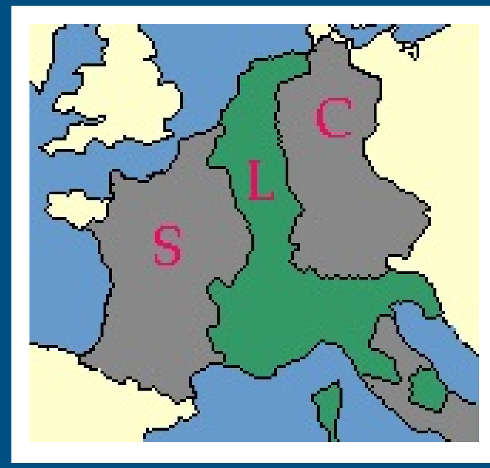
SLC 89
89th Séminaire Lotharingien de Combinatoire
26-29 March 2023
Centro Residenziale Universitario di Bertinoro, Italy



Alternating Sign Matrices With Reflective Symmetry and Plane Partitions: $n+3$ Pairs of Equivalent Statistics

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Correspondence between ASMs and DPPs (Andrews 1979, Zeilberger 1996)

ASMs and DPPs of the same order are equinumerous.

Alternating sign matrices (ASMs)

An **alternating sign matrix (ASM) of order n** is an $n \times n$-matrix with entries -1, 0 or $+1$ such that

- the entries in each row and each column sum to 1, and
- the nonzero entries alternate in sign along each row and each column.

Descending plane partitions (DPPs)

A **descending plane partition (DPP) of order n** is a filling of a shifted Young diagram with positive integers less than or equal to n such that

- the entries weakly decrease along rows
- and strictly decrease down columns, and
- the first part in each row is strictly larger than the length of the row
- but less than or equal to the length of the previous row.

ASM of order 4 and DPP of order $\geqslant 6$

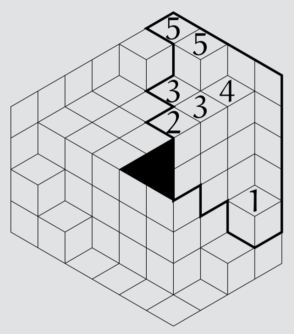
Monotone triangle (MT)

$$\begin{array}{ccccccc} & & & 2 & & & \\ & & 2 & & 3 & & \\ & 1 & & 2 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array} \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Cyclically symmetric rhombus tiling

5	5	4	1
	3	3	
		2	

$\longleftrightarrow$



Reflective symmetry in ASMs

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \circlearrowleft \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 2 & & 4 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 3 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Correspondence between ASMs and MTs (Mills, Robbins, Rumsey 1983)

There is a bijective correspondence between

- ASMs of order n and MTs with bottom row $1, 2, \dots, n$ and
- vertically symmetrice ASMs of order $2n+1$ and MTs with bottom row $0, 2, \dots, 2n-2$.

Arrowed monotone triangles (AMTs) with $n+3$ statistics

An **arrowed monotone triangle (AMT) of order n** is a MT of order n where each entry e carries a decoration from $\{\nwarrow, \nearrow, \nwarrow\!\!\!\nearrow\}$ such that the following two conditions are satisfied:

- If e has a $\nwarrow$-neighbor and is equal to it, then e must carry $\nearrow$.
- If e has a $\nearrow$-neighbor and is equal to it, then e must carry $\nwarrow$.

We assign the weight:

$$u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\nearrow} \prod_{i=1}^{n} X_i^{(\sum \text{row } i)-(\sum \text{row } i-1)+(\#\nearrow \text{ in row } i)-(\#\nwarrow \text{ in row } i)}.$$

AMT of order 3

The weight of

$$\begin{array}{ccccc} & & \nwarrow 3 \nearrow & & \\ & \nwarrow 1 & & \nwarrow 4 & \\ \nwarrow 0 \nearrow & & \nwarrow 2 & & 4 \nearrow \end{array}$$

is $uv^3w^2X_1^3X_3$.

Generating function

The generating function of AMTs with bottom row $0, 2, \dots, 2n-2$ is given by

$$\prod_{i=1}^{n} \frac{X_i^{n-2}}{u - vX_i^{-2}} \cdot \frac{\det_{1 \leqslant i,j \leqslant n} \left(\left(u^2X_i^2 + uwX_i\right)^j - \left(v^2X_i^{-2} + vwX_i^{-1}\right)^j \right)}{\prod_{1 \leqslant i < j \leqslant n} (X_j - X_i) \left(u - vX_i^{-1}X_j^{-1}\right)}.$$

Which families of (non-intersecting) lattice paths or plane partition objects have the same generating function?

- three signed models in terms of lattice paths
- one signless model in terms of pairs of plane partitions

$\longrightarrow$ different proofs by algebraic manipulations and lattice path combinatorics

Pair of plane partitions with $n+3$ statistics

Pairs (P, Q) of plane partitions of the same shape such that

- P and Q have n rows allowing rows of length 0;
- P is column-strict;
- Q is row-strict;
- row restrictions on P and Q.

The weight is

$$w^{\binom{n+1}{2} - \#\,\text{entries in } Q} \prod_{i=1}^{n} X_i^{n-1} (uX_i)^{\#2i-1 \text{ in } P} (vX_i^{-1})^{\#2i \text{ in } P}.$$

(P, Q) with three rows and weight $u^3v^2wX_1X_2^4X_3^2$

Symplectic tableau

$\bar{1}$	2
2	2
$\bar{3}$	

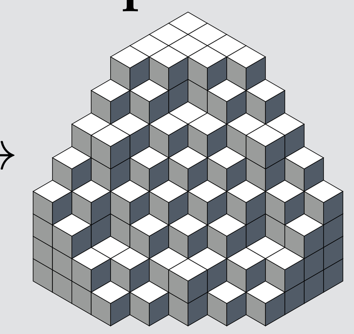
$\longleftrightarrow$

$$\begin{array}{r|cc|} & \multicolumn{2}{c}{\geqslant} \\ 6 \geqslant & 6 & 5 \\ 4 \geqslant & 3 & 3 \\ 2 \geqslant & 2 & \end{array} \vee$$

Totally symmetric self-complementary plane partition

$$\begin{array}{r|cc|} & \multicolumn{2}{c}{>} \\ 3 \geqslant & 3 & 1 \\ 2 \geqslant & 2 & 1 \\ 1 \geqslant & 1 & \end{array} \veebar$$

$\longleftrightarrow$



Expansion into symmetric functions

Expansion of the generating function into symplectic Schur functions with coefficients given by totally symmetric self-complementary plane partitions

Proof idea

- Transform generating function into Jacobi–Trudi-type formula
- Interpret formula as signed enumeration of lattice paths via Lindström–Gessel–Viennot lemma
- Apply sign-reversing involutions
- Read off pair of plane partitions