Lattice paths and negatively indexed weight-dependent binomial coefficients

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joint work with
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Der Wissenschaftsfonds.

Motivation

By the end of the talk we want to be able to prove Frenkel and Turaev's ${}_{10}V_9$ summation from elliptic hypergeometric series

Theorem

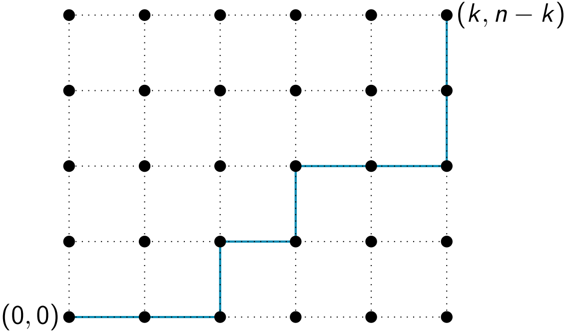
Let n be a nonnegative integer. When $bcde = a^2 q^{n+1}$, then

$$\frac{(aq, aq/bc, aq/bd, aq/cd; q, p)_k}{(aq/b, aq/c, aq/d, aq/bcd; q, p)_k} = \sum_{j=0}^{k} \frac{\theta(aq^{2j}; p)}{\theta(a; p)} \frac{(a, b, c, d, e, q^{-k}; q, p)_j}{(q, aq/b, aq/c, aq/d, aq/e, aq^{k+1}; q, p)_j} q^j$$

holds.

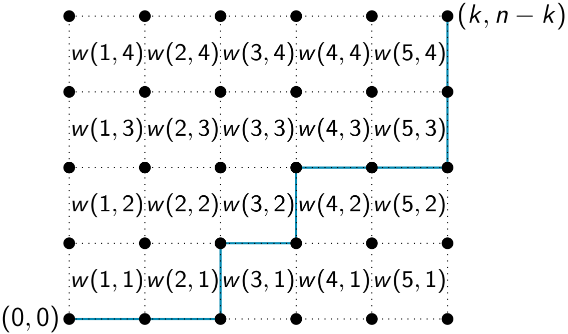
combinatorially.

Weight-dependent binomial coefficient



$$\binom{n}{k}$$

Weight-dependent binomial coefficient



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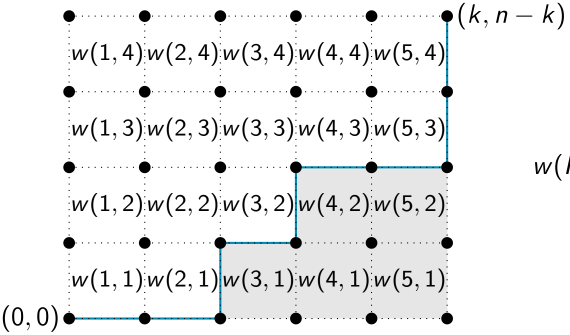
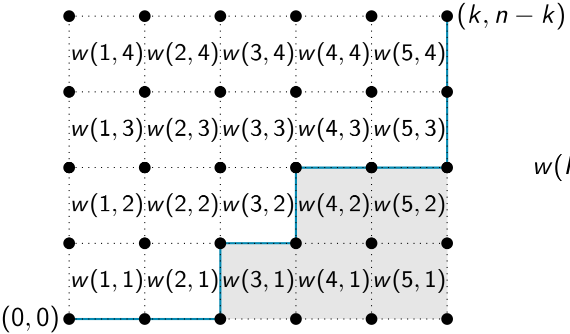


$$w(P) = w(3,1)w(4,1)w(4,2)w(5,1)w(5,2)$$

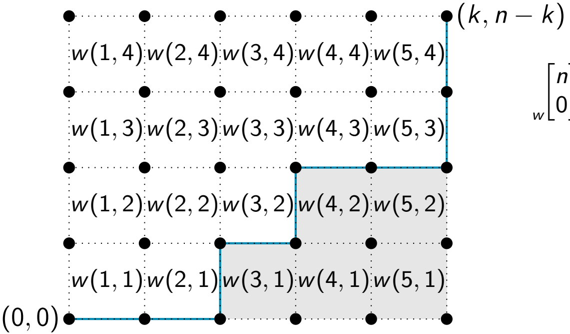
Weight-dependent binomial coefficient



$$w(P) = w(3,1)w(4,1)w(4,2)w(5,1)w(5,2)$$

Let ${}_w\begin{bmatrix} n \\ k \end{bmatrix} = \sum_P w(P)$ be the weighted counting of paths from $(0,0)$ to $(k, n-k)$.

Weight-dependent binomial coefficient



$${}_w\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$$

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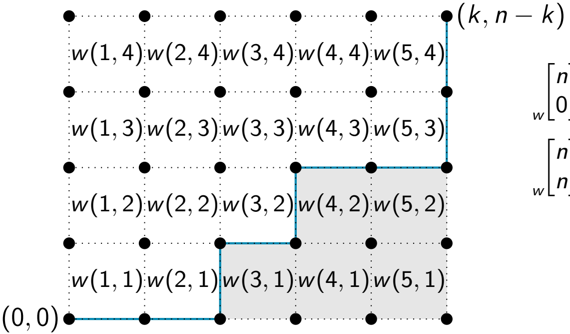

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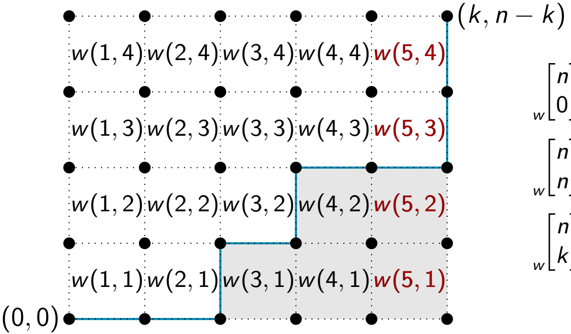

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Let ${}_w\begin{bmatrix} n \\ k \end{bmatrix} = \sum_P w(P)$ be the weighted counting of paths from $(0,0)$ to $(k, n-k)$.

Negative values

We want to define ${}_w\begin{bmatrix} n \\ k \end{bmatrix}$ for all integers $n, k \in \mathbb{Z}$.

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For the binomial coefficients (Loeb): For $n, k \in \mathbb{Z}$,

$$\binom{n}{k} = \lim_{a \mapsto 0} \frac{\Gamma(n+1+a)}{\Gamma(k+1+a)\Gamma(n-k+1+a)},$$

where $\Gamma(z)$ is the *gamma function*.

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where $\Gamma(z)$ is the *gamma function*.

For the q-binomial coefficients (Formichella–Straub): For $n, k \in \mathbb{Z}$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \lim_{a \mapsto q} \frac{(a;q)_n}{(a;q)_k (a;q)_{n-k}}$$

where $(a;q)_n$ is the *q-Pochhammer symbol*.

Negative values for binomial coefficients

For $n, k \in \mathbb{Z}$, the binomial coefficient satisfies $\binom{n}{k}$ satisfies the recursion

$$\binom{n}{0} = \binom{n}{n} = 1 \qquad \text{for } n \in \mathbb{Z} \ ,$$

and for $n, k \in \mathbb{Z}$, provided that $(n+1, k) \neq (0, 0)$,

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Weight-dependent binomial coefficients

For $n, k \in \mathbb{Z}$, the *weight-dependent binomial coefficient* ${}_w\begin{bmatrix} n \\ k \end{bmatrix}$ is defined recursively by

$$ {}_w\begin{bmatrix} n \\ 0 \end{bmatrix} = {}_w\begin{bmatrix} n \\ n \end{bmatrix} = 1 \qquad \text{for } n \in \mathbb{Z}\ , $$

and for $n, k \in \mathbb{Z}$, provided that $(n+1, k) \neq (0, 0)$,

$$ {}_w\begin{bmatrix} n+1 \\ k \end{bmatrix} = {}_w\begin{bmatrix} n \\ k \end{bmatrix} + {}_w\begin{bmatrix} n \\ k-1 \end{bmatrix} W(k, n+1-k) $$

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with

$$ W(s,t) = \prod_{j=1}^{t} w(s,j) = \begin{cases} w(s,1)w(s,2)w(s,3)\dots w(s,t) & \text{if } t > 0 \\ 1 & \text{if } t = 0 \\ w(s,0)^{-1}w(s,-1)^{-1}w(s,-2)^{-1}\dots w(s,t+1)^{-1} & \text{if } t < 0 \end{cases} . $$

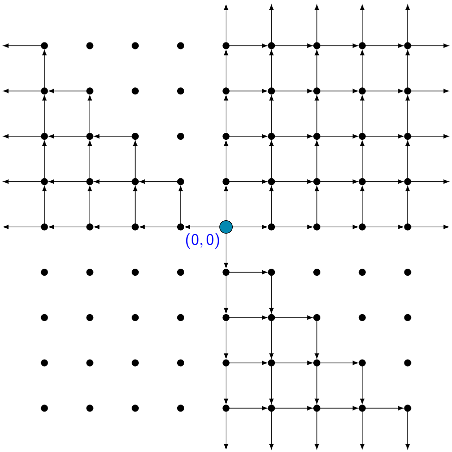
for a sequence of invertible weights $w(s,t)$.

Binomial coefficients ${}_w\begin{bmatrix} n+k \\ k \end{bmatrix}$

$n\backslash k$	-3	-2	-1	0	1	2
3	0	0	0	1	$W(1,1)+W(1,2)+W(1,3)+1$	$W(1,1)W(2,1)+W(2,1)+W(1,1)W(2,2)+W(1,2)W(2,2)+W(2,2)+\dots$
2	$\frac{1}{W(-2,2)W(-1,1)}$	0	0	1	$W(1,1)+W(1,2)+1$	$W(1,1)W(2,1)+W(2,1)+\dots$
1	$-\frac{W(-1,1)+1}{W(-2,1)W(-1,1)}$	$-\frac{1}{W(-1,1)}$	0	1	$W(1,1)+1$	$W(1,1)W(2,1)+W(2,1)+1$
0	1	1	1	1	1	1
-1	0	0	0	1	0	0
-2	0	0	0	1	$-W(1,-1)$	0
-3	0	0	0	1	$-W(1,-2)-W(1,-1)$	$W(1,-1)W(2,-2)$
-4	0	0	0	1	$-W(1,-3)-W(1,-2)-W(1,-1)$	$W(1,-2)W(2,-3)+W(1,-1)W(2,-3)+W(1,-1)W(2,-2)$

Lattice Path Model

${}_w\begin{bmatrix} n \\ k \end{bmatrix}$ counts lattice paths from $(0,0)$ to $(k, n-k)$ with general weights:



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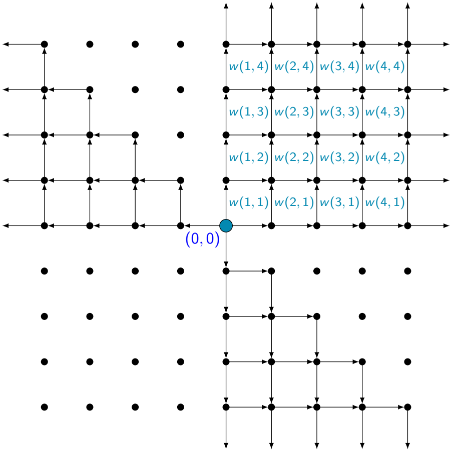

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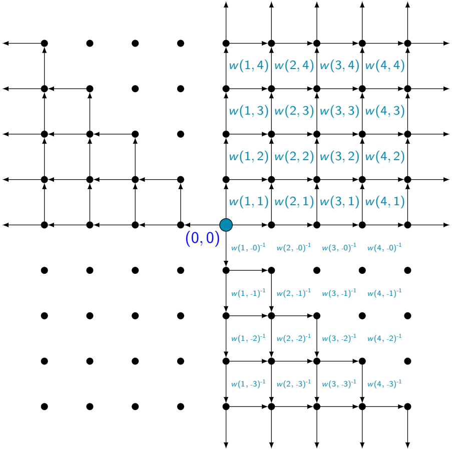

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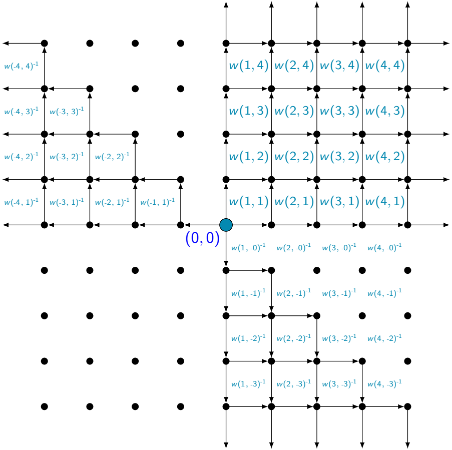

Let

$$\text{sgn}(n) = \begin{cases} 1 & n \geq 0 \\ -1 & n < 0 \end{cases}.$$

The weight of a path is defined as

$$w(P) = \prod_{(i,j)} w(i,j)^{sgn(i-1)sgn(j-1)}$$

where the product runs over all cells (i,j) between the path and the x-axis

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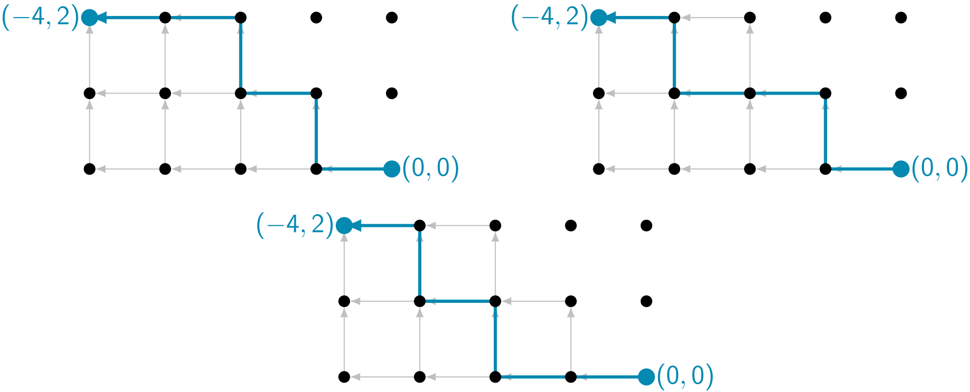
where the product runs over all cells (i,j) between the path and the x-axis and we have

$${}_w\begin{bmatrix} n \\ k \end{bmatrix} = \sum_P (-1)^P w(P)$$

where the sum runs over all paths from $(0,0)$ to $(k, n-k)$.

Example



$$
\begin{aligned}
{}_w\!\left[\begin{smallmatrix}-2\\-4\end{smallmatrix}\right] &= \big(w(-3,1)w(-3,2)w(-2,1)w(-1,1)\big)^{-1} \\
&+ \big(w(-3,1)w(-3,2)w(-2,1)w(-2,2)w(-1,1)\big)^{-1} \\
&+ \big(w(-3,1)w(-3,2)w(-2,1)\big)^{-1}
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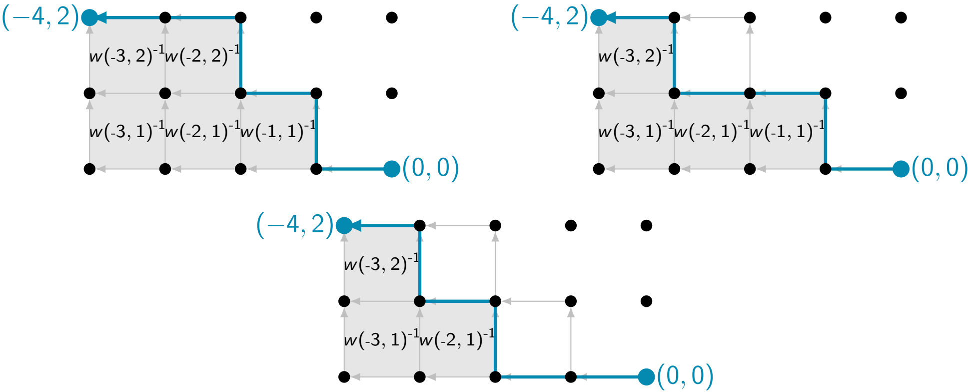

$$
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{}_w\!\begin{bmatrix} -2 \\ -4 \end{bmatrix} &= \left(w(-3,1)w(-3,2)w(-2,1)w(-1,1)\right)^{-1} \\
&+ \left(w(-3,1)w(-3,2)w(-2,1)w(-2,2)w(-1,1)\right)^{-1} \\
&+ \left(w(-3,1)w(-3,2)w(-2,1)\right)^{-1}
\end{aligned}
$$

Binomial theorem

Theorem (Noncommutative weight-dependent binomial theorem)

Suppose x and y are noncommuting variables satisfying the relations

$$
\begin{aligned}
yx &= w(1,1)xy, \\
xw(s,t) &= w(s+1,t)x, \\
yw(s,t) &= w(s,t+1)y,
\end{aligned}
$$

then for all $n \in \mathbb{Z}$:

$$(x+y)^n = \sum_{k \geq 0} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}, \tag{1}$$

$$(x+y)^n = \sum_{k \leq n} {}_w\begin{bmatrix} n \\ k \end{bmatrix} x^k y^{n-k}. \tag{2}$$

Convolution formula I

Corollary

Let $n, m \in \mathbb{Z}$ *and* $k \geq 0$*, then*

$$
{}_w\!\begin{bmatrix} n+m \\ k \end{bmatrix} = \sum_{j=0}^{k} {}_w\!\begin{bmatrix} n \\ j \end{bmatrix} \left(x^j y^{n-j} \, {}_w\!\begin{bmatrix} m \\ k-j \end{bmatrix} y^{j-n} x^{-j} \right) \prod_{i=1}^{k-j} W(i+j, n-j).
$$

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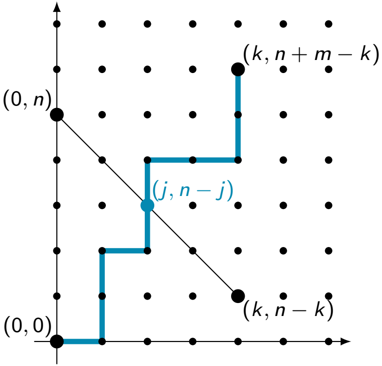
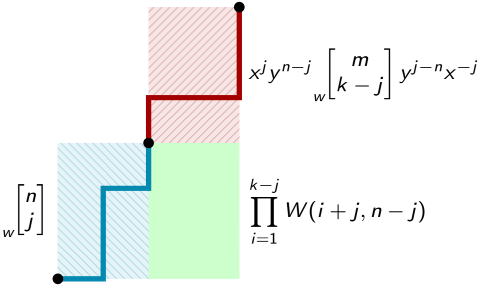
For $m, n \geq 0$, this identity can be proven by a convolution over weighted paths:



$\leftrightarrow$

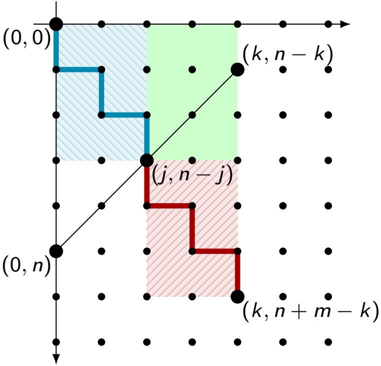


Corollary

Let $n, m \in \mathbb{Z}$ and $k \geq 0$, then

$$ {}_w\begin{bmatrix} n+m \\ k \end{bmatrix} = \sum_{j=0}^{k} {}_w\begin{bmatrix} n \\ j \end{bmatrix} \left(x^j y^{n-j} \, {}_w\begin{bmatrix} m \\ k-j \end{bmatrix} y^{j-n} x^{-j} \right) \prod_{i=1}^{k-j} W(i+j, n-j). $$

For $m, n < 0$, this identity can be proven by a convolution over weighted paths:



Convolution formula I

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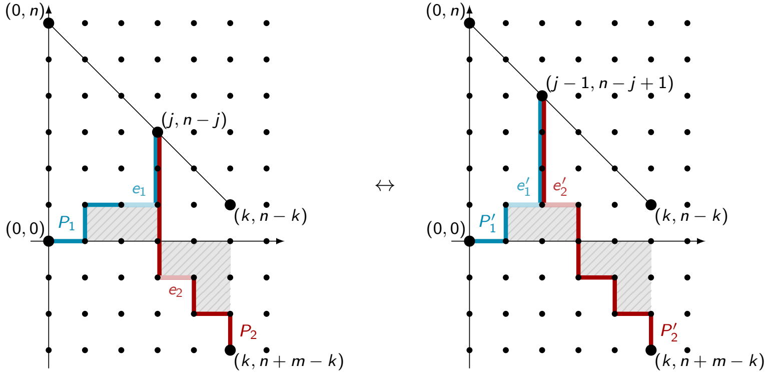
For mixed signs, the identity can be proven by a sign-reversing weight-preserving involution.



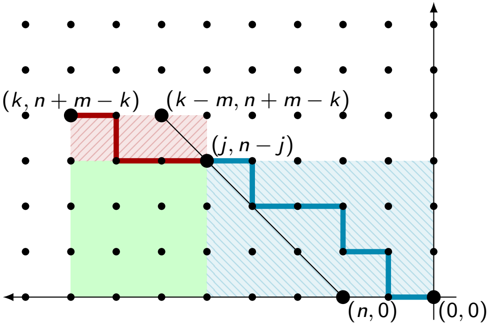
Convolution formula II

Corollary

Let $n, m \in \mathbb{Z}$ and $k \leq n+m$, then

$$ {}_w\begin{bmatrix} n+m \\ k \end{bmatrix} = \sum_{j=k-m}^{n} {}_w\begin{bmatrix} n \\ j \end{bmatrix} \left(x^j y^{n-j} \, {}_w\begin{bmatrix} m \\ k-j \end{bmatrix} y^{j-n} x^{-j} \right) \prod_{i=1}^{k-j} W(i+j, n-j). $$

For $m, n < 0$, this identity can be interpreted as convolution over weighted paths.



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Summary

What we know about the weight-dependent binomial coefficients with integer values:

- recursion formula
- lattice path model
- binomial theorem
- convolution formulas
- (reflection formulas)

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What we know about the weight-dependent binomial coefficients with integer values:

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These results transfer to all specializations of the weights.

Specializations

Choose $w(s,t) = 1$, then for $n, k \in \mathbb{Z}$,

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Choose $w(s,t)=q$, then for $n,k\in\mathbb{Z}$,

$$ {}_w\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q. $$

Symmetric functions

Choose $w(s,t) = \frac{a_{s+t}}{a_{s+t-1}}$. For $n, k \in \mathbb{Z}$ we have

$$_w\begin{bmatrix} n \\ k \end{bmatrix} = e_k(n) \prod_{i=1}^{k} a_i^{-1},$$

where $e_k(n)$ is the *elementary symmetric function* of order k in variables a_i.

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Choose $w(s,t) = \frac{a_{t+1}}{a_t}$. For $n, k \in \mathbb{Z}$ we have

$$ {}_w\begin{bmatrix} n \\ k \end{bmatrix} = h_k(n-k+1) \prod_{i=1}^{k} a_i^{-1}, $$

where $h_k(n)$ is the *complete homogeneous symmetric function* of order k in variables a_i.

Elliptic analogues

What about the combinatorial proof of Frenkel and Turaev's ${}_{10}V_9$ summation from elliptic hypergeometric series?

Theorem

Let n be a nonnegative integer. When $bcde = a^2q^{n+1}$, then

$$\frac{(aq, aq/bc, aq/bd, aq/cd; q, p)_k}{(aq/b, aq/c, aq/d, aq/bcd; q, p)_k} = \sum_{j=0}^{k} \frac{\theta(aq^{2j}; p)}{\theta(a; p)} \frac{(a, b, c, d, e, q^{-k}; q, p)_j}{(q, aq/b, aq/c, aq/d, aq/e, aq^{k+1}; q, p)_j} q^j$$

holds.

Elliptic analogues

Definition (Elliptic function)

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- Elliptic enumeration of *non-intersecting lattice paths* and *plane partitions* (Schlosser 2007, Borodin-Gorin-Rains 2010)
- Elliptic *binomial coefficients*, *Stirling numbers* of the first and second kind, *Lucas sequences*, *Lah numbers*, *rook numbers*, *Fibonacci numbers*, *Fibonomial numbers* ... (Kereskényiné Balogh, Schlosser, Yoo, Bergeron-Ceballos-K.)

Elliptic binomial coefficient

By choosing

$$w(s,t) = \frac{\theta(aq^{s+2t}, bq^{2s+t-2}, aq^{t-s-1}/b; p)}{\theta(aq^{s+2t-2}, bq^{2s+t}, aq^{t-s+1}/b; p)} q$$

we obtain the *elliptic binomial coefficient*

$$_w\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} := \frac{(q^{n-k+1}, aq^{n-k+1}, bq^{1+k}, \frac{a}{b}q^{1-k}; q, p)_k}{(q, aq, bq^{n+1}, \frac{a}{b}q^{n-2k+1}; q, p)_k},$$

where

$$\theta(x;p) := \prod_{j\geq 0}\left((1-p^j x)(1-\frac{p^{j+1}}{x})\right), \quad \theta(x_1,\dots,x_\ell;p) = \prod_{k=1}^{\ell}\theta(x_k;p),$$

and

$$(a;q,p)_k = \prod_{i=0}^{k-1}\theta(aq^i;p), \quad (a_1,a_2,\dots,a_l;q,p)_k = \prod_{i=1}^{l}(a_i;q,p)_k.$$

Elliptic binomial coefficient

The binomial theorem yields the noncommutative elliptic binomial theorem:

Theorem (Elliptic binomial theorem)

Suppose x and y are nonommuting variables satisfying the relations

$$\begin{aligned} yx &= \frac{\theta(aq^3, bq, a/bq; p)}{\theta(aq, bq^3, aq/b; p)} qxy, \\ xf(a,b) &= f(aq, bq^2)x, \\ yf(a,b) &= f(aq^2, bq)y, \end{aligned}$$

then for all $n \in \mathbb{Z}$:

$$\begin{aligned} (x+y)^n &= \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} x^k y^{n-k}, \\ (x+y)^n &= \sum_{k \leq n} \begin{bmatrix} n \\ k \end{bmatrix}_{a,b;q,p} x^k y^{n-k}. \end{aligned}$$

Elliptic binomial coefficient

And from the convolution formulas we obtain...

Corollary

Let $n, m \in \mathbb{Z}$ and $k \geq 0$, then

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_{a,b;q,p} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p} \begin{bmatrix} m \\ k-j \end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p} \prod_{i=1}^{k-j} W(i+j, n-j).$$

Corollary

Let $n, m \in \mathbb{Z}$ and $k \leq n+m$, then

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_{a,b;q,p} = \sum_{j=k-m}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p} \begin{bmatrix} m \\ k-j \end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p} \prod_{i=1}^{k-j} W(i+j, n-j).$$

Elliptic analogues

By substituting $(a, b, q^{-n}, q^{-m}, k) \mapsto (e/b, a/b, e, aq/bce, n)$ and manipulating the convolution (we proved combinatorially)

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_{a,b;q,p} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix}_{a,b;q,p} \begin{bmatrix} m \\ k-j \end{bmatrix}_{aq^{2n-j},bq^{n+j};q,p} \prod_{i=1}^{k-j} W(i+j, n-j)$$

we obtain Frenkel and Turaev's ${}_{10}V_9$ summation

$$\frac{(aq, aq/bc, aq/bd, aq/cd; q,p)_k}{(aq/b, aq/c, aq/d, aq/bcd; q,p)_k} = \sum_{j=0}^{k} \frac{\theta(aq^{2j};p)}{\theta(a;p)} \frac{(a,b,c,d,e,q^{-k};q,p)_j}{(q, aq/b, aq/c, aq/d, aq/e, aq^{k+1}; q,p)_j} q^j$$

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