Categorifying combinatorial Hopf algebras

Nat Thiem
University of Colorado Boulder

Joint with Farid Aliniaeifard
The University of British Columbia

Supported in part by the Simons Foundation

Some examples.

Some examples.

$\mathbf{NCSym}^*$

NCSym

FQSym

$\mathbf{CQSym}^*$

NSym

Sym

QSym

Some examples.

$\mathbf{NCSym}^*$

NCSym **FQSym**

$\mathbf{CQSym}^*$ **NSym**

Sym **QSym**

Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

Some examples.

$NCSym^*$

set partitions
NCSym

FQSym
permutations

integer compositions
NSym

$CQSym^*$
Dyck paths

Sym
integer partitions

QSym
integer compositions

Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

Some examples.

set partitions
NCSym^*

set partitions
NCSym

FQSym
permutations

integer compositions
NSym

CQSym^*
Dyck paths

Sym
integer partitions

QSym
integer compositions

Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

Combinatorial Hopf algebras

Some examples.

set partitions
$\mathbf{NCSym}^*$

set partitions
NCSym

FQSym
permutations

integer
compositions
NSym

$\mathbf{CQSym}^*$

Dyck paths

Sym
integer
partitions

QSym
integer
compositions

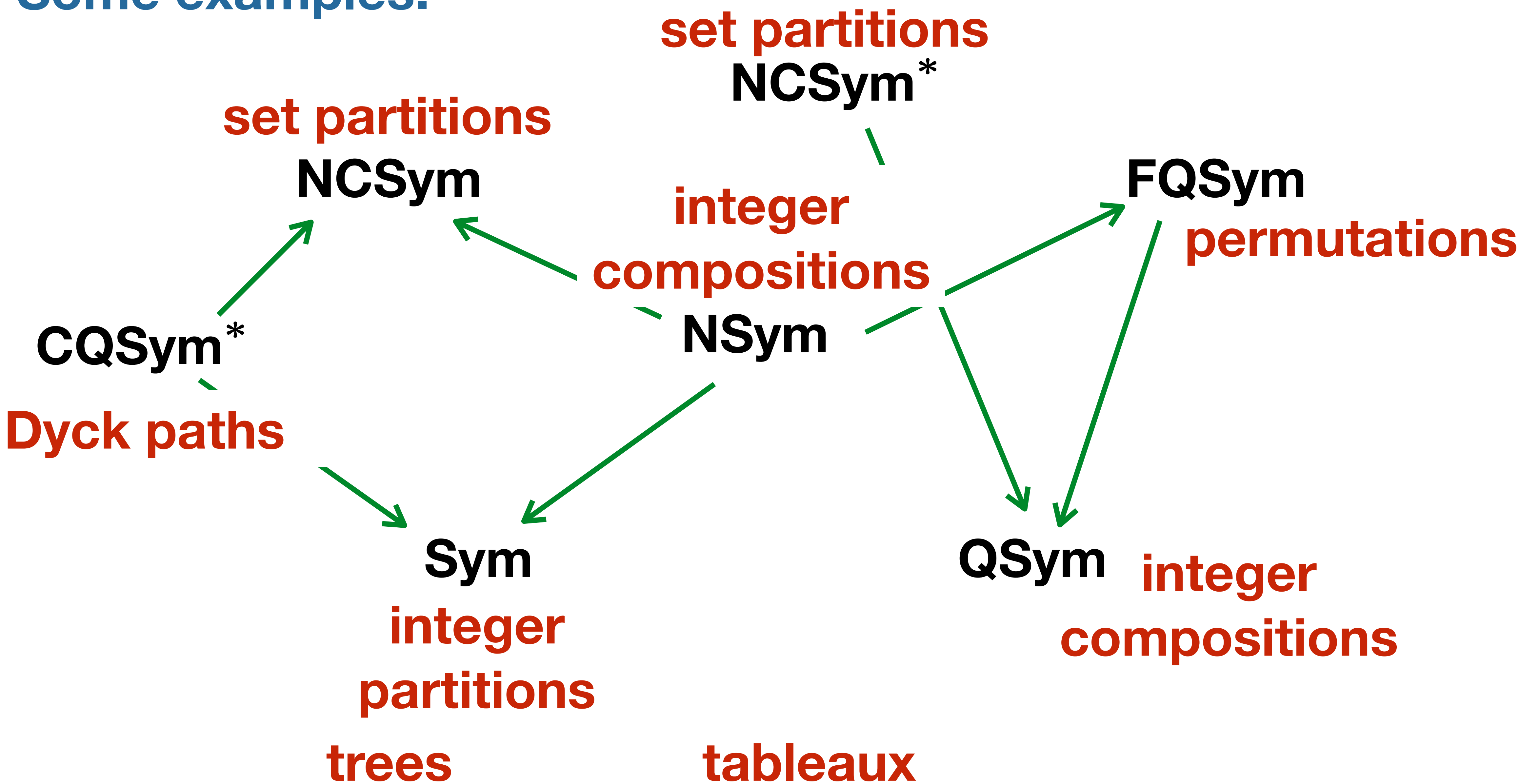
trees

tableaux

Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

Combinatorial Hopf algebras

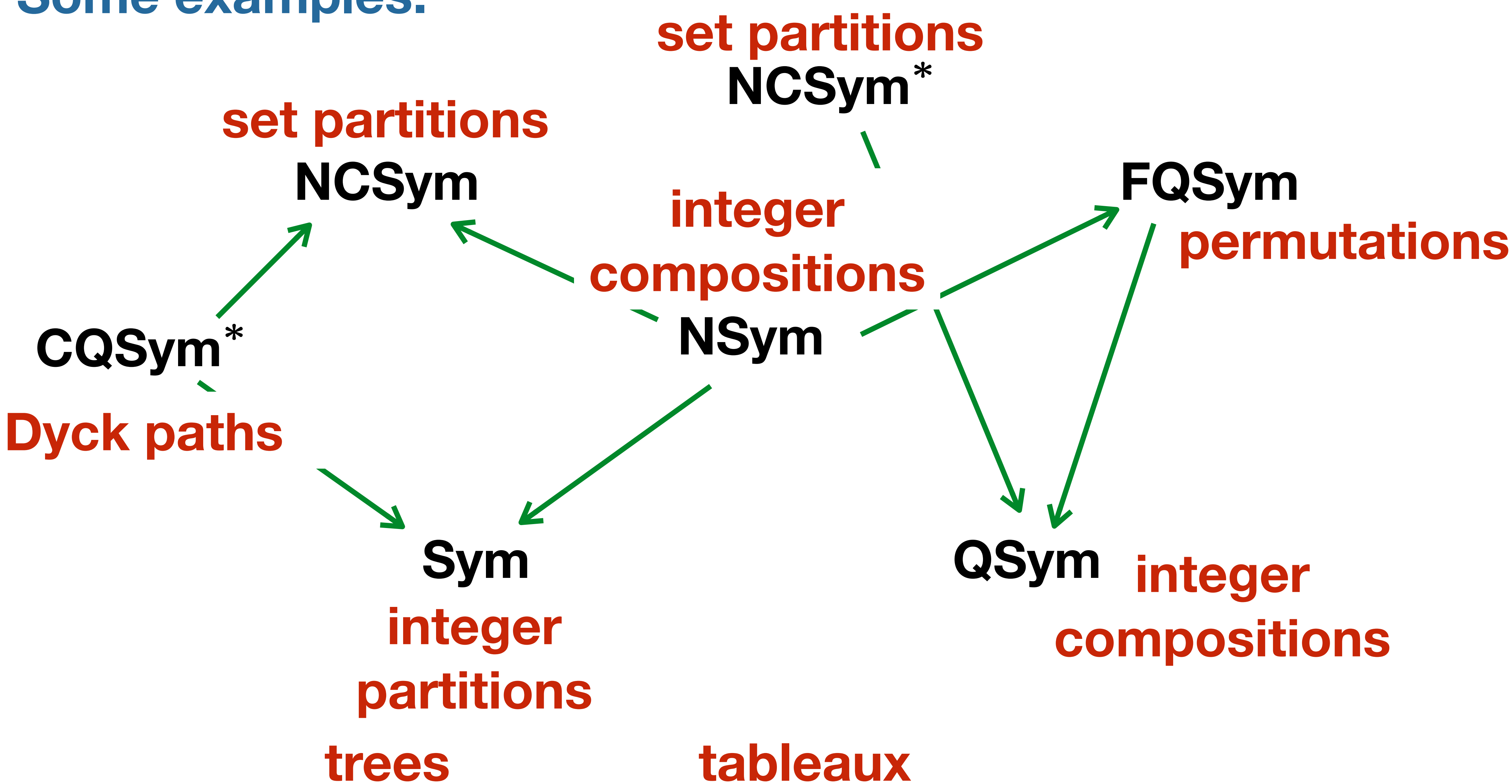
Some examples.



Also. **Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.**

Combinatorial Hopf algebras

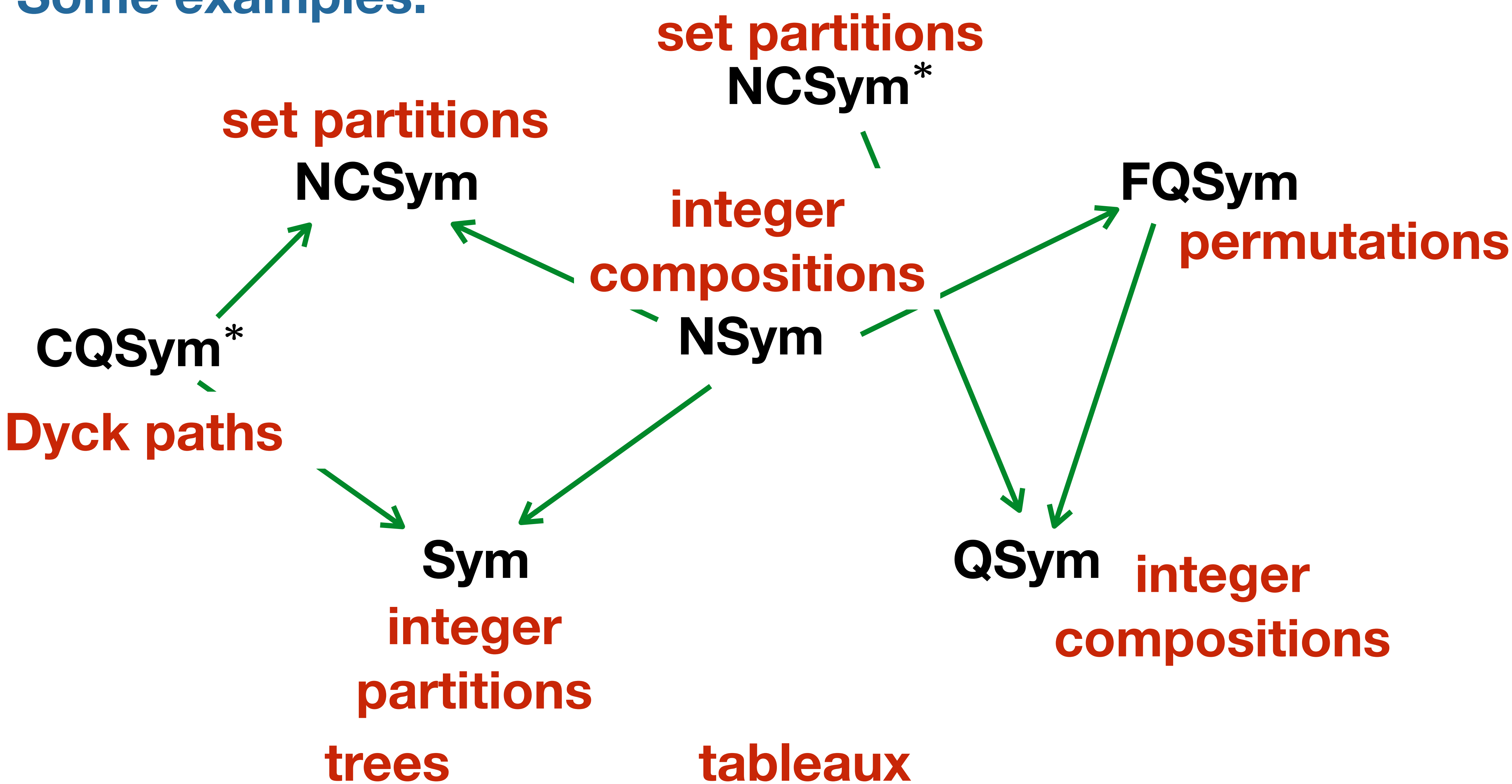
Some examples.



Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

Also. I believe my notation is regionally incorrect.

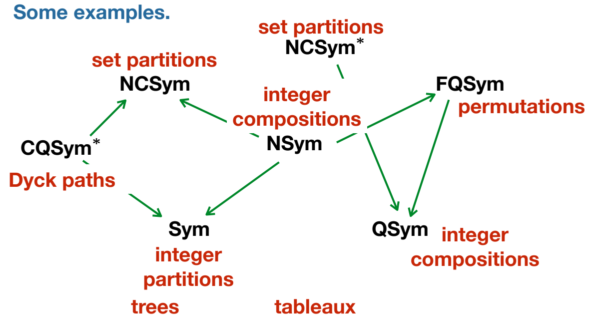
Some examples.



Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

Combinatorial Hopf algebras

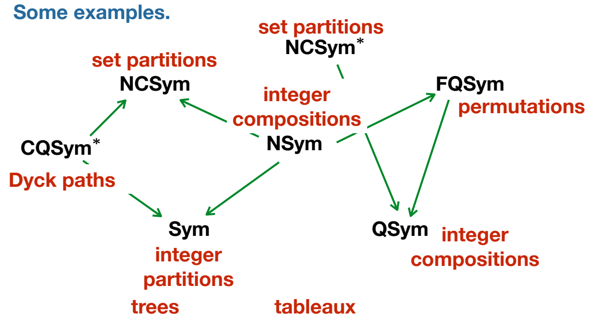
Some examples.



Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

General Structure.

Some examples.



Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

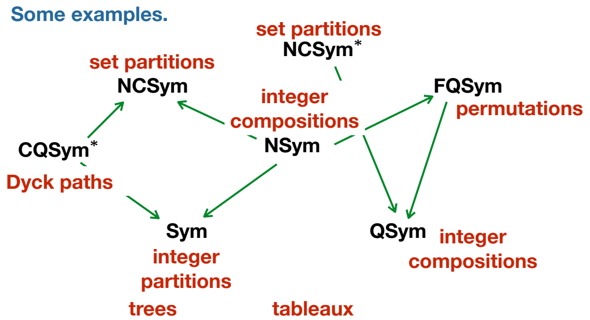
Combinatorial Hopf algebras

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

Some examples.



Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

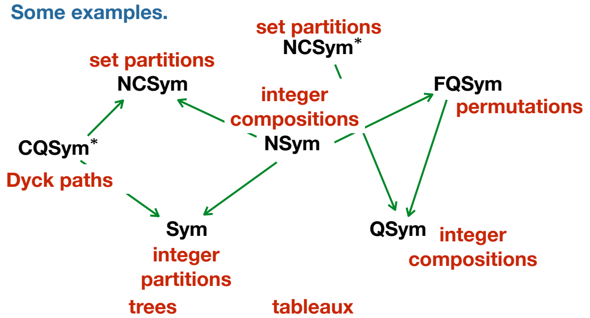
General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

basis of combinatorial objects

Some examples.



Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

Combinatorial Hopf algebras

General Structure.

. graded vector space

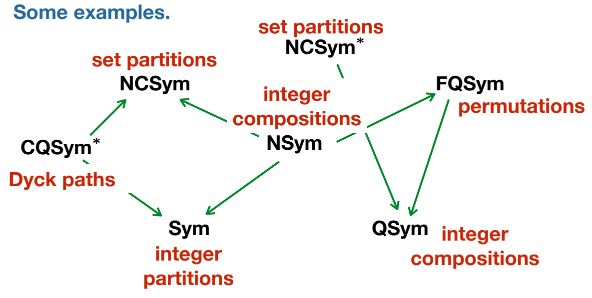
basis of combinatorial objects

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

Some examples.



Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

Combinatorial Hopf algebras

General Structure.

. graded vector space

basis of combinatorial objects

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

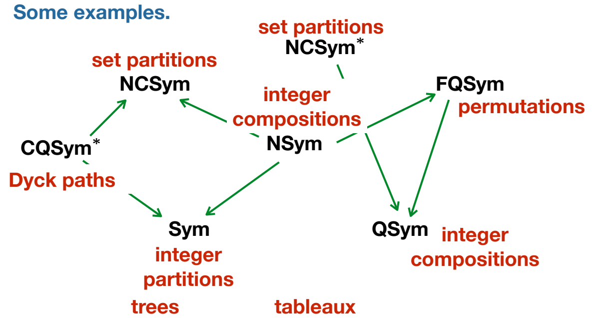
. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

compatible

(think functors + Mackey formula)

Some examples.



Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

Combinatorial Hopf algebras

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

basis of combinatorial objects

. bialgebra structure

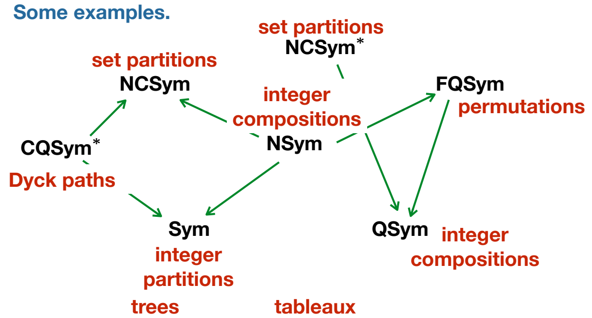
$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

compatible

(think functors + Mackey formula)

Problem. There are many choices of basis. Which are the good ones? Why?

Some examples.



Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

basis of combinatorial objects

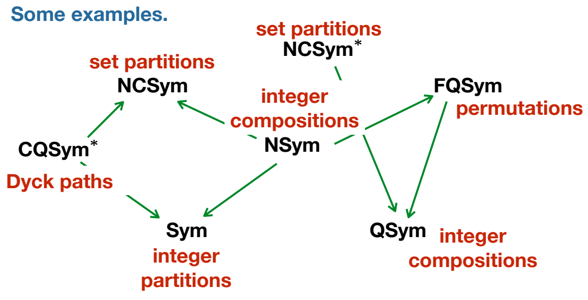
. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

compatible

(think functors + Mackey formula)

Some examples.



Also. Loday—Ronco, Poirier—Reutenauer, Reading, Lam—Pylyavsky, Connes—Kreimer, Steenrod, etc.

Problem. There are many choices of basis. Which are the good ones? Why?

Representation theory gives one approach to addressing these questions.

Combinatorial Hopf algebras

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

basis of combinatorial objects

Towards representation theory.

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

basis of combinatorial objects

Towards representation theory.

For an equivalence relation $\sim$ on a group G, let

$$\mathrm{f}_{\sim}(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\}$$

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

basis of combinatorial objects

Towards representation theory.

For an equivalence relation $\sim$ on a group G, let

$$\mathrm{f}_{\sim}(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\}$$

so

$$f_{=}(G) = \{\text{all functions}\} \supseteq \{\text{class function}\} = f_{\text{conjugacy}}(G).$$

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

basis of combinatorial objects

Towards representation theory.

For an equivalence relation $\sim$ on a group G, let

$$\mathsf{f}_{\sim}(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\}$$

so

$$f_{=}(G) = \{\text{all functions}\} \supseteq \{\text{class function}\} = f_{\text{conjugacy}}(G).$$

Given a tower of groups

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$$

with an associated equivalence relation $\sim$, let

$$\mathsf{f}_{\sim} = \bigoplus_{n \geq 0} \mathsf{f}_{\sim}(G_n).$$

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

basis of combinatorial objects

Towards representation theory.

For an equivalence relation $\sim$ on a group G, let

$$\mathsf{f}_{\sim}(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\}$$

so

$$f_{=}(G) = \{\text{all functions}\} \supseteq \{\text{class function}\} = f_{\text{conjugacy}}(G).$$

Given a tower of groups

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$$

with an associated equivalence relation $\sim$, let

$$\mathsf{f}_{\sim} = \bigoplus_{n \geq 0} \mathsf{f}_{\sim}(G_n).$$

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

basis of combinatorial objects

Towards representation theory.

For an equivalence relation $\sim$ on a group G, let

$$\mathsf{f}_\sim(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\}$$

so

$$f_=(G) = \{\text{all functions}\} \supseteq \{\text{class function}\} = f_{\text{conjugacy}}(G).$$

traditional choice

Given a tower of groups

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots$$

with an associated equivalence relation $\sim$, let

$$\mathsf{f}_\sim = \bigoplus_{n\geq 0} \mathsf{f}_\sim(G_n).$$

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

Towards representation theory.

$$\mathbf{f}_{\sim} = \bigoplus_{n \geq 0} \mathbf{f}_{\sim}(G_n).$$

General Structure.

$$\mathbf{f}_{\sim} = \bigoplus_{n \geq 0} \mathbf{f}_{\sim}(G_n).$$

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

Towards representation theory.

General Structure.

$$\mathbf{f}_\sim = \bigoplus_{n\geq 0} \mathbf{f}_\sim(G_n).$$

. graded vector space

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

Towards representation theory.

General Structure.

$$\mathsf{f}_{\sim} = \bigoplus_{n \geq 0} \mathsf{f}_{\sim}(G_n).$$

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

Towards representation theory.

Here we want compatible functors

$$I : \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n) \longrightarrow \mathsf{f}_{\sim}(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_{\sim}(G_{m+n}) \longrightarrow \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

General Structure.

- graded vector space

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n \qquad \mathsf{f}_\sim = \bigoplus_{n\geq 0} \mathsf{f}_\sim(G_n).$$

- bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

Towards representation theory.

Here we want compatible functors

$$I : \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n) \longrightarrow \mathsf{f}_\sim(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_\sim(G_{m+n}) \longrightarrow \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

$$\mathsf{f}_\sim = \bigoplus_{n \geq 0} \mathsf{f}_\sim(G_n).$$

Sym = symmetric functions

. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

Towards representation theory.

Here we want compatible functors

$$I : \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n) \longrightarrow \mathsf{f}_\sim(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_\sim(G_{m+n}) \longrightarrow \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

Traditional examples

General Structure.

$$\mathsf{f}_{\sim} = \bigoplus_{n\geq 0} \mathsf{f}_{\sim}(G_n).$$

. graded vector space

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

$\mathsf{f}_{\text{conjugacy}}(S_n)$

Sym = symmetric functions

. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

Towards representation theory.

Here we want compatible functors

$$I : \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n) \longrightarrow \mathsf{f}_{\sim}(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_{\sim}(G_{m+n}) \longrightarrow \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

Traditional examples

General Structure.

. graded vector space

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

$$\mathsf{f}_\sim = \bigoplus_{n\geq 0} \mathsf{f}_\sim(G_n).$$

Sym = symmetric functions

$\mathsf{f}_{\text{conjugacy}}(S_n)$

. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

Towards representation theory.

Here we want compatible functors

$I : \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n) \longrightarrow \mathsf{f}_\sim(G_{m+n})$ Induction

Restriction $R_{m,n} : \mathsf{f}_\sim(G_{m+n}) \longrightarrow \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n)$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

Traditional examples

General Structure.

. graded vector space

$$\mathsf{f}_{\sim} = \bigoplus_{n\geq 0} \mathsf{f}_{\sim}(G_n).$$

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

$\mathsf{f}_{\text{conjugacy}}(S_n)$

Sym = symmetric functions

. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

G_n	$\sim$	I	$R_{m,n}$	$\mathcal{H}$
S_n	conjugacy	$\text{Ind}_{S_m \times S_n}^{S_{m+n}}$	$\text{Res}_{S_m \times S_n}^{S_{m+n}}$	Sym
$G(r, 1, n)$	conjugacy	Ind	Res	$\bigotimes$ Sym
$\text{GL}_n(\mathbb{F}_q)$	conjugacy	Indf	Resf	$\bigotimes$ Sym
$\text{U}_n(\mathbb{F}_q)$	conjugacy	R	R*	$\bigotimes$ Sym

General Structure.

$$\mathsf{f}_\sim = \bigoplus_{n\geq 0} \mathsf{f}_\sim(G_n).$$

. graded vector space

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

$\mathsf{f}_{\text{conjugacy}}(S_n)$

Sym = symmetric functions

. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

G_n	$\sim$	I	$R_{m,n}$	$\mathcal{H}$
S_n	conjugacy	$\mathrm{Ind}_{S_m\times S_n}^{S_{m+n}}$	$\mathrm{Res}_{S_m\times S_n}^{S_{m+n}}$	Sym
$G(r,1,n)$	conjugacy	Ind	Res	$\bigotimes$ Sym
$\mathrm{GL}_n(\mathbb{F}_q)$	conjugacy	Indf	Resf	$\bigotimes$ Sym
$\mathrm{U}_n(\mathbb{F}_q)$	conjugacy	R	R*	$\bigotimes$ Sym

Deligne—Lusztig induction and restriction

General Structure.

$$\mathfrak{f}_\sim = \bigoplus_{n\geq 0} \mathfrak{f}_\sim(G_n).$$

. graded vector space

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

$\mathfrak{f}_{\text{conjugacy}}(S_n)$

Sym = symmetric functions

. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

G_n	$\sim$	I	$R_{m,n}$	$\mathcal{H}$
S_n	conjugacy	$\text{Ind}_{S_m \times S_n}^{S_{m+n}}$	$\text{Res}_{S_m \times S_n}^{S_{m+n}}$	Sym
$G(r,1,n)$	conjugacy	Ind	Res	$\bigotimes$ Sym
$\text{GL}_n(\mathbb{F}_q)$	conjugacy	Indf	Resf	$\bigotimes$ Sym
$\text{U}_n(\mathbb{F}_q)$	conjugacy	R	R*	$\bigotimes$ Sym

Deligne—Lusztig induction and restriction

Traditional examples

G_n	$\sim$	I	$R_{m,n}$	$\mathcal{H}$
S_n	conjugacy	$\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}$	$\mathrm{Res}_{S_m \times S_n}^{S_{m+n}}$	Sym
$G(r, 1, n)$	conjugacy	Ind	Res	$\bigotimes$ Sym
$\mathrm{GL}_n(\mathbb{F}_q)$	conjugacy	Indf	Resf	$\bigotimes$ Sym
$\mathrm{U}_n(\mathbb{F}_q)$	conjugacy	R	R*	$\bigotimes$ Sym

Traditional examples

G_n	$\sim$	I	$R_{m,n}$	$\mathcal{H}$
S_n	conjugacy	$\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}$	$\mathrm{Res}_{S_m \times S_n}^{S_{m+n}}$	Sym
$G(r, 1, n)$	conjugacy	Ind	Res	$\bigotimes$ Sym
$\mathrm{GL}_n(\mathbb{F}_q)$	conjugacy	Indf	Resf	$\bigotimes$ Sym
$\mathrm{U}_n(\mathbb{F}_q)$	conjugacy	R	R*	$\bigotimes$ Sym

Newer examples

G_n	$\sim$	I	$R_{m,n}$	$\mathcal{H}$
S_n	conjugacy	$\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}$	$\mathrm{Res}_{S_m \times S_n}^{S_{m+n}}$	Sym
$G(r,1,n)$	conjugacy	Ind	Res	$\bigotimes$ Sym
$\mathrm{GL}_n(\mathbb{F}_q)$	conjugacy	Indf	Resf	$\bigotimes$ Sym
$\mathrm{U}_n(\mathbb{F}_q)$	conjugacy	R	R^*	$\bigotimes$ Sym
$\mathrm{UT}_n(\mathbb{F}_q)$	super	Inf	Res	NCSym

$$\mathrm{UT}_n(\mathbb{F}_q) = \begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

Newer examples

G_n	$\sim$	I	$R_{m,n}$	$\mathcal{H}$
S_n	conjugacy	$\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}$	$\mathrm{Res}_{S_m \times S_n}^{S_{m+n}}$	Sym
$G(r, 1, n)$	conjugacy	Ind	Res	$\bigotimes$ Sym
$\mathrm{GL}_n(\mathbb{F}_q)$	conjugacy	Indf	Resf	$\bigotimes$ Sym
$\mathrm{U}_n(\mathbb{F}_q)$	conjugacy	R	R*	$\bigotimes$ Sym
$\mathrm{UT}_n(\mathbb{F}_q)$	super	Inf	Res	NCSym

Newer examples

G_n	$\sim$	I	$R_{m,n}$	$\mathcal{H}$
S_n	conjugacy	$\text{Ind}_{S_m \times S_n}^{S_{m+n}}$	$\text{Res}_{S_m \times S_n}^{S_{m+n}}$	Sym
$G(r, 1, n)$	conjugacy	Ind	Res	$\bigotimes$ Sym
$\text{GL}_n(\mathbb{F}_q)$	conjugacy	Indf	Resf	$\bigotimes$ Sym
$\text{U}_n(\mathbb{F}_q)$	conjugacy	R	R^*	$\bigotimes$ Sym
$\text{UT}_n(\mathbb{F}_q)$	super	Inf	Res	NCSym

and with Aliniaeifard,

$\text{UT}_n(\mathbb{F}_q)$	super	Inf	Res	CQSym^*

Newer examples

G_n	$\sim$	I	$R_{m,n}$	$\mathcal{H}$
S_n	conjugacy	$\text{Ind}_{S_m \times S_n}^{S_{m+n}}$	$\text{Res}_{S_m \times S_n}^{S_{m+n}}$	Sym
$G(r,1,n)$	conjugacy	Ind	Res	$\bigotimes$ Sym
$\text{GL}_n(\mathbb{F}_q)$	conjugacy	Indf	Resf	$\bigotimes$ Sym
$\text{U}_n(\mathbb{F}_q)$	conjugacy	R	R^*	$\bigotimes$ Sym
$\text{UT}_n(\mathbb{F}_q)$	super	Inf	Res	NCSym

and with Aliniaeifard,

$\text{UT}_n(\mathbb{F}_q)$	super	Inf	Res	CQSym^*
$\mathfrak{ut}_n(\mathbb{F}_q)$	super	Stfl	Dela	FQSym

Newer examples

G_n	$\sim$	I	$R_{m,n}$	$\mathcal{H}$
S_n	conjugacy	$\text{Ind}_{S_m \times S_n}^{S_{m+n}}$	$\text{Res}_{S_m \times S_n}^{S_{m+n}}$	Sym
$G(r, 1, n)$	conjugacy	Ind	Res	$\bigotimes$ Sym
$\text{GL}_n(\mathbb{F}_q)$	conjugacy	Indf	Resf	$\bigotimes$ Sym
$\text{U}_n(\mathbb{F}_q)$	conjugacy	R	R^*	$\bigotimes$ Sym
$\text{UT}_n(\mathbb{F}_q)$	super	Inf	Res	NCSym

and with Aliniaeifard,

$\text{UT}_n(\mathbb{F}_q)$	super	Inf	Res	CQSym^*
$\mathfrak{ut}_n(\mathbb{F}_q)$	super	Stfl	Dela	FQSym
G^{n-1}	super	various	various	NSym

Newer examples

G_n	$\sim$	I	$R_{m,n}$	$\mathcal{H}$
S_n	conjugacy	$\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}}$	$\mathrm{Res}_{S_m \times S_n}^{S_{m+n}}$	Sym
$G(r,1,n)$	conjugacy	Ind	Res	$\bigotimes$ Sym
$\mathrm{GL}_n(\mathbb{F}_q)$	conjugacy	Indf	Resf	$\bigotimes$ Sym
$\mathrm{U}_n(\mathbb{F}_q)$	conjugacy	R	R^*	$\bigotimes$ Sym
$\mathrm{UT}_n(\mathbb{F}_q)$	super	Inf	Res	NCSym

and with Aliniaeifard,

$\mathrm{UT}_n(\mathbb{F}_q)$	super	Inf	Res	CQSym^*
$\mathfrak{ut}_n(\mathbb{F}_q)$	super	Stfl	Dela	FQSym
G^{n-1}	super	various	various	NSym

a two-parameter family

What is super?

Given a finite group G, a supercharacter theory $\sim$ is an equivalence relation on G such that

$$\mathfrak{f}_{\sim}(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\} \subseteq \mathfrak{f}_{\text{conjugacy}}(G)$$

is a subspace containing both the regular character and a basis of orthogonal characters.

$\mathrm{UT}_n(\mathbb{F}_q)$	super	Inf	Res	CQSym*
$\mathfrak{ut}_n(\mathbb{F}_q)$	super	Stfl	Dela	FQSym
G^{n-1}	super	various	various	NSym

a two-parameter family

Given a finite group G, a supercharacter theory $\sim$ is an equivalence relation on G such that (blocks = superclasses)

$$\mathsf{f}_\sim(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\} \subseteq \mathsf{f}_{\text{conjugacy}}(G)$$

is a subspace containing both the regular character and a basis of orthogonal characters.

$\mathrm{UT}_n(\mathbb{F}_q)$	super	Inf	Res	CQSym*
$\mathfrak{ut}_n(\mathbb{F}_q)$	super	Stfl	Dela	FQSym
G^{n-1}	super	various	various	NSym

a two-parameter family

What is super?

Given a finite group G, a supercharacter theory $\sim$ is an equivalence relation on G such that (blocks = superclasses)

$$\mathsf{f}_{\sim}(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\} \subseteq \mathsf{f}_{\text{conjugacy}}(G)$$

is a subspace containing both the regular character and a basis of orthogonal characters. ⟵ supercharacters

$\mathrm{UT}_n(\mathbb{F}_q)$	super	Inf	Res	CQSym*
$\mathfrak{ut}_n(\mathbb{F}_q)$	super	Stfl	Dela	FQSym
G^{n-1}	super	various	various	NSym

a two-parameter family

What is super?

Given a finite group G, a supercharacter theory $\sim$ is an equivalence relation on G such that (blocks = superclasses)

$$\mathsf{f}_\sim(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\} \subseteq \mathsf{f}_{\text{conjugacy}}(G)$$

is a subspace containing both the regular character and a basis of orthogonal characters. ⟵ supercharacters

Remark. The superclasses and the supercharacters behave largely like conjugacy classes and irreducible characters.

$\mathrm{UT}_n(\mathbb{F}_q)$	super	Inf	Res	CQSym*
$\mathfrak{ut}_n(\mathbb{F}_q)$	super	Stfl	Dela	FQSym
G^{n-1}	super	various	various	NSym

a two-parameter family

Given a finite group G, a supercharacter theory $\sim$ is an equivalence relation on G such that (blocks = superclasses)

$$\mathsf{f}_{\sim}(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\} \subseteq \mathsf{f}_{\text{conjugacy}}(G)$$

is a subspace containing both the regular character and a basis of orthogonal characters. ⟵ supercharacters

Remark. The superclasses and the supercharacters behave largely like conjugacy classes and irreducible characters.

Given a finite group G, a supercharacter theory $\sim$ is an equivalence relation on G such that (blocks = superclasses)

$$\mathsf{f}_{\sim}(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\} \subseteq \mathsf{f}_{\text{conjugacy}}(G)$$

is a subspace containing both the regular character and a basis of orthogonal characters. ⟵ supercharacters

Remark. The superclasses and the supercharacters behave largely like conjugacy classes and irreducible characters.

Favorite example (lattice supercharacter theories).

Let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{N}$ of normal subgroups of G.

What is super?

Given a finite group G, a supercharacter theory $\sim$ is an equivalence relation on G such that (blocks = superclasses)

$$\mathrm{f}_\sim(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\} \subseteq \mathrm{f}_{\text{conjugacy}}(G)$$

is a subspace containing both the regular character and a basis of orthogonal characters. ⟵ supercharacters

Remark. The superclasses and the supercharacters behave largely like conjugacy classes and irreducible characters.

Favorite example (lattice supercharacter theories).

Let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{N}$ of normal subgroups of G. For $g \in G$, let

$$g^{\mathcal{L}} = \min\{N \in \mathcal{L} \mid g \in N\}.$$

What is super?

Given a finite group G, a supercharacter theory $\sim$ is an equivalence relation on G such that (blocks = superclasses)

$$\mathsf{f}_{\sim}(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\} \subseteq \mathsf{f}_{\text{conjugacy}}(G)$$

is a subspace containing both the regular character and a basis of orthogonal characters. ⟵ supercharacters

Remark. The superclasses and the supercharacters behave largely like conjugacy classes and irreducible characters.

Favorite example (lattice supercharacter theories).

Let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{N}$ of normal subgroups of G. For $g \in G$, let

$$g^{\mathcal{L}} = \min\{N \in \mathcal{L} \mid g \in N\}.$$

Define

$$g \sim h \quad \text{if and only if} \quad g^{\mathcal{L}} = h^{\mathcal{L}}.$$

What is super?

Given a finite group G, a supercharacter theory $\sim$ is an equivalence relation on G such that (blocks = superclasses)

$$\mathsf{f}_{\sim}(G) = \{\psi : G \to \mathbb{C} \mid \psi(g) = \psi(h) \text{ if } g \sim h\} \subseteq \mathsf{f}_{\text{conjugacy}}(G)$$

is a subspace containing both the regular character and a basis of orthogonal characters. ⟵ supercharacters

Remark. The superclasses and the supercharacters behave largely like conjugacy classes and irreducible characters.

Favorite example (lattice supercharacter theories).

Let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{N}$ of normal subgroups of G. For $g \in G$, let

$$g^{\mathcal{L}} = \min\{N \in \mathcal{L} \mid g \in N\}.$$

Define

$$g \sim h \quad \text{if and only if} \quad g^{\mathcal{L}} = h^{\mathcal{L}}.$$

Thm (Aliniaeifard). The equivalence relation $\sim$ is a supercharacter theory.

Favorite example (lattice supercharacter theories).

Let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{N}$ of normal subgroups of G. For $g \in G$, let

$$g^{\mathcal{L}} = \min\{N \in \mathcal{L} \mid g \in N\}.$$

Define

$$g \sim h \quad \text{if and only if} \quad g^{\mathcal{L}} = h^{\mathcal{L}}.$$

Favorite example (lattice supercharacter theories).

Let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{N}$ of normal subgroups of G. For $g \in G$, let

$$g^{\mathcal{L}} = \min\{N \in \mathcal{L} \mid g \in N\}.$$

Define

$$g \sim h \quad \text{if and only if} \quad g^{\mathcal{L}} = h^{\mathcal{L}}.$$

Favorite example (lattice supercharacter theories).

Let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{N}$ of normal subgroups of G. For $g \in G$, let

$$g^{\mathcal{L}} = \min\{N \in \mathcal{L} \mid g \in N\}.$$

Define

$$g \sim h \quad \text{if and only if} \quad g^{\mathcal{L}} = h^{\mathcal{L}}.$$

Example. Fix a finite group H, and for $n \in \mathbb{Z}_{\geq 0}$, let

$$G_n = \underbrace{H \times H \times \cdots \times H}_{n-1 \text{ terms}} \times \{1\}.$$

Favorite example (lattice supercharacter theories).

Let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{N}$ of normal subgroups of G. For $g \in G$, let

$$g^{\mathcal{L}} = \min\{N \in \mathcal{L} \mid g \in N\}.$$

Define

$$g \sim h \quad \text{if and only if} \quad g^{\mathcal{L}} = h^{\mathcal{L}}.$$

Example. Fix a finite group H, and for $n \in \mathbb{Z}_{\geq 0}$, let

$$G_n = \underbrace{H \times H \times \cdots \times H}_{n-1 \text{ terms}} \times \{1\}.$$

For a binary sequence $\underline{b} = (b_1, \ldots, b_{n-1}) \in \{0,1\}^{n-1}$, let

$$G_{\underline{b}} = \{(g_1, \ldots, g_{n-1}, 1) \in G_n \mid g_j \neq 1 \text{ implies } b_j = 1\}.$$

Favorite example (lattice supercharacter theories).
Let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{N}$ of normal subgroups of G. For $g \in G$, let

$$g^{\mathcal{L}} = \min\{N \in \mathcal{L} \mid g \in N\}.$$

Define

$$g \sim h \quad \text{if and only if} \quad g^{\mathcal{L}} = h^{\mathcal{L}}.$$

Example. Fix a finite group H, and for $n \in \mathbb{Z}_{\geq 0}$, let

$$G_n = \underbrace{H \times H \times \cdots \times H}_{n-1 \text{ terms}} \times \{1\}.$$

For a binary sequence $\underline{b} = (b_1, \ldots, b_{n-1}) \in \{0,1\}^{n-1}$, let

$$G_{\underline{b}} = \{(g_1, \ldots, g_{n-1}, 1) \in G_n \mid g_j \neq 1 \text{ implies } b_j = 1\}.$$

Let

$$\mathcal{L}_n = \{G_{\underline{b}} \mid \underline{b} \in \{0,1\}^{n-1}\}.$$

Favorite example (lattice supercharacter theories).

Let $\mathcal{L}$ be a sublattice of the lattice $\mathcal{N}$ of normal subgroups of G. For $g \in G$, let

$$g^{\mathcal{L}} = \min\{N \in \mathcal{L} \mid g \in N\}.$$

Define

$$g \sim h \quad \text{if and only if} \quad g^{\mathcal{L}} = h^{\mathcal{L}}.$$

Example. Fix a finite group H, and for $n \in \mathbb{Z}_{\geq 0}$, let

$$G_n = \underbrace{H \times H \times \cdots \times H}_{n-1 \text{ terms}} \times \{1\}.$$

For a binary sequence $\underline{b} = (b_1, \ldots, b_{n-1}) \in \{0,1\}^{n-1}$, let

$$G_{\underline{b}} = \{(g_1, \ldots, g_{n-1}, 1) \in G_n \mid g_j \neq 1 \text{ implies } b_j = 1\}.$$

Let

$$\mathcal{L}_n = \{G_{\underline{b}} \mid \underline{b} \in \{0,1\}^{n-1}\}.$$

Here

$$g^{\mathcal{L}_n} = \{(h_1, \ldots, h_{n-1}, 1) \in G_n \mid h_j \neq 1 \text{ implies } g_j \neq 1\}.$$

$n = 4$

Example. Fix a finite group H, and for $n \in \mathbb{Z}_{\geq 0}$, let

$$G_n = \underbrace{H \times H \times \cdots \times H}_{n-1 \text{ terms}} \times \{1\}.$$

For a binary sequence $\underline{b} = (b_1, \ldots, b_{n-1}) \in \{0,1\}^{n-1}$, let

$$G_{\underline{b}} = \{(g_1, \ldots, g_{n-1}, 1) \in G_n \mid g_j \neq 1 \text{ implies } b_j = 1\}.$$

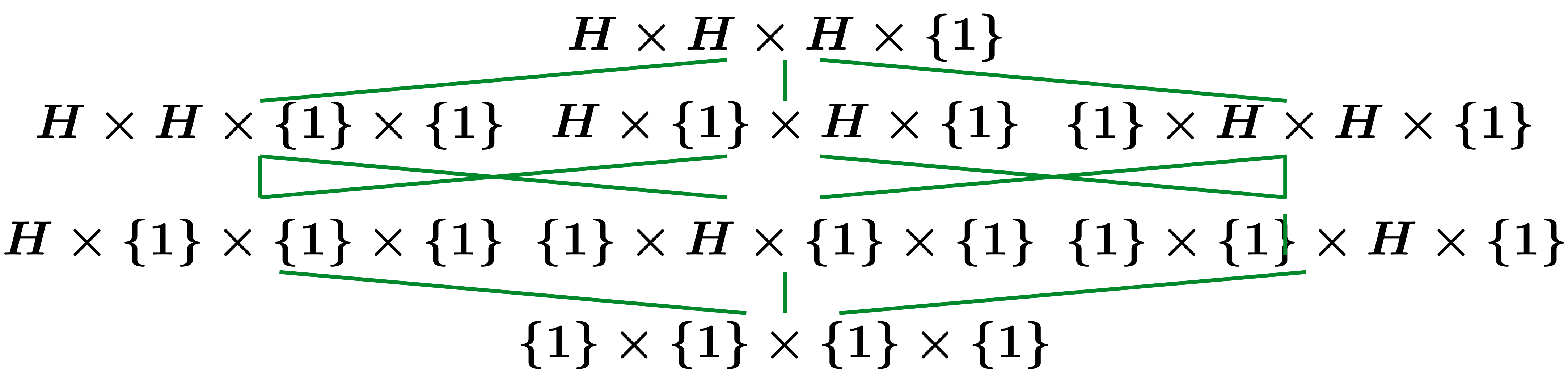
Let

$$\mathcal{L}_n = \{G_{\underline{b}} \mid \underline{b} \in \{0,1\}^{n-1}\}.$$

Here

$$g^{\mathcal{L}_n} = \{(h_1, \ldots, h_{n-1}, 1) \in G_n \mid h_j \neq 1 \text{ implies } g_j \neq 1\}.$$

$n = 4$



Example. **Fix a finite group H, and for $n \in \mathbb{Z}_{\geq 0}$, let**

$$G_n = \underbrace{H \times H \times \cdots \times H}_{n-1 \text{ terms}} \times \{1\}.$$

For a binary sequence $\underline{b} = (b_1, \ldots, b_{n-1}) \in \{0,1\}^{n-1}$, let

$$G_{\underline{b}} = \{(g_1, \ldots, g_{n-1}, 1) \in G_n \mid g_j \neq 1 \text{ implies } b_j = 1\}.$$

Let

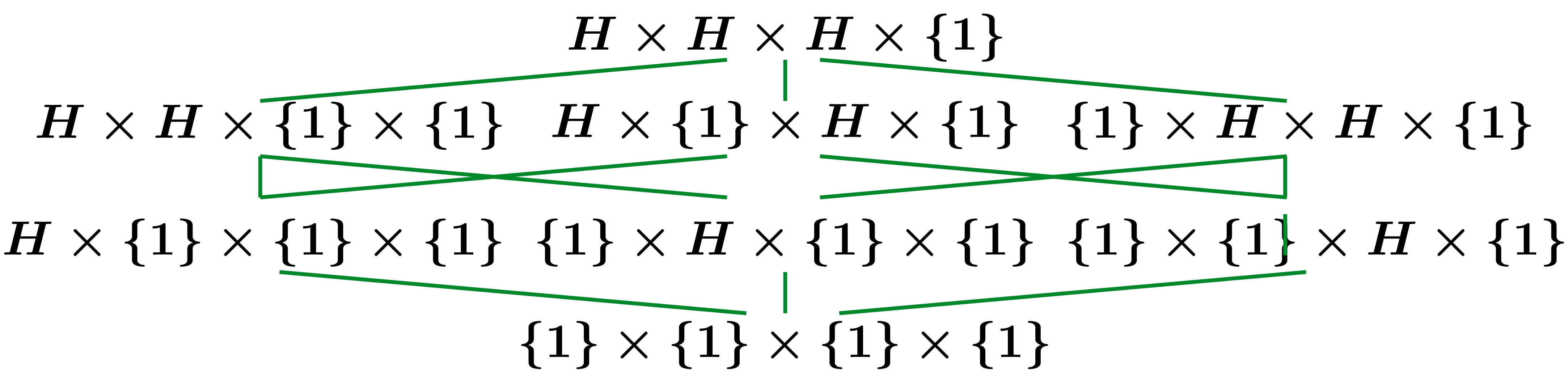
$$\mathcal{L}_n = \{G_{\underline{b}} \mid \underline{b} \in \{0,1\}^{n-1}\}.$$

Here

$$g^{\mathcal{L}_n} = \{(h_1, \ldots, h_{n-1}, 1) \in G_n \mid h_j \neq 1 \text{ implies } g_j \neq 1\}.$$

To noncommutative symmetric functions

$n = 4$



To noncommutative symmetric functions

$n = 4$

$$H \times H \times H \times \{1\}$$

$$H \times H \times \{1\} \times \{1\} \quad H \times \{1\} \times H \times \{1\} \quad \{1\} \times H \times H \times \{1\}$$

$$H \times \{1\} \times \{1\} \times \{1\} \quad \{1\} \times H \times \{1\} \times \{1\} \quad \{1\} \times \{1\} \times H \times \{1\}$$

$$\{1\} \times \{1\} \times \{1\} \times \{1\}$$

To noncommutative symmetric functions

$n = 4$

$$H \times H \times H \times \{1\}$$

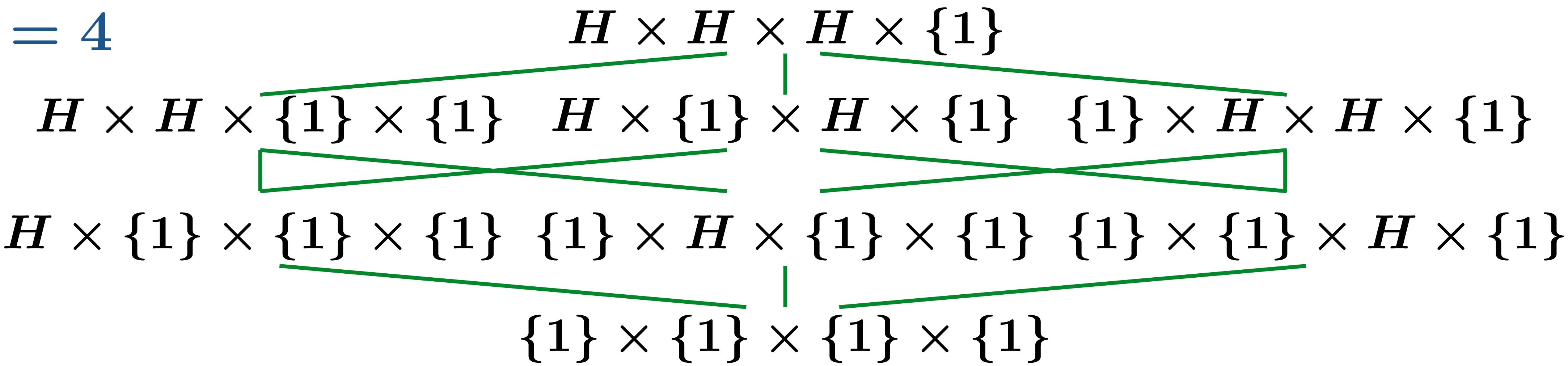
$$H \times H \times \{1\} \times \{1\} \quad H \times \{1\} \times H \times \{1\} \quad \{1\} \times H \times H \times \{1\}$$

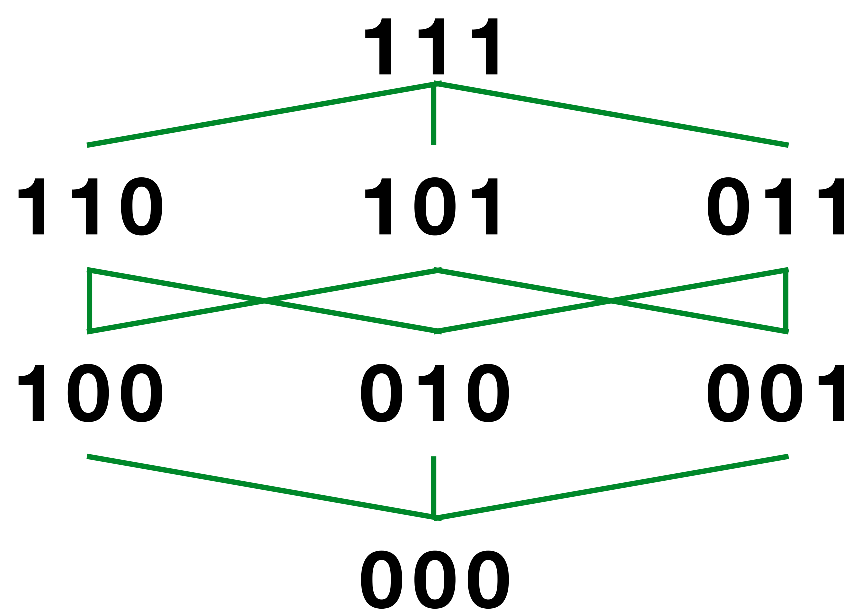
$$H \times \{1\} \times \{1\} \times \{1\} \quad \{1\} \times H \times \{1\} \times \{1\} \quad \{1\} \times \{1\} \times H \times \{1\}$$

$$\{1\} \times \{1\} \times \{1\} \times \{1\}$$

To noncommutative symmetric functions

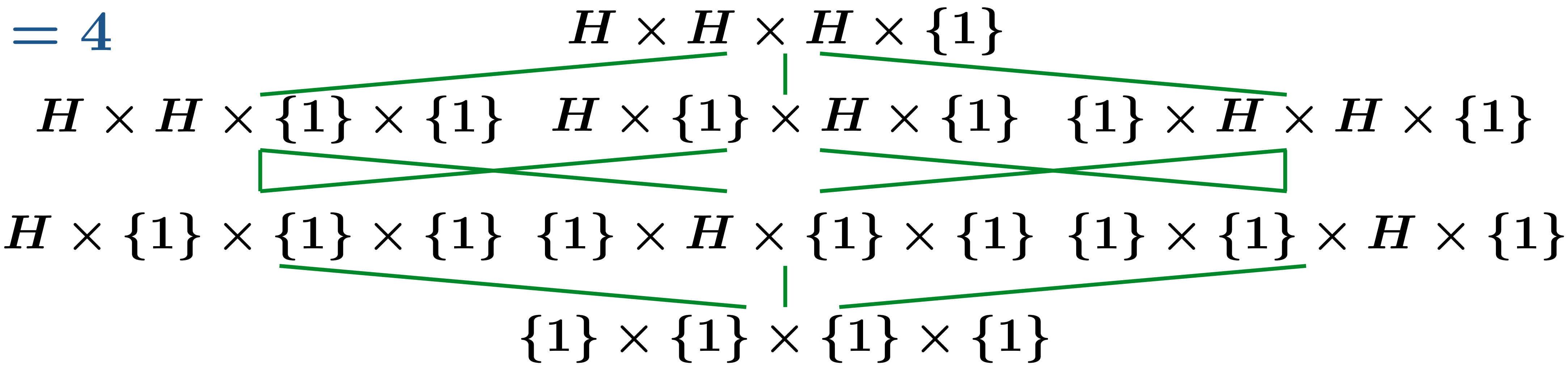
$n = 4$

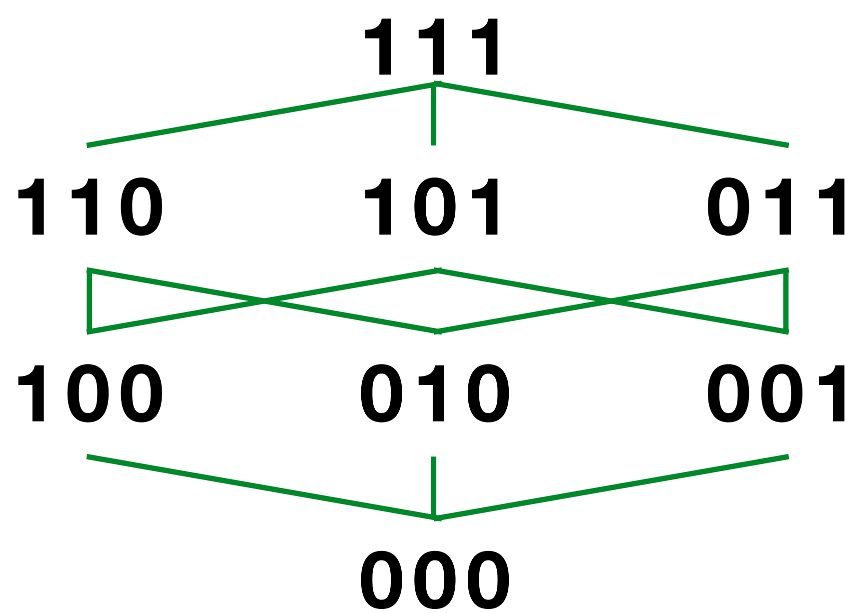




To noncommutative symmetric functions

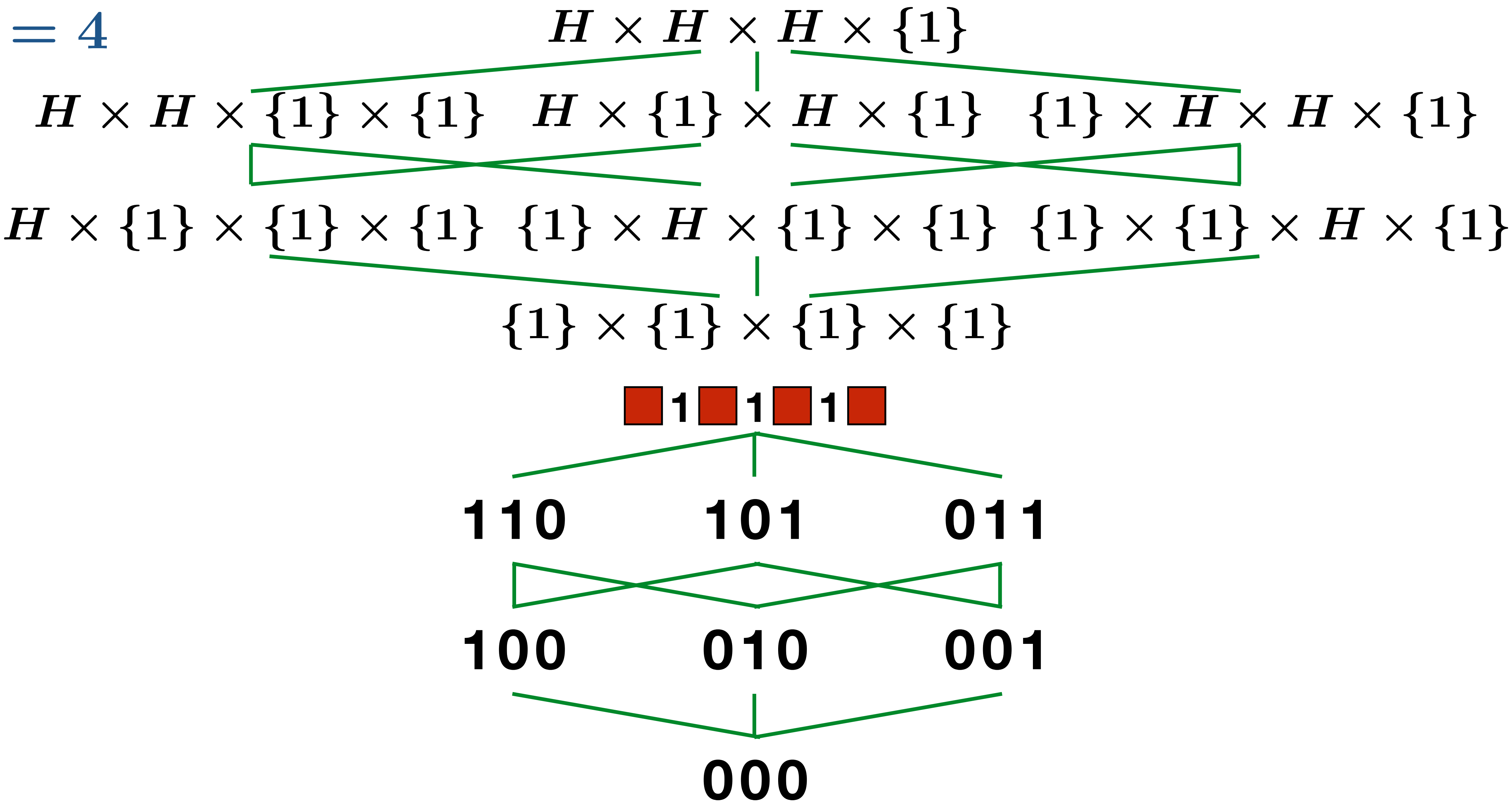
$n = 4$





To noncommutative symmetric functions

$n = 4$



To noncommutative symmetric functions

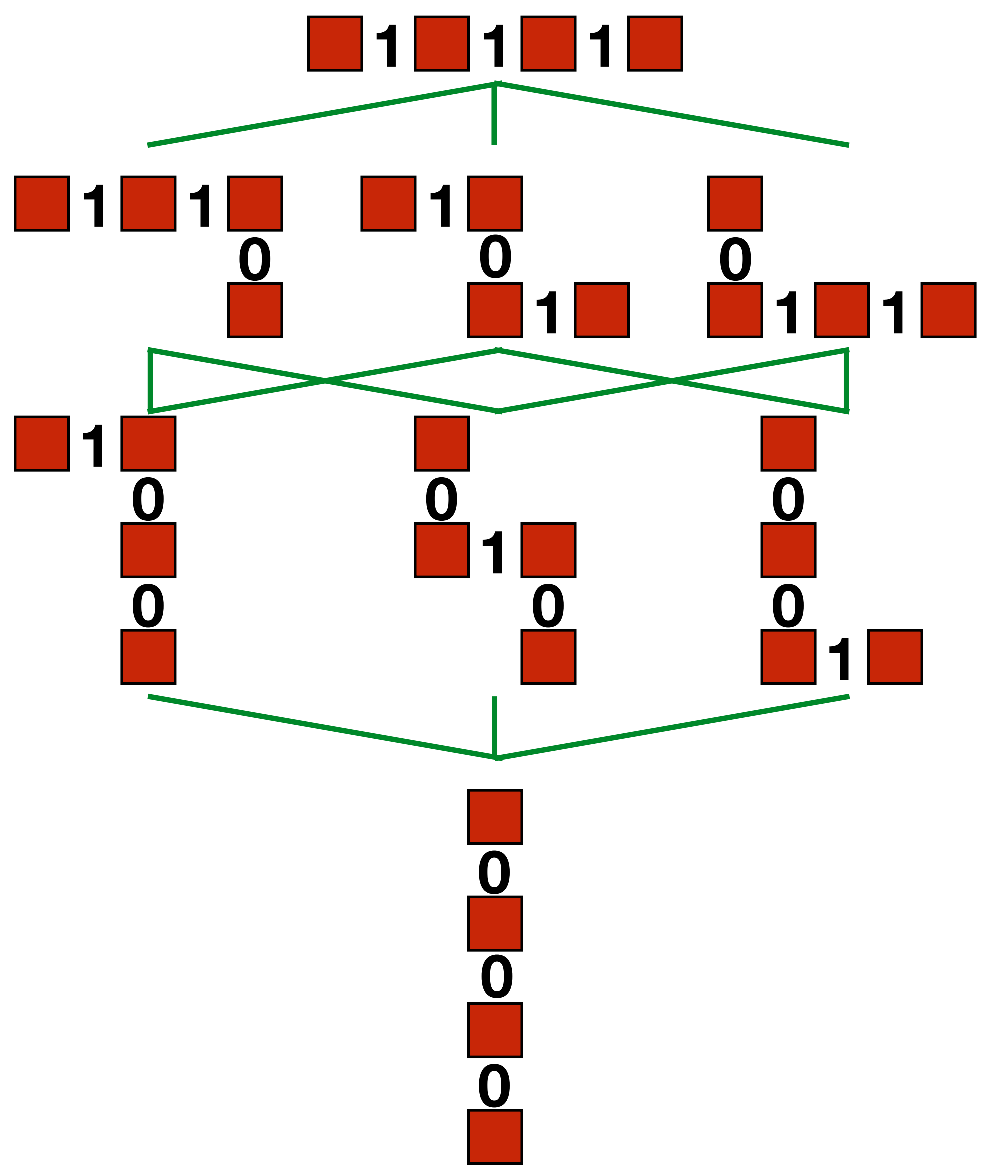
$n = 4$

$$H \times H \times H \times \{1\}$$

$$H \times H \times \{1\} \times \{1\} \quad H \times \{1\} \times H \times \{1\} \quad \{1\} \times H \times H \times \{1\}$$

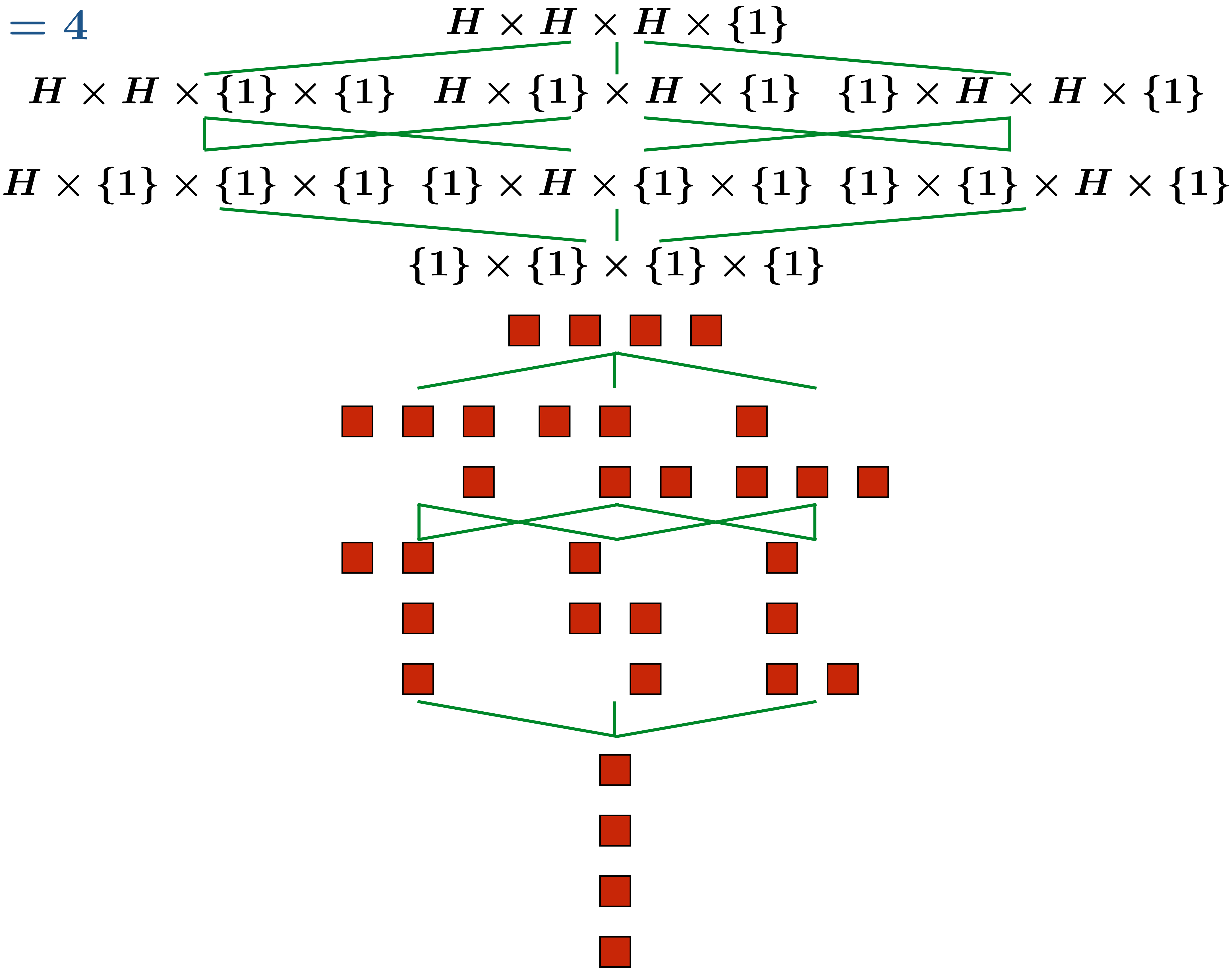
$$H \times \{1\} \times \{1\} \times \{1\} \quad \{1\} \times H \times \{1\} \times \{1\} \quad \{1\} \times \{1\} \times H \times \{1\}$$

$$\{1\} \times \{1\} \times \{1\} \times \{1\}$$



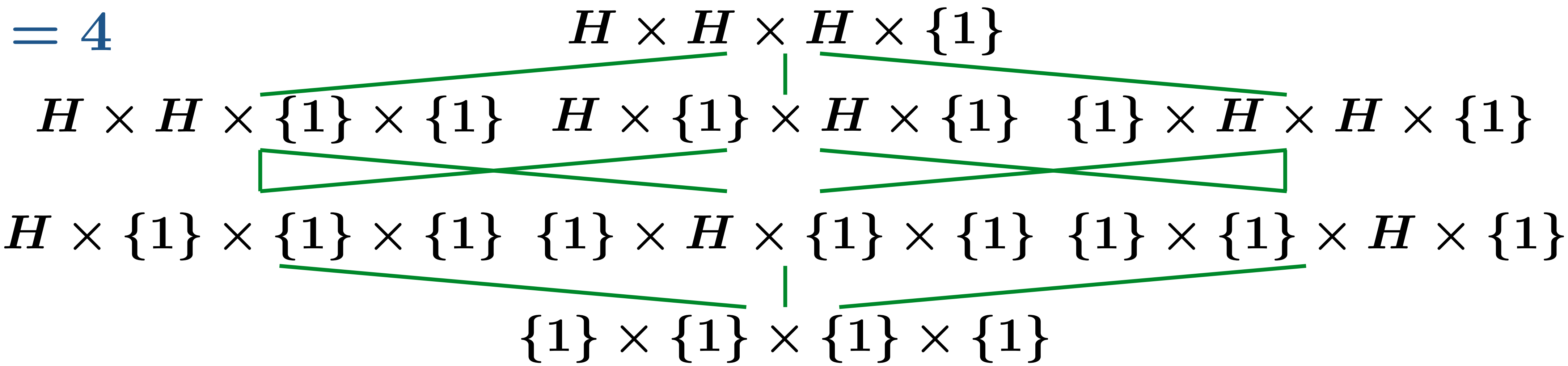
To noncommutative symmetric functions

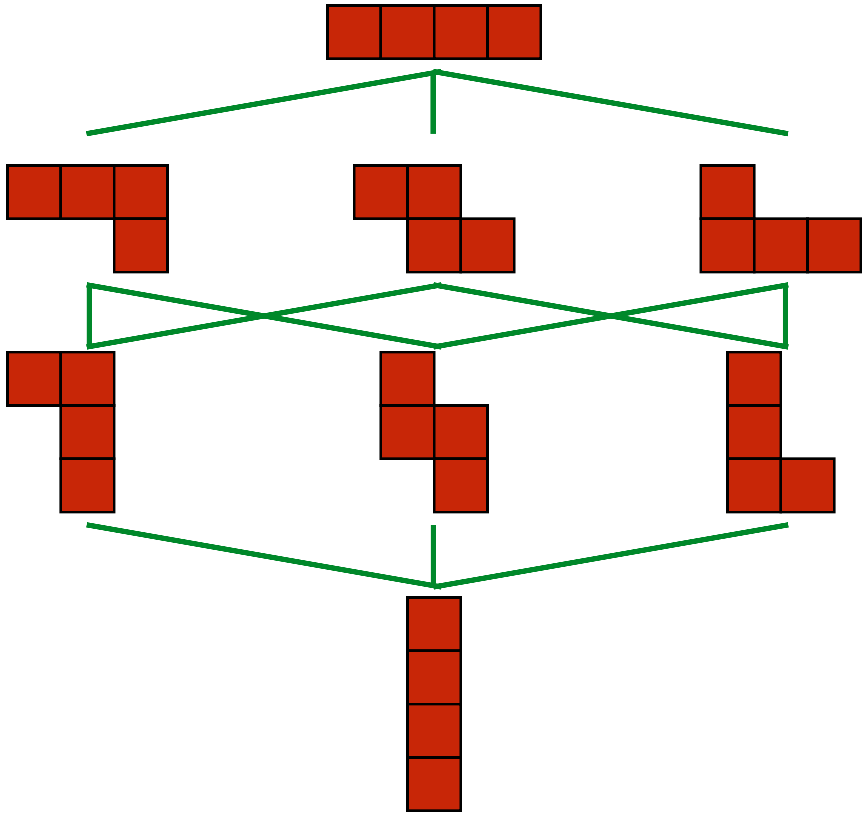
$n = 4$



To noncommutative symmetric functions

$n = 4$





To noncommutative symmetric functions

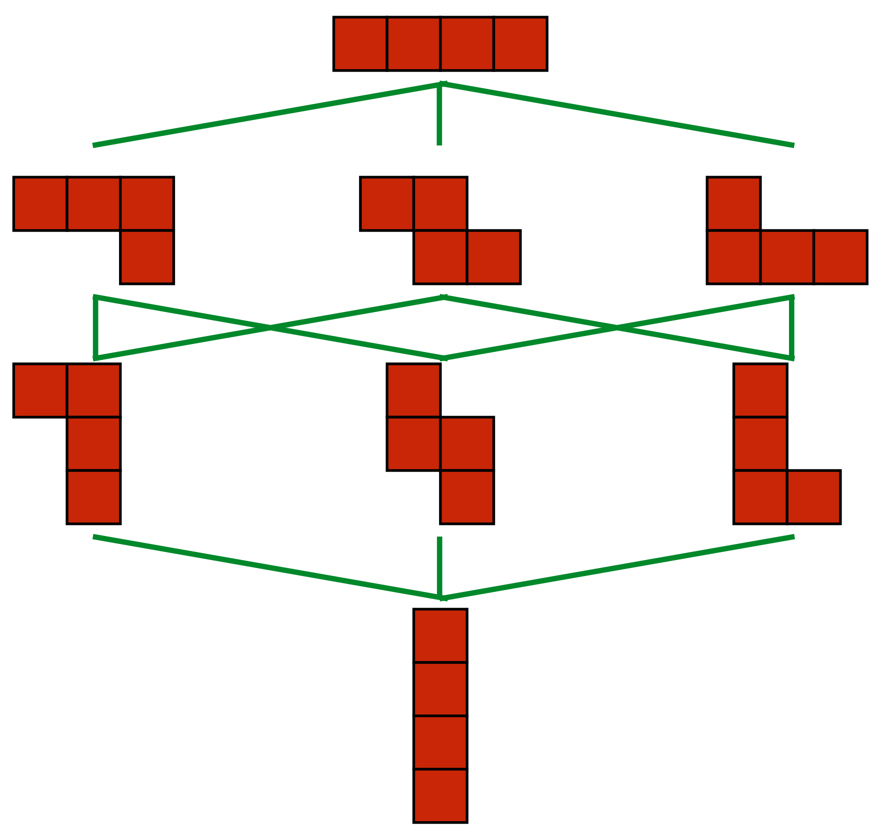
$n = 4$

$$H \times H \times H \times \{1\}$$

$$H \times H \times \{1\} \times \{1\} \quad H \times \{1\} \times H \times \{1\} \quad \{1\} \times H \times H \times \{1\}$$

$$H \times \{1\} \times \{1\} \times \{1\} \quad \{1\} \times H \times \{1\} \times \{1\} \quad \{1\} \times \{1\} \times H \times \{1\}$$

$$\{1\} \times \{1\} \times \{1\} \times \{1\}$$



compositions of n

To noncommutative symmetric functions

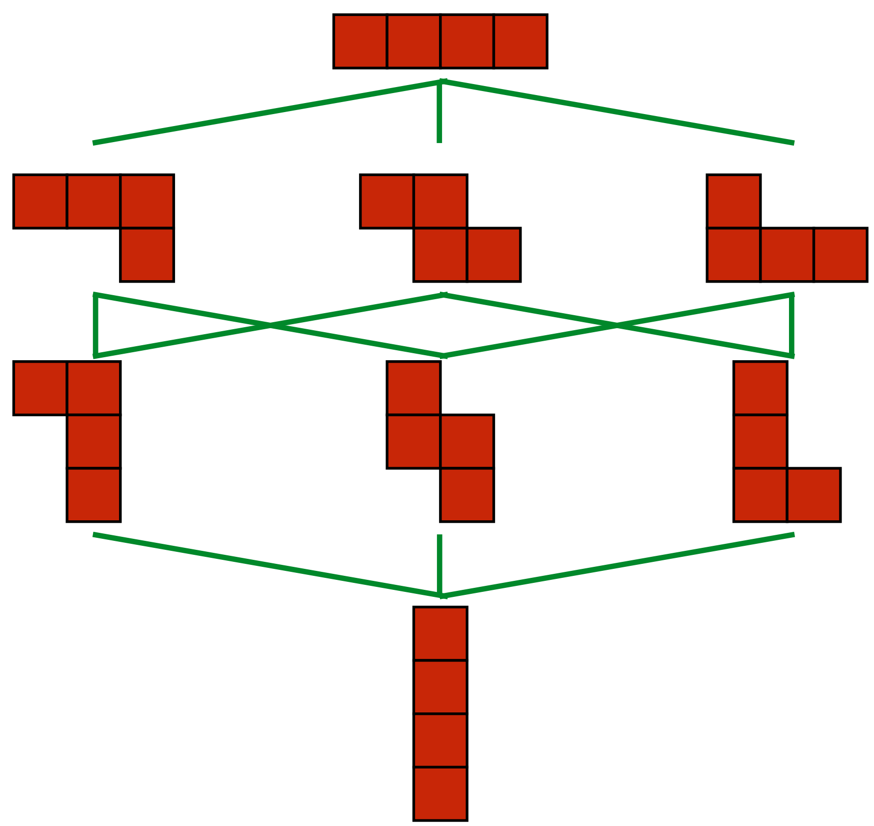
$n = 4$

$$H \times H \times H \times \{1\}$$

$$H \times H \times \{1\} \times \{1\} \quad H \times \{1\} \times H \times \{1\} \quad \{1\} \times H \times H \times \{1\}$$

$$H \times \{1\} \times \{1\} \times \{1\} \quad \{1\} \times H \times \{1\} \times \{1\} \quad \{1\} \times \{1\} \times H \times \{1\}$$

$$\{1\} \times \{1\} \times \{1\} \times \{1\}$$



compositions of n

We have an equivalence relation whose classes are indexed by integer compositions, and whose containment lattice is the usual refinement order.

To noncommutative symmetric functions

General Structure.

$$\mathbf{f}_\sim = \bigoplus_{n \geq 0} \mathbf{f}_\sim(G_n).$$

. graded vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

Towards representation theory.

Here we want compatible functors

$$I : \mathbf{f}_\sim(G_m) \otimes \mathbf{f}_\sim(G_n) \longrightarrow \mathbf{f}_\sim(G_{m+n})$$

$$R_{m,n} : \mathbf{f}_\sim(G_{m+n}) \longrightarrow \mathbf{f}_\sim(G_m) \otimes \mathbf{f}_\sim(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

To noncommutative symmetric functions

General Structure.

$$\mathsf{f}_\sim = \bigoplus_{n\geq 0} \mathsf{f}_\sim(G_n).$$

. graded vector space

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

. bialgebra structure

$$\mathcal{H}_m \otimes \mathcal{H}_n \longrightarrow \mathcal{H}_{m+n} \qquad \mathcal{H}_n \longrightarrow \bigoplus_{j=0}^{n} \mathcal{H}_j \otimes \mathcal{H}_{n-j}$$

Towards representation theory.

Here we want compatible functors

$$I : \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n) \longrightarrow \mathsf{f}_\sim(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_\sim(G_{m+n}) \longrightarrow \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

To noncommutative symmetric functions

Towards representation theory.

Here we want compatible functors

$$I : \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n) \longrightarrow \mathsf{f}_{\sim}(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_{\sim}(G_{m+n}) \longrightarrow \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n)$$

such that I **and** $\sum_{j=0}^{n} R_{j,n-j}$ **give a Hopf algebra.**

Towards representation theory.

Here we want compatible functors

$$I : \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n) \longrightarrow \mathsf{f}_\sim(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_\sim(G_{m+n}) \longrightarrow \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

To noncommutative symmetric functions

Towards representation theory.

$$G_n = \underbrace{H \times \cdots \times H}_{n-1 \text{ terms}} \times \{1\}$$

Here we want compatible functors

$$I : \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n) \longrightarrow \mathsf{f}_{\sim}(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_{\sim}(G_{m+n}) \longrightarrow \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

Towards representation theory.

Here we want compatible functors

$$G_n = \underbrace{H \times \cdots \times H}_{n-1 \text{ terms}} \times \{1\}$$

$$I : \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n) \longrightarrow \mathsf{f}_{\sim}(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_{\sim}(G_{m+n}) \longrightarrow \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

Example. **Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.**

To noncommutative symmetric functions

Towards representation theory.

$$G_n = \underbrace{H \times \cdots \times H}_{n-1 \text{ terms}} \times \{1\}$$

Here we want compatible functors

$$I : \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n) \longrightarrow \mathsf{f}_{\sim}(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_{\sim}(G_{m+n}) \longrightarrow \mathsf{f}_{\sim}(G_m) \otimes \mathsf{f}_{\sim}(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

trivial regular

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

To noncommutative symmetric functions

Towards representation theory.

$$G_n = \underbrace{H \times \cdots \times H}_{n-1 \text{ terms}} \times \{1\}$$

Here we want compatible functors

$$I : \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n) \longrightarrow \mathsf{f}_\sim(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_\sim(G_{m+n}) \longrightarrow \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

Example. **Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\underset{\text{trivial}}{\mathbb{1}_H}, \underset{\text{regular}}{\text{reg}_H}\} \subseteq f_{\text{conjugacy}}(H)$.**

$$\begin{array}{rccc} I : & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

To noncommutative symmetric functions

Towards representation theory.

$$G_n = \underbrace{H \times \cdots \times H}_{n-1 \text{ terms}} \times \{1\}$$

Here we want compatible functors

$$I : \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n) \longrightarrow \mathsf{f}_\sim(G_{m+n})$$

$$R_{m,n} : \mathsf{f}_\sim(G_{m+n}) \longrightarrow \mathsf{f}_\sim(G_m) \otimes \mathsf{f}_\sim(G_n)$$

such that I and $\sum_{j=0}^{n} R_{j,n-j}$ give a Hopf algebra.

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\underbrace{\mathbb{1}_H}_{\text{trivial}}, \underbrace{\text{reg}_H}_{\text{regular}}\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I : & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

To noncommutative symmetric functions

Example. **Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.**

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. **If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.**

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_{\sim}(G_m) \otimes f_{\sim}(G_n) & \longrightarrow & f_{\sim}(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \mathrm{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \mathrm{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A: f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \iota^A \odot \mathrm{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \mathrm{Def}_{G_{(A,B)}}^{G_n}((\alpha \times \beta)_A \odot \psi) \end{array}$$

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A: f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \text{Def}_{G_{(A,B)}}^{G_n}((\alpha \times \beta)_A \odot \psi) \end{array}$$

$$\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4 \otimes \psi_5 \otimes \psi_6 \otimes \psi_7 \otimes \mathbb{1}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A : f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \text{Def}_{G_{(A,B)}}^{G_n}((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$

$$\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4 \otimes \psi_5 \otimes \psi_6 \otimes \psi_7 \otimes \mathbb{1}$$

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A: f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \text{Def}_{G_{(A,B)}}^{G_n}((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ $\quad A \quad \boxed{1} \quad \boxed{2} \quad \boxed{4} \quad \boxed{5} \quad \boxed{8}$

$$\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4 \otimes \psi_5 \otimes \psi_6 \otimes \psi_7 \otimes \mathbb{1}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A : f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \text{Def}_{G_{(A,B)}}^{G_n}((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ $\quad A \quad \boxed{1} \quad \boxed{2} \quad \boxed{3} \quad \boxed{4} \quad \boxed{5} \quad \boxed{6} \quad \boxed{7} \quad \boxed{8} \quad B$

$$\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4 \otimes \psi_5 \otimes \psi_6 \otimes \psi_7 \otimes \mathbb{1}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A : f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \text{Def}_{G_{(A,B)}}^{G_n} \left((\alpha \times \beta)_A \odot \psi \right) \end{array}$$

$n = 8$ $\quad A \quad$ 1 2 3 4 5 6 7 8 $\quad B$

$$\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4 \otimes \psi_5 \otimes \psi_6 \otimes \psi_7 \otimes \mathbb{1}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}\text{-span}\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rccc} R_A: & f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ & \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|}\times G_{|B|}} \circ \text{Def}_{G_{(A,B)}}^{G_n}\left(\boxed{(\alpha \times \beta)_A \odot \psi}\right) \end{array}$$

$n = 8$ A $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ B

$$\psi_1 \otimes \psi_2 \otimes \psi_3 \otimes \psi_4 \otimes \psi_5 \otimes \psi_6 \otimes \psi_7 \otimes \mathbb{1}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A: f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \text{Def}_{G_{(A,B)}}^{G_n} \left(\boxed{(\alpha \times \beta)_A \odot \psi} \right) \end{array}$$

$n = 8$ A [1] [2] [3] [4] [5] [6] [7] [8] B

$$\psi_1 \otimes \psi_2 \odot \alpha \otimes \psi_3 \odot \beta \otimes \psi_4 \otimes \psi_5 \odot \alpha \otimes \psi_6 \otimes \psi_7 \odot \beta \otimes \mathbb{1}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}\text{-span}\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rccc} R_A: & f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ & \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|}\times G_{|B|}} \circ \boxed{\text{Def}_{G_{(A,B)}}^{G_n}}((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ A $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ B

$$\psi_1 \otimes \psi_2 \odot \alpha \otimes \psi_3 \odot \beta \otimes \psi_4 \otimes \psi_5 \odot \alpha \otimes \psi_6 \otimes \psi_7 \odot \beta \otimes \mathbb{1}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A : f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \boxed{\text{Def}_{G_{(A,B)}}^{G_n}}((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ A $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ B

$$\psi_1 \otimes \psi_2 \odot \alpha \otimes \psi_3 \odot \beta \otimes \psi_4 \otimes \psi_5 \odot \alpha \otimes \psi_6 \otimes \psi_7 \odot \beta \otimes \mathbb{1}$$

$$G_{(A,B)} = H \times \{1\} \times \{1\} \times H \times \{1\} \times H \times \{1\} \times \{1\}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}\text{-span}\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rccc} R_A: & f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ & \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \boxed{\text{Def}_{G_{(A,B)}}^{G_n}}((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ A $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ B

$$\psi_1 \otimes \psi_2 \odot \alpha \otimes \psi_3 \odot \beta \otimes \psi_4 \otimes \psi_5 \odot \alpha \otimes \psi_6 \otimes \psi_7 \odot \beta \otimes \mathbb{1}$$

$$\langle \psi_2, \alpha \rangle \mathbb{1}$$

$$G_{(A,B)} = H \times \{1\} \times \{1\} \times H \times \{1\} \times H \times \{1\} \times \{1\}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rcl} I: \quad f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A : f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \boxed{\text{Def}_{G_{(A,B)}}^{G_n}} ((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ A: $\boxed{1}$ $\boxed{2}$ $\boxed{4}$ $\boxed{5}$ $\boxed{8}$; B: $\boxed{3}$ $\boxed{6}$ $\boxed{7}$

$$\psi_1 \otimes \psi_2 \odot \alpha \otimes \psi_3 \odot \beta \otimes \psi_4 \otimes \psi_5 \odot \alpha \otimes \psi_6 \otimes \psi_7 \odot \beta \otimes \mathbb{1}$$

$\langle \psi_2, \alpha \rangle \mathbb{1}$ $\quad$ $\langle \psi_3, \beta \rangle \mathbb{1}$

$$G_{(A,B)} = H \times \{1\} \times \{1\} \times H \times \{1\} \times H \times \{1\} \times \{1\}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rccc} R_A: & f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ & \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \boxed{\text{Def}_{G_{(A,B)}}^{G_n}} ((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ A: $\boxed{1}\ \boxed{2}\ \boxed{4}\ \boxed{5}\ \boxed{8}$; B: $\boxed{3}\ \boxed{6}\ \boxed{7}$

$$\psi_1 \otimes \psi_2 \odot \alpha \otimes \psi_3 \odot \beta \otimes \psi_4 \otimes \psi_5 \odot \alpha \otimes \psi_6 \otimes \psi_7 \odot \beta \otimes \mathbb{1}$$

$$\langle \psi_2, \alpha \rangle \mathbb{1} \qquad \langle \psi_3, \beta \rangle \mathbb{1} \qquad \langle \psi_5, \alpha \rangle \mathbb{1}$$

$$G_{(A,B)} = H \times \{1\} \times \{1\} \times H \times \{1\} \times H \times \{1\} \times \{1\}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rccc} R_A: & f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ & \psi & \mapsto & \iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}} \circ \boxed{\text{Def}_{G_{(A,B)}}^{G_n}} ((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ A: 1, 2, 4, 5 B: 3, 6, 7, 8

$$\psi_1 \otimes \psi_2 \odot \alpha \otimes \psi_3 \odot \beta \otimes \psi_4 \otimes \psi_5 \odot \alpha \otimes \psi_6 \otimes \psi_7 \odot \beta \otimes \mathbb{1}$$

$$\langle \psi_2, \alpha \rangle \mathbb{1} \quad \langle \psi_3, \beta \rangle \mathbb{1} \quad \langle \psi_5, \alpha \rangle \mathbb{1} \quad \langle \psi_7, \beta \rangle \mathbb{1}$$

$$G_{(A,B)} = H \times \{1\} \times \{1\} \times H \times \{1\} \times H \times \{1\} \times \{1\}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \text{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \text{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \dots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A: f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \boxed{\iota^A \odot \text{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}}} \circ \text{Def}_{G_{(A,B)}}^{G_n}((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ A $\quad$ B

$$\underset{\boxed{1}}{\psi_1} \otimes \underset{\boxed{2}}{\psi_2 \odot \alpha} \otimes \underset{\boxed{3}}{\psi_3 \odot \beta} \otimes \underset{\boxed{4}}{\psi_4} \otimes \underset{\boxed{5}}{\psi_5 \odot \alpha} \otimes \underset{\boxed{6}}{\psi_6} \otimes \underset{\boxed{7}}{\psi_7 \odot \beta} \otimes \underset{\boxed{8}}{\mathbb{1}}$$

$$\langle \psi_2, \alpha \rangle \mathbb{1} \quad \langle \psi_3, \beta \rangle \mathbb{1} \quad \langle \psi_5, \alpha \rangle \mathbb{1} \quad \langle \psi_7, \beta \rangle \mathbb{1}$$

$$G_{(A,B)} = H \times \{1\} \times \{1\} \times H \times \{1\} \times H \times \{1\} \times \{1\}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}$-span$\{\mathbb{1}_H, \mathrm{reg}_H\} \subseteq f_{\mathrm{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \mathrm{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A : f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \boxed{\iota^A \odot \mathrm{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}}} \circ \mathrm{Def}_{G_{(A,B)}}^{G_n}((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ A: $\boxed{1}$ $\boxed{2}$ $\boxed{4}$ $\boxed{5}$ $\boxed{8}$; B: $\boxed{3}$ $\boxed{6}$ $\boxed{7}$

$$\psi_1 \otimes \psi_2 \odot \alpha \otimes \psi_3 \odot \beta \otimes \psi_4 \otimes \psi_5 \odot \alpha \otimes \psi_6 \otimes \psi_7 \odot \beta \otimes \mathbb{1}$$

$\psi_2 \odot \alpha \to \langle \psi_2, \alpha \rangle \mathbb{1}$, $\psi_3 \odot \beta \to \langle \psi_3, \beta \rangle \mathbb{1}$, $\psi_5 \odot \alpha \to \langle \psi_5, \alpha \rangle \mathbb{1}$, $\psi_7 \odot \beta \to \langle \psi_7, \beta \rangle \mathbb{1}$

$$G_{(A,B)} = H \times \{1\} \times \{1\} \times H \times \{1\} \times H \times \{1\} \times \{1\}$$

$$G_5 \times G_3 \cong H \times H \times H \times H \times H \times H \times \{1\} \times \{1\}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}\text{-span}\{\mathbb{1}_H, \mathrm{reg}_H\} \subseteq f_{\mathrm{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m)\otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi\otimes\mathbb{1})\otimes(\gamma\otimes\mathbb{1}) & \mapsto & \psi\otimes\iota\otimes\gamma\otimes\mathbb{1}\end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \mathrm{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rccl} R_A: & f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|})\otimes f_\sim(G_{|B|}) \\ & \psi & \mapsto & \boxed{\iota^A \odot \mathrm{Inf}_{G_{(A,B)}}^{G_{|A|}\times G_{|B|}}} \circ \mathrm{Def}_{G_{(A,B)}}^{G_n}((\alpha\times\beta)_A\odot\psi)\end{array}$$

$n = 8$ A: $1, 2, 4, 5, 8$ B: $3, 6, 7$

$$\psi_1\otimes\psi_2\odot\alpha\otimes\psi_3\odot\beta\otimes\psi_4\otimes\psi_5\odot\alpha\otimes\psi_6\otimes\psi_7\odot\beta\otimes\mathbb{1}$$

$$\langle\psi_2,\alpha\rangle\mathbb{1} \quad \langle\psi_3,\beta\rangle\mathbb{1} \quad \langle\psi_5,\alpha\rangle\mathbb{1} \quad \langle\psi_7,\beta\rangle\mathbb{1}$$

$$G_{(A,B)} = H\times\{1\}\times\{1\}\times H\times\{1\}\times H\times\{1\}\times\{1\}$$

$$\psi_1\otimes\langle\psi_2,\alpha\rangle\iota\otimes\langle\psi_3,\beta\rangle\iota\otimes\psi_4\otimes\langle\psi_5,\alpha\rangle\iota\otimes\psi_6\otimes\langle\psi_7,\beta\rangle\mathbb{1}\otimes\mathbb{1}$$

$$G_5\times G_3 \cong H\times H\times H\times H\times H\times H\times\{1\}\times\{1\}$$

To noncommutative symmetric functions

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}\text{-span}\{\mathbb{1}_H, \mathrm{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \mathrm{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A : f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \boxed{\iota^A \odot \mathrm{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}}} \circ \mathrm{Def}_{G_{(A,B)}}^{G_n}((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ A $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ B

$$\psi_1 \otimes \psi_2 \odot \alpha \otimes \psi_3 \odot \beta \otimes \psi_4 \otimes \psi_5 \odot \alpha \otimes \psi_6 \otimes \psi_7 \odot \beta \otimes \mathbb{1}$$

$$\langle \psi_2, \alpha \rangle \mathbb{1} \quad \langle \psi_3, \beta \rangle \mathbb{1} \quad \langle \psi_5, \alpha \rangle \mathbb{1} \quad \langle \psi_7, \beta \rangle \mathbb{1}$$

$$G_{(A,B)} = H \times \{1\} \times \{1\} \times H \times \{1\} \times H \times \{1\} \times \{1\}$$

$$\psi_1 \otimes \langle \psi_2, \alpha \rangle \iota \otimes \langle \psi_3, \beta \rangle \iota \otimes \psi_4 \otimes \langle \psi_5, \alpha \rangle \iota \otimes \psi_6 \otimes \langle \psi_7, \beta \rangle \mathbb{1} \otimes \mathbb{1}$$

$$G_5 \times G_3 \cong H \times H \times H \times H \times H \times H \times \{1\} \times \{1\}$$

Example. Fix $\iota, \alpha, \beta \in \mathbb{C}\text{-span}\{\mathbb{1}_H, \mathrm{reg}_H\} \subseteq f_{\text{conjugacy}}(H)$.

$$\begin{array}{rccc} I: & f_\sim(G_m) \otimes f_\sim(G_n) & \longrightarrow & f_\sim(G_{m+n}) \\ & (\psi \otimes \mathbb{1}) \otimes (\gamma \otimes \mathbb{1}) & \mapsto & \psi \otimes \iota \otimes \gamma \otimes \mathbb{1} \end{array}$$

Remark. If $\iota = \mathbb{1}_H$, then I is the usual inflation functor, and if $\iota = \mathrm{reg}_H$, then I is the induction functor.

For $A \subseteq \{1, 2, \ldots, n\}$ with complement B, let

$$\begin{array}{rcl} R_A : f_\sim(G_n) & \longrightarrow & f_\sim(G_{|A|}) \otimes f_\sim(G_{|B|}) \\ \psi & \mapsto & \boxed{\iota^A \odot \mathrm{Inf}_{G_{(A,B)}}^{G_{|A|} \times G_{|B|}}} \circ \mathrm{Def}_{G_{(A,B)}}^{G_n}((\alpha \times \beta)_A \odot \psi) \end{array}$$

$n = 8$ A [1] [2] [3] [4] [5] [6] [7] [8] B

$$\psi_1 \otimes \psi_2 \odot \alpha \otimes \psi_3 \odot \beta \otimes \psi_4 \otimes \psi_5 \odot \alpha \otimes \psi_6 \otimes \psi_7 \odot \beta \otimes \mathbb{1}$$

$$\langle \psi_2, \alpha \rangle \mathbb{1} \quad \langle \psi_3, \beta \rangle \mathbb{1} \quad \langle \psi_5, \alpha \rangle \mathbb{1} \quad \langle \psi_7, \beta \rangle \mathbb{1}$$

$$G_{(A,B)} = H \times \{1\} \times \{1\} \times H \times \{1\} \times H \times \{1\} \times \{1\}$$

$$\psi_1 \otimes \langle \psi_2, \alpha \rangle \iota \otimes \langle \psi_3, \beta \rangle \iota \otimes \psi_4 \otimes \langle \psi_5, \alpha \rangle \iota \otimes \psi_6 \otimes \langle \psi_7, \beta \rangle \mathbb{1} \otimes \mathbb{1}$$

$$G_5 \times G_3 \cong H \times H \times H \times H \times H \times H \times \{1\} \times \{1\}$$

A noncanonical categorification

Thm. [A–T] Let $\mathsf{f}_\sim(H^\bullet)$ denote the direct product structure. Then

- $\mathsf{f}_\sim(H^\bullet)$ is a Hopf algebra if and only if $\langle \iota, \alpha \rangle = \langle \iota, \beta \rangle = 1$.
- if $\mathsf{f}_\sim(H^\bullet)$ is a Hopf algebra, then

$$\mathsf{f}_\sim(H^\bullet) \cong \mathrm{NSym}.$$

A noncanonical categorification

Thm. [A–T] Let $f_\sim(H^\bullet)$ denote the direct product structure. Then

- $f_\sim(H^\bullet)$ is a Hopf algebra if and only if $\langle \iota, \alpha \rangle = \langle \iota, \beta \rangle = 1$.
- if $f_\sim(H^\bullet)$ is a Hopf algebra, then

$$f_\sim(H^\bullet) \cong \mathrm{NSym}.$$

Remarks.

- if $\alpha \neq \beta$, then $\iota = \alpha^* + \beta^*$.

A noncanonical categorification

Thm. [A–T] Let $\mathsf{f}_{\sim}(H^{\bullet})$ denote the direct product structure. Then

- $\mathsf{f}_{\sim}(H^{\bullet})$ is a Hopf algebra if and only if $\langle \iota, \alpha \rangle = \langle \iota, \beta \rangle = 1$.
- if $\mathsf{f}_{\sim}(H^{\bullet})$ is a Hopf algebra, then

$$\mathsf{f}_{\sim}(H^{\bullet}) \cong \mathsf{NSym}.$$

Remarks.

- if $\alpha \neq \beta$, then $\iota = \alpha^* + \beta^*$.
- By varying α and β we get different isomorphisms, that send characters of G_n to different bases (e.g. h-basis, ribbon basis, etc.).

A noncanonical categorification

Thm. [A–T] Let $\mathsf{f}_{\sim}(H^{\bullet})$ denote the direct product structure. Then

. $\mathsf{f}_{\sim}(H^{\bullet})$ is a Hopf algebra if and only if $\langle \iota, \alpha \rangle = \langle \iota, \beta \rangle = 1$.

. if $\mathsf{f}_{\sim}(H^{\bullet})$ is a Hopf algebra, then

$$\mathsf{f}_{\sim}(H^{\bullet}) \cong \mathsf{NSym}.$$

Remarks.

. if $\alpha \neq \beta$, then $\iota = \alpha^{*} + \beta^{*}$.

. By varying α and β we get different isomorphisms, that send characters of G_n to different bases (e.g. h-basis, ribbon basis, etc.).

.. In fact, the normal lattice theory comes with 3 canonical bases: superclass identifiers, supercharacters, and normal subgroup identifiers.

A noncanonical categorification

Thm. [A–T] Let $f_\sim(H^\bullet)$ denote the direct product structure. Then

. $f_\sim(H^\bullet)$ is a Hopf algebra if and only if $\langle \iota, \alpha\rangle = \langle \iota, \beta\rangle = 1$.

. if $f_\sim(H^\bullet)$ is a Hopf algebra, then

$$f_\sim(H^\bullet) \cong \text{NSym}.$$

Remarks.

. if $\alpha \neq \beta$, then $\iota = \alpha^* + \beta^*$.

. By varying α and β we get different isomorphisms, that send characters of G_n to different bases (e.g. h-basis, ribbon basis, etc.).

$|G| - 1$ becomes a parameter

.. In fact, the normal lattice theory comes with 3 canonical bases: superclass identifiers, supercharacters, and normal subgroup identifiers.

A noncanonical categorification

Thm. [A–T] Let $\mathsf{f}_\sim(H^\bullet)$ denote the direct product structure. Then

- $\mathsf{f}_\sim(H^\bullet)$ is a Hopf algebra if and only if $\langle \iota, \alpha\rangle = \langle \iota, \beta\rangle = 1$.
- if $\mathsf{f}_\sim(H^\bullet)$ is a Hopf algebra, then

$$\mathsf{f}_\sim(H^\bullet) \cong \mathsf{NSym}.$$

Remarks.

- if $\alpha \neq \beta$, then $\iota = \alpha^* + \beta^*$.
- By varying α and β we get different isomorphisms, that send characters of G_n to different bases (e.g. h-basis, ribbon basis, etc.).

  $|G| - 1$ becomes a parameter

  - In fact, the normal lattice theory comes with 3 canonical bases: superclass identifiers, supercharacters, and normal subgroup identifiers.
- By varying α and β we get different "natural" homomorphisms to other Hopf algebras such as NCSym and FQSym.

A noncanonical categorification

Thm. [A–T] Let $\mathsf{f}_{\sim}(H^{\bullet})$ denote the direct product structure. Then

. $\mathsf{f}_{\sim}(H^{\bullet})$ is a Hopf algebra if and only if $\langle \iota, \alpha \rangle = \langle \iota, \beta \rangle = 1$.

Remarks.

. By varying α and β we get different isomorphisms, that send characters of G_n to different bases (e.g. h-basis, ribbon basis, etc.).

.. In fact, the normal lattice theory comes with 3 canonical bases: superclass identifiers, supercharacters, and normal subgroup identifiers.

. By varying α and β we get different "natural" homomorphisms to other Hopf algebras such as NCSym and FQSym.

A noncanonical categorification

Thm. [A–T] Let $\mathsf{f}_{\sim}(H^{\bullet})$ denote the direct product structure. Then

- $\mathsf{f}_{\sim}(H^{\bullet})$ is a Hopf algebra if and only if $\langle \iota, \alpha \rangle = \langle \iota, \beta \rangle = 1$.

Remarks.

- By varying α and β we get different isomorphisms, that send characters of G_n to different bases (e.g. h-basis, ribbon basis, etc.).
  - In fact, the normal lattice theory comes with 3 canonical bases: superclass identifiers, supercharacters, and normal subgroup identifiers.
- By varying α and β we get different "natural" homomorphisms to other Hopf algebras such as NCSym and FQSym.

A noncanonical categorification

Thm. [A–T] Let $f_{\sim}(H^{\bullet})$ denote the direct product structure. Then

- $f_{\sim}(H^{\bullet})$ is a Hopf algebra if and only if $\langle \iota, \alpha \rangle = \langle \iota, \beta \rangle = 1$.

Remarks.

- By varying α and β we get different isomorphisms, that send characters of G_n to different bases (e.g. h-basis, ribbon basis, etc.).
  - In fact, the normal lattice theory comes with 3 canonical bases: superclass identifiers, supercharacters, and normal subgroup identifiers.
- By varying α and β we get different "natural" homomorphisms to other Hopf algebras such as NCSym and FQSym.
- There are a family of "canonical" homomorphisms from NSym to Sym that depend on an algebra homomorphism NSym$\rightarrow \mathbb{C}$. Representation theory gives of a natural set of choices for such algebra homomorphisms.

A noncanonical categorification

Thm. [A–T] Let $\mathsf{f}_{\sim}(H^{\bullet})$ denote the direct product structure. Then

- $\mathsf{f}_{\sim}(H^{\bullet})$ is a Hopf algebra if and only if $\langle \iota, \alpha \rangle = \langle \iota, \beta \rangle = 1$.

Remarks.

- By varying α and β we get different isomorphisms, that send characters of G_n to different bases (e.g. h-basis, ribbon basis, etc.).

- By varying α and β we get different "natural" homomorphisms to other Hopf algebras such as NCSym and FQSym.
- There are a family of "canonical" homomorphisms from NSym to Sym that depend on an algebra homomorphism NSym$\to \mathbb{C}$. Representation theory gives of a natural set of choices for such algebra homomorphisms.

A noncanonical categorification

Thm. [A–T] Let $f_{\sim}(H^{\bullet})$ denote the direct product structure. Then

. $f_{\sim}(H^{\bullet})$ is a Hopf algebra if and only if $\langle \iota, \alpha \rangle = \langle \iota, \beta \rangle = 1$.

Remarks.

. By varying α and β we get different isomorphisms, that send characters of G_n to different bases (e.g. h-basis, ribbon basis, etc.).

. By varying α and β we get different "natural" homomorphisms to other Hopf algebras such as NCSym and FQSym.

. There are a family of "canonical" homomorphisms from NSym to Sym that depend on an algebra homomorphism NSym$\rightarrow \mathbb{C}$. Representation theory gives of a natural set of choices for such algebra homomorphisms.

A noncanonical categorification

Thm. **[A–T]** Let $f_\sim(H^\bullet)$ denote the direct product structure. Then

. $f_\sim(H^\bullet)$ is a Hopf algebra if and only if $\langle \iota, \alpha \rangle = \langle \iota, \beta \rangle = 1$.

Remarks.

. By varying α and β we get different isomorphisms, that send characters of G_n to different bases (e.g. h-basis, ribbon basis, etc.).

. By varying α and β we get different "natural" homomorphisms to other Hopf algebras such as NCSym and FQSym.

. There are a family of "canonical" homomorphisms from NSym to Sym that depend on an algebra homomorphism NSym$\rightarrow \mathbb{C}$. Representation theory gives of a natural set of choices for such algebra homomorphisms.

. By varying the supercharacter theory on H, we obtain a family of Hopf algebras that behave NSym-like.