissible word is a sequence $(c_i)_{i\geq 1}$ of non-negative integers such that :

- there are finitely many positive terms,
- $c_i \in \{0, \dots, o_i^{(k,l)} + 1\}$,
- there is no pair $1 \leq i < j$ such that

$$c_h = o_h^{(k,l)} + \chi(h \in \{i,j\}) \qquad \text{for} \quad h \in \{i, i+1, \dots, j\}.$$

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$$c_h = o_h^{(k,l)} + \chi(h \in \{i, j\}) \qquad \text{for} \quad h \in \{i, i+1, \dots, j\}.$$

Let $\mathcal{C}^{(k,l)}$ be the set of (k, l)-admissible words. Let $n \geq 1$.
The set ${}_n\mathcal{C}^{(k,l)}$: (k, l)-admissible words with the $(n - 1)$ first terms equal to 0.
${}_n(c_i)_{i\geq 1}$: replace $c_1, \dots, c_{n-1}$ by 0.

Order on (k, l)-admissible words

Let $\prec$ be the lexicographic strict order on the set of integer sequences:

$(c_i) \prec (d_i)$ if and only if there exists $n > 0$ such that $c_n < d_n$ and $c_i = d_i$ for $i > n$.

Proposition 1: Fraenkel's numeration system

The function

$$\Gamma_{(k,l)} : \mathcal{C}^{(k,l)} \to \mathbb{Z}_{\geq 0}$$
$$(c_i)_{i\geq 1} \mapsto \sum_{i\geq 1} c_i \cdot a_i^{(k,l)}$$

describes a bijection from $\mathcal{C}^{(k,l)}$ to $\mathbb{Z}_{\geq 0}$ and

$$(c_i) \prec (d_i) \Longleftrightarrow \Gamma_{(k,l)}((c_i)) < \Gamma_{(k,l)}((d_i)).$$

For all $m \in \mathbb{Z}_{\geq 0}$, we write $[m]^{(k,l)} = \Gamma_{(k,l)}^{-1}(m)$.

The transformation $0\cdot$

For $t \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and all integer sequence $c = (c_i)_{i=1}^{t}$, $0\cdot c$ denotes the sequence $d = (d_i)_{i=1}^{t+1}$ satisfying $d_1 = 0$ and $d_{i+1} = c_i$ for $1 \leq i \leq t$.

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Proposition 4: The shifting

Let $k, l \geq 2$. For positive integers n and $n+1 \geq j \geq 1$, $0\cdot$ induces a bijection from ${}_{n}\mathcal{C}^{(l,k)}$ to ${}_{n+1}\mathcal{C}^{(k,l)}$.

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Let $k, l \geq 2$. For positive integers n and $n+1 \geq j \geq 1$, $0\cdot$ induces a bijection from ${}_n\mathcal{C}^{(l,k)}$ to ${}_{n+1}\mathcal{C}^{(k,l)}$.

Proposition 7: Order in terms of (k, l)-admissible words

For a sequence $\lambda = (\lambda_1, \ldots, \lambda_{2t})$ such that $t \geq 1$, $0 = \lambda_{2t} \leq \lambda_{2t-1}$ and $\lambda_i > 0$ for $1 \leq i \leq 2t-2$,

$$\lambda \in \mathcal{L}^{(k,l)} \iff [\lambda_{2i-1}]^{(k,l)} \succeq 0\cdot[\lambda_{2i}]^{(l,k)} \succeq 00\cdot[\lambda_{2i+1}]^{(k,l)} \text{ for all } 1 \leq i \leq t-1.$$

The bijection in terms of (k, l)-admissible words

Let $(S, \preceq)$ be a countable and total ordered set. For $m \in \mathbb{Z}_{\geq 0}$, c is the m^{th} element that precedes d in S or d is the m^{th} element that follows c in S, if the intervalle $[c, d]$ have $m+1$ elements in S, and we note

$$d = \mathcal{F}(m, S, c) = \mathcal{F}(m, S) \cdot c.$$

We set the following notations.

- $\left(\lambda_{2j-1}^{(i)}, \lambda_{2j}^{(i)}\right)$: the pairs $(\lambda_{2j-1}, \lambda_{2j})$ after the insertion of all the parts $b_i^{(k,l)}$.
- $m_i^{(j)}$: the number of parts $b_i^{(k,l)}$ inserted into the pair $(\lambda_{2j-1}, \lambda_{2j})$.

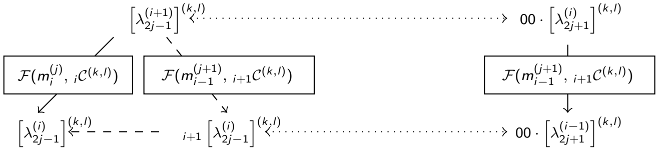
Hence, $m_i^{(1)}$ equals the number of occurrences of $b_i^{(k,l)}$ in ν, and the image of ν by $\Phi^{(k,l)}$ consists of $\left(\lambda_j^{(1)}\right)_{j=1}^{2t}$, where t is the smallest j such that $\lambda_{2j}^{(1)} = 0$.

The bijection in terms of (k, l)-admissible words

For $t \geq j \geq 1$,

- for $i \geq 2$, we have

$$\left[\lambda^{(i)}_{2j-1}\right]^{(k,l)} = 0\cdot \left[\lambda^{(i)}_{2j}\right]^{(l,k)} \in {}_{i}\mathcal{C}^{(k,l)}\,.$$



- Finally, $\lambda^{(1)}_{2j} = \lambda^{(2)}_{2j}$ and $\left[\lambda^{(1)}_{2j-1}\right]^{(k,l)} = \mathcal{F}\left(m^{(j)}_1, \mathcal{C}^{(k,l)}, \left[\lambda^{(2)}_{2j-1}\right]^{(k,l)}\right)$.
  Equivalently, this means that $m^{(j)}_1 = \lambda^{(1)}_{2j-1} - 1 - \left\lfloor s^{(k,l)}_0 \lambda^{(1)}_{2j} \right\rfloor$ if $\lambda^{(1)}_{2j} > 0$ and $m^{(j)}_1 = \lambda^{(1)}_{2j-1}$ if $\lambda^{(1)}_{2j} = 0$.

The little Göllnitz theorem

Theorem 5: Little Göllnitz' identities 1963

Let n be a non-negative integer. Then,

- ⋆ the number of partitions of n into parts differing by at least 2 and no consecutive odd parts equals the number of partitions of n into parts congruent to $1, 5, 6 \mod 8$,
- ⋆ the number of partitions of n into parts differing by at least 2, no consecutive odd parts, and no ones equals the number of partitions of n into parts congruent to $2, 3, 7 \mod 8$.

In terms of q-series, we have

$$\sum_{n\geq 0} \frac{(-q^{-1};q^2)_n q^{n^2+n}}{(q^2;q^2)_n} = \frac{1}{(q,q^5,q^6;q^8)_\infty},$$

$$\sum_{n\geq 0} \frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n} = \frac{1}{(q^2,q^3,q^7;q^8)_\infty},$$

where $(a_1,\ldots,a_t;q)_n = \prod_{i\geq 0}\prod_{j=1}^t (1-a_j q^i)$ for $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

The $(1,4)$ and $(4,1)$-Euler theorems

Theorem 6: The Savage–Sills identities 2011

Let n be a non-negative integer. Then,

- ⋆ the number of partitions of n into distinct parts such that the positive parts at even positions are even equals the number of partitions of n into parts congruent to $1, 5, 6 \mod 8$,
- ⋆ the number of partitions of n into distinct parts such that the positive parts at odd positions are even equals the number of partitions of n into parts congruent to $2, 3, 7 \mod 8$.

In terms of q-series, we have

$$\sum_{n\geq 0} \frac{(-q^{3-4\lceil n/2\rceil}; q^4)_{\lceil n/2\rceil} q^{n^2+n}}{(q^2;q^2)_n} = \frac{1}{(q,q^5,q^6;q^8)_\infty},$$

$$\sum_{n\geq 0} \frac{(-q^{1-4\lfloor n/2\rfloor}; q^4)_{\lfloor n/2\rfloor} q^{n^2+n}}{(q^2;q^2)_n} = \frac{1}{(q^2,q^3,q^7;q^8)_\infty}.$$

Open question

Theorem 7

We have

$$\sum_{n\geq 0} \frac{(-q^{-1};q^2)_n q^{n^2+n}}{(q^2;q^2)_n} = \sum_{n\geq 0} \frac{(-q^{3-4\lceil n/2\rceil};q^4)_{\lceil n/2\rceil} q^{n^2+n}}{(q^2;q^2)_n},$$

$$\sum_{n\geq 0} \frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n} = \sum_{n\geq 0} \frac{(-q^{1-4\lfloor n/2\rfloor};q^4)_{\lfloor n/2\rfloor} q^{n^2+n}}{(q^2;q^2)_n}.$$

Bijective proofs of the above identities induce bijective proofs of the little Göllnitz identities. How do we build them?

THANK YOU!!!

Example for $\Phi^{(k,l)}$ with $(k,l) = (3,2)$

$$\begin{aligned}\nu &= \left(b_1^{(3,2)}\right)^5 \left(b_2^{(3,2)}\right)^4 \left(b_3^{(3,2)}\right)^2 \left(b_4^{(3,2)}\right)^3 \left(b_5^{(3,2)}\right) \left(b_6^{(3,2)}\right)^3 \\ &= (1+0)^5(2+1)^4(5+3)^2(8+5)^3(19+12)(30+19)^3.\end{aligned}$$

For the insertion into the pair (λ_1, λ_2), we have the following.

- Insertions of $b_6^{(3,2)}$: we successively apply (2), (2) and (1) to obtain $(\lambda_1, \lambda_2) = (71, 45)$, and store once $b_5^{(3,2)}$ for the pair (λ_3, λ_4).
- Insertions of $b_5^{(3,2)}$: we apply (2) to obtain $(\lambda_1, \lambda_2) = (90, 57)$.
- Insertions of $b_4^{(3,2)}$: we successively apply (2), (1) and (2) to obtain $(\lambda_1, \lambda_2) = (109, 69)$, and store once $b_3^{(3,2)}$ for the pair (λ_3, λ_4).
- Insertions of $b_3^{(3,2)}$: we successively apply (1) and (2) to obtain $(\lambda_1, \lambda_2) = (117, 74)$, and store once $b_2^{(3,2)}$ for the pair (λ_3, λ_4).
- Insertions of $b_2^{(3,2)}$: we successively apply (2), (1) , (2) and (2) to obtain $(\lambda_1, \lambda_2) = (124, 78)$, and store once $b_1^{(3,2)}$ for the pair (λ_3, λ_4).
- Insertions of $b_1^{(3,2)}$: we apply five times (2) to obtain $(\lambda_1, \lambda_2) = (129, 78)$.

Hence, we store once $b_5^{(3,2)}, b_3^{(3,2)}, b_2^{(3,2)}, b_1^{(3,2)}$ for the insertion into the pair (λ_3, λ_4). We then do (2) for $i = 5, 3, 2, 1$ to obtain $(\lambda_3, \lambda_4) = (27, 16)$. As there is no part stored for the insertion in (λ_5, λ_6), we have $(\lambda_5, \lambda_6) = (0, 0)$. Set $\Phi^{(3,2)}(\nu) = (129, 78, 27, 16, 0, 0) \in \mathcal{L}^{(3,2)}$.

Example for $\Phi^{(k,l)}$ with $(k, l) = (3, 2)$

i	$m_i^{(1)}$	$\left[\lambda_1^{(i)}\right]^{(3,2)}$	$m_i^{(2)}$	$\left[\lambda_3^{(i)}\right]^{(3,2)}$	$m_i^{(3)}$	$\left[\lambda_5^{(i)}\right]^{(3,2)}$
7	0	$(0,0,0,0,0,0,0,0,0,\dots)$	0	$(0,0,0,0,0,0,0,\dots)$	0	$(0,0,0,0,0,0,0,\dots)$
6	3	$(0,0,0,0,0,0,1,0,0,\dots)$	0	$(0,0,0,0,0,0,0,\dots)$	0	$(0,0,0,0,0,0,0,\dots)$
5	1	$(0,0,0,0,1,0,1,0,0,\dots)$	1	$(0,0,0,0,1,0,0,\dots)$	0	$(0,0,0,0,0,0,0,\dots)$
4	3	$(0,0,0,1,0,1,1,0,0,\dots)$	0	$(0,0,0,0,1,0,0,\dots)$	0	$(0,0,0,0,0,0,0,\dots)$
3	2	$(0,0,1,0,0,0,0,1,0,\dots)$	1	$(0,0,1,0,1,0,0,\dots)$	0	$(0,0,0,0,0,0,0,\dots)$
2	4	$(0,2,0,1,0,0,0,1,0,\dots)$	1	$(0,1,1,0,1,0,0,\dots)$	0	$(0,0,0,0,0,0,0,\dots)$
1	5	$(1,0,0,2,0,0,0,1,0,\dots)$	1	$(0,0,0,1,1,0,0,\dots)$	0	$(0,0,0,0,0,0,0,\dots)$