A local characterization of quasi-crystal graphs

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Motivation

Plactic monoid
[Lascoux, Schützenberger '81]

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- Young tableaux, Schensted insertion

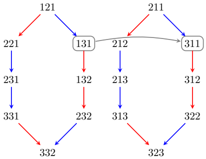
1	1	1	1
2	2		
3			

- Knuth relations

$$acb \equiv cab,\ a \le b < c$$

$$bac \equiv bca,\ a < b \le c$$

- Crystals



- Schur functions s_λ.

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Hypoplactic monoid
[Krob, Thibon '97], [Novelli '00]

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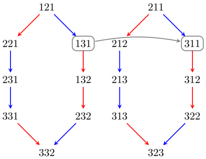

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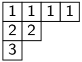

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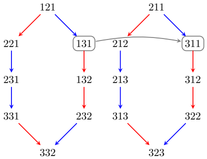
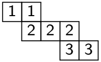


- Schur functions s_λ.

Hypoplactic monoid
[Krob, Thibon '97], [Novelli '00]

- Quasi-ribbon tableaux, Krob–Thibon insertion

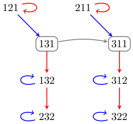


- Knuth + quartic relations

$$cadb \equiv acbd, \; a \leq b < c \leq d$$

$$bdac \equiv dbca, \; a < b \leq c < d$$

- Quasi-crystals



- Fundamental quasisymmetric functions F_α.

Crystals

Definition

A **crystal** of type A_{n-1} is a non-empty set $\mathcal{C}$ together with maps

$$\tilde{e}_i, \tilde{f}_i : \mathcal{C} \longrightarrow \mathcal{C} \sqcup \{\perp\} \qquad \text{(Kashiwara operators)}$$

$$\tilde{\varepsilon}_i, \tilde{\varphi}_i : \mathcal{C} \longrightarrow \mathbb{Z} \sqcup \{-\infty\} \qquad \text{(length functions)}$$

$$wt : \mathcal{C} \longrightarrow \mathbb{Z}^n \qquad \text{(weight function)}$$

for $i \in I := \{1, \dots, n-1\}$, satisfying the following:

C1. For any $x, y \in \mathcal{C}$, $\tilde{e}_i(x) = y$ iff $x = \tilde{f}_i(y)$, and in that case

$$wt(y) = wt(x) + \alpha_i, \quad \tilde{\varepsilon}_i(y) = \tilde{\varepsilon}_i(x) + 1, \quad \tilde{\varphi}_i(y) = \tilde{\varphi}_i(x) - 1$$

C2. $\tilde{\varphi}_i(x) = \tilde{\varepsilon}_i(x) + \langle wt(x), \alpha_i \rangle$

where $\alpha_i = (0, \dots, 0, 1, -1, 0, \dots, 0)$.

(This definition is generalized for other Cartan types)

- $\mathcal{A}_n = \{1 < \cdots < n\}$.
- A word $w = w_1 \cdots w_k \in \mathcal{A}_n^*$ has an i-**inversion**, if $(i+1)i$ occurs as a subword of w.

1 2 3 1 1 3 2 4

- To compute $\tilde{f}_i(w)$ and $\tilde{e}_i(w)$ on a word $w \in \mathcal{A}_n^*$:
  - consider the subword with only symbols i and $i+1$, replace each i with) and each $i+1$ with (.
  - cancel all pairs ( ), until there are no pairs left.
  - $\tilde{e}_i$ changes the *leftmost* (to), if possible; if not, it is $\perp$.
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2-inversion

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Crystals

Kashiwara operators

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Crystals

- A crystal is **seminormal** if, for all $i \in I$ and $x \in \mathcal{C}$,

$$\tilde{\varepsilon}_i(x) = \max\{k : \tilde{e}_i(x)^k \neq \bot\}, \qquad \tilde{\varphi}_i(x) = \max\{k : \tilde{f}_i(x)^k \neq \bot\}$$

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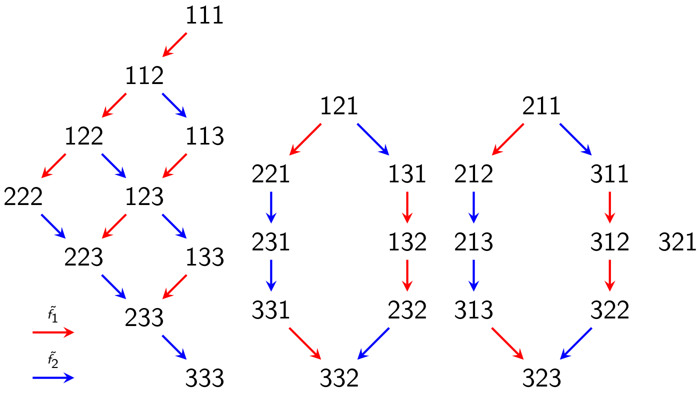


Stembridge crystals

- A **Stembridge crystal** is a seminormal crystal of simply-laced type that satisfies some local axioms [Stembridge '03].
- These are the crystal graphs that correspond to representations of Lie algebras.
- Nice properties on the connected components:
  - Uniqueness of highest weight element (source vertex).
  - All vertices can be reached from the highest weight element.
  - In type A, the highest weight is dominant and, if it is a partition, the character of the component is a Schur function s_λ.
  - All components whose highest weight elements have the same weight are isomorphic.

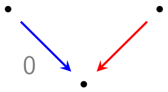
Stembridge crystals

Local axioms

S1. If $\tilde{e}_i(x) = y$, then $\tilde{\varepsilon}_j(y)$ is equal to $\tilde{\varepsilon}_j(x)$ or $\tilde{\varepsilon}_j(x) + 1$ (the second case is possible only if $|i - j| = 1$).

S2. If $\tilde{e}_i(x) = y$ and $\tilde{e}_j(x) = z$, and $\tilde{\varepsilon}_i(z) = \tilde{\varepsilon}_i(x)$ then

$$\tilde{e}_i\tilde{e}_j(x) = \tilde{e}_j\tilde{e}_i(x) \neq \perp.$$



(and dual axioms for $\tilde{f}_i$, $\tilde{f}_j$)

S3. If $\tilde{e}_i(x) = y$ and $\tilde{e}_j(x) = z$, and $\tilde{\varepsilon}_i(z) = \tilde{\varepsilon}_i(x) + 1$ and $\tilde{\varepsilon}_j(y) = \tilde{\varepsilon}_j(x) + 1$ then

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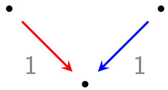

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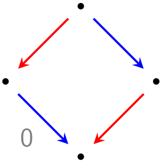

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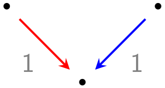

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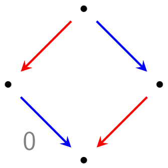

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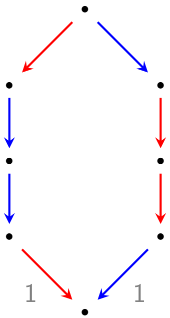

Quasi-crystals

- Quasi-crystals were first introduced by Cain and Malheiro (2017), to provide another characterization of the hypoplactic monoid of type A, where $u \equiv_{\text{hypo}} v$ iff u and v are in the same position of isomorphic quasi-crystal components.
- Each connected component has a unique highest weight element, and is isomorphic to a quasi-crystal of quasi-ribbon tableaux, which are indexed by compositions.
- The characters are fundamental quasisymmetric functions F_α.
- Noting the decomposition of Schur functions into fundamental quasi-symmetric functions, Maas-Gariépy (2023) independently introduced quasi-crystals, as subgraphs of a connected component of a crystal graph.
- Cain, Guilherme and Malheiro (2023) recently provided a definition of abstract quasi-crystals for other Cartan types.

Quasi-crystals

Definition (Cain, Guilherme, Malheiro '23)

A **quasi-crystal** of type A_{n-1} is a non-empty set $\mathcal{Q}$ together with maps

$$\ddot{e}_i, \ddot{f}_i : \mathcal{Q} \longrightarrow \mathcal{Q} \sqcup \{\bot\} \qquad \text{(quasi-Kashiwara operators)}$$
$$\ddot{\varepsilon}_i, \ddot{\varphi}_i : \mathcal{Q} \longrightarrow \mathbb{Z} \sqcup \{-\infty, +\infty\}$$
$$wt : \mathcal{Q} \longrightarrow \mathbb{Z}^n$$

for $i \in \{1, \ldots, n-1\}$, satisfying the following:

QC1. For any $x, y \in \mathcal{C}$, $\ddot{e}_i(x) = y$ iff $x = \ddot{f}_i(y)$, and in that case

$$wt(y) = wt(x) + \alpha_i, \quad \ddot{\varepsilon}_i(y) = \ddot{\varepsilon}_i(x) + 1, \quad \ddot{\varphi}_i(y) = \ddot{\varphi}_i(x) - 1$$

QC2. $\ddot{\varphi}_i(x) = \ddot{\varepsilon}_i(x) + \langle wt(x), \alpha_i \rangle$

QC3. If $\ddot{\varepsilon}_i(x) = +\infty$, then $\ddot{e}_i(x) = \ddot{f}_i(x) = \bot$.

Quasi-crystals

- A crystal is a quasi-crystal $\mathcal{Q}$ where $\ddot{\varepsilon}_i(x) \neq +\infty$ and $\ddot{\varphi}_i(x) \neq +\infty$, for all $i \in I, x \in \mathcal{Q}$.
- A quasi-crystal is **seminormal** if, for all $i \in I$ and $x \in \mathcal{Q}$,

$$\tilde{\varepsilon}_i(x) = \max\{k : \tilde{e}_i(x)^k \neq \bot\}$$
$$\tilde{\varphi}_i(x) = \max\{k : \tilde{f}_i(x)^k \neq \bot\}$$

  whenever $\ddot{\varepsilon}_i(x) \neq +\infty$.
- A crystal is seminormal (as a crystal) iff it is seminormal as a quasi-crystal.
- For the quasi-crystal of words:
  - $\ddot{\varepsilon}_i(w) = +\infty$ iff w has an i-inversion.
  - $\ddot{e}_i(w)$ coincides with $\tilde{e}_i(w)$ if w has no i-inversions, otherwise $\ddot{e}_i(w) = \bot$.

Quasi-crystals

The **quasi-crystal graph** associated to a quasi-crystal $\mathcal{Q}$ is the directed weighted graph where:

- $x \xrightarrow{i} y$ iff $\ddot{f}_i(x) = y$.
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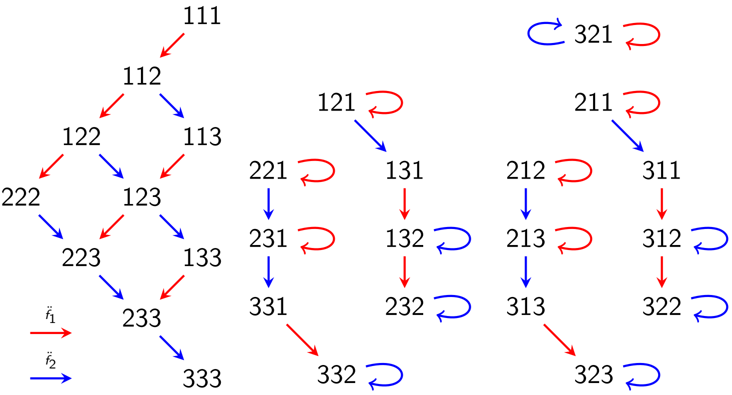

Quasi-tensor product

Cain, Guilherme, and Malheiro (2023) introduced a notion of quasi-tensor product of seminormal quasi-crystals, denoted $\mathcal{Q} \ddot{\otimes} \mathcal{Q}'$, which has $\mathcal{Q} \times \mathcal{Q}'$ as underlying set and maps:

- $wt(x \ddot{\otimes} x') = wt(x) + wt(x')$.
- If $\ddot{\varphi}_i(x) > 0$ and $\ddot{\varepsilon}_i(x') > 0$, $\ddot{e}_i(x \ddot{\otimes} x') = \ddot{f}_i(x \ddot{\otimes} x') = \perp$ and $\ddot{\varepsilon}_i(x \ddot{\otimes} x') = \ddot{\varphi}_i(x \ddot{\otimes} x') = +\infty$, otherwise,

$$\ddot{e}_i(x \ddot{\otimes} x') = \begin{cases} \ddot{e}_i(x) \ddot{\otimes} x' & \text{if } \ddot{\varphi}_i(x) \geq \ddot{\varepsilon}_i(x') \\ x \ddot{\otimes} \ddot{e}_i(x') & \text{if } \ddot{\varphi}_i(x) < \ddot{\varepsilon}_i(x') \end{cases}$$

$$\ddot{f}_i(x \ddot{\otimes} x') = \begin{cases} \ddot{f}_i(x) \ddot{\otimes} x' & \text{if } \ddot{\varphi}_i(x) > \ddot{\varepsilon}_i(x') \\ x \ddot{\otimes} \ddot{f}_i(x') & \text{if } \ddot{\varphi}_i(x) \leq \ddot{\varepsilon}_i(x') \end{cases}$$

$$\ddot{\varepsilon}_i(x) = \max\{\ddot{\varepsilon}_i(x), \ddot{\varepsilon}_i(x') - \langle wt(x), \alpha_i \rangle\}$$
$$\ddot{\varphi}_i(x) = \max\{\ddot{\varphi}_i(x) + \langle wt(x'), \alpha_i \rangle, \ddot{\varphi}_i(x')\}$$

(With this convention $x \ddot{\otimes} y$ is identified with the word yx.)

Quasi-tensor product

- $\mathcal{B}_n$ is the standard crystal of type A_{n-1}:

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \quad \cdots \quad \xrightarrow{n-1} n$$

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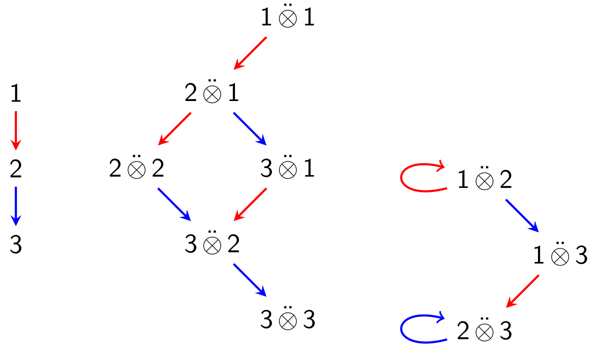

$\mathcal{B}_3$ $\qquad$ $\mathcal{B}_3 \ddot{\otimes} \mathcal{B}_3$

Local characterization of quasi-crystals

Local quasi-crystal axioms

LQC1. $\ddot{\varepsilon}_i(x) = 0$ iff $\ddot{\varphi}_{i+1}(x) = 0$, for $i \in \{1, \dots, n-2\}$.

LQC2. If $\ddot{e}_i(x) = y$, then:

- For $|i-j| > 1$, $\ddot{\varepsilon}_j(x) = \ddot{\varepsilon}_j(y)$.
- For $j = i+1$,

$$\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y) \Leftrightarrow \big(\ddot{\varepsilon}_{i+1}(x) = +\infty \wedge \ddot{\varepsilon}_i(y) = 0\big) \Rightarrow \ddot{\varepsilon}_{i+1}(y) > 0.$$

- For $j = i-1$,

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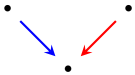
LQC3. If both $\ddot{e}_i(x)$ and $\ddot{e}_j(x)$ are defined, for $i \neq j$, then $\ddot{e}_i\ddot{e}_j(x) = \ddot{e}_j\ddot{e}_i(x) \neq \perp$ (and dual axiom for $\ddot{f}_i$, $\ddot{f}_j$.)



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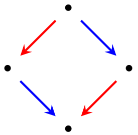

Local characterization of quasi-crystals

Theorem (Cain, Malheiro, Rodrigues, R. '23)

If $\mathcal{Q}$ is a quasi-crystal of type A (not necessarily seminormal) satisfying the local axioms, and such that $\ddot{\varepsilon}_i(x) \neq +\infty$ and $\ddot{\varphi}_i(x) \neq +\infty$, for all $i \in I, x \in \mathcal{Q}$, then $\mathcal{Q}$ is a weak Stembridge crystal (i.e. not necessarily seminormal).

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Theorem (Cain, Malheiro, Rodrigues, R. '23)

Let $\mathcal{Q}$ and $\mathcal{Q}'$ be connected components of seminormal quasi-crystal graphs of type A satisfying the local axioms, with highest weight elements u and v. If $wt(u) = wt(v)$, then there exists a weight-preserving isomorphism between $\mathcal{Q}$ and $\mathcal{Q}'$.

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Let $\mathcal{Q}$ and $\mathcal{Q}'$ be seminormal quasi-crystal graphs satisfying the local axioms. Then, $\mathcal{Q} \ddot{\otimes} \mathcal{Q}'$ is a seminormal quasi-crystal that satisfies the same axioms.

- The standard crystal $\mathcal{B}_n$ satisfies the local axioms.
- In particular, the quasi-crystal of words satisfies the local axioms.
- As a consequence, every connected component of a seminormal quasi-crystal satisfying the local axioms is isomorphic a quasi-crystal of quasi-ribbon tableaux.

From crystals to quasi-crystals

Let $(\mathcal{C}, \tilde{f}_i, \tilde{e}_i, \tilde{\varepsilon}_i, \tilde{\varphi}_i)$ be a connected component of a Stembridge crystal, weighted in $\mathbb{Z}^n_{\geq 0}$, and define $(\mathcal{Q}, \ddot{f}_i, \ddot{e}_i, \ddot{\varepsilon}_i, \ddot{\varphi}_i)$ to have the same underlying set as $\mathcal{C}$ and define:

$$\ddot{\varepsilon}_i(x) := \begin{cases} \tilde{\varepsilon}_i(x) & \text{if } \tilde{\varepsilon}_i(x) = wt_{i+1}(x) \\ +\infty & \text{otherwise} \end{cases}$$

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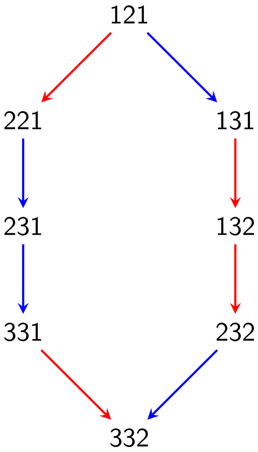
$\mathcal{Q}$ is a seminormal quasi-crystal that satisfies the local axioms.

From crystals to quasi-crystals



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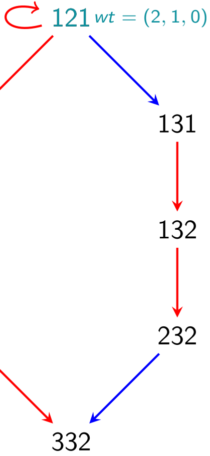

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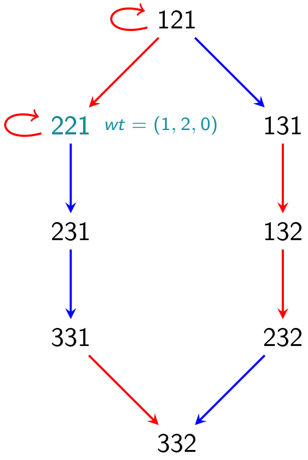

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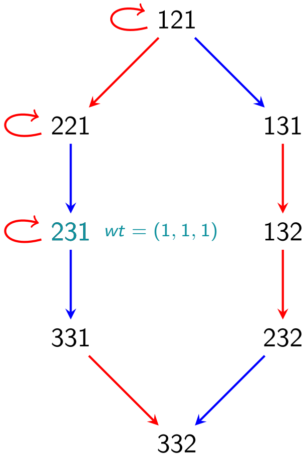

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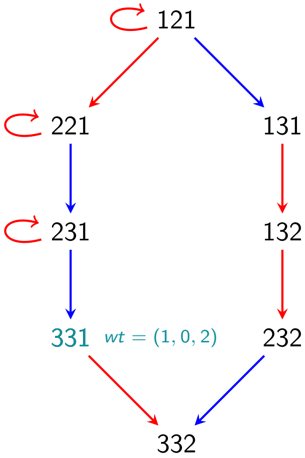

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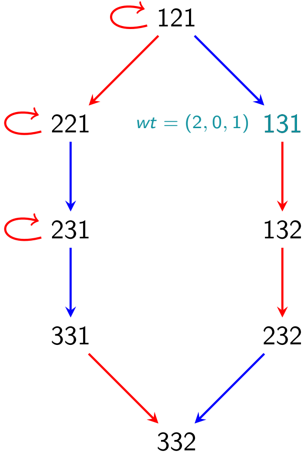

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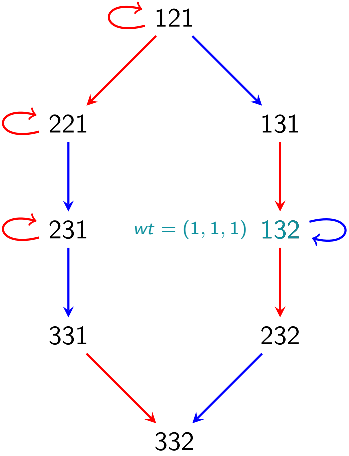

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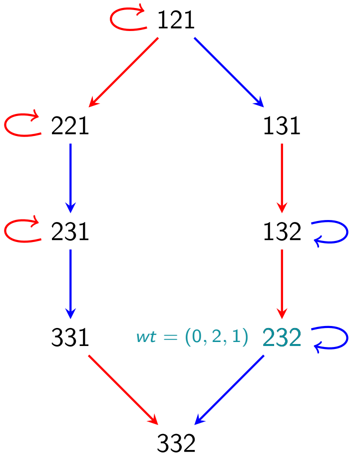

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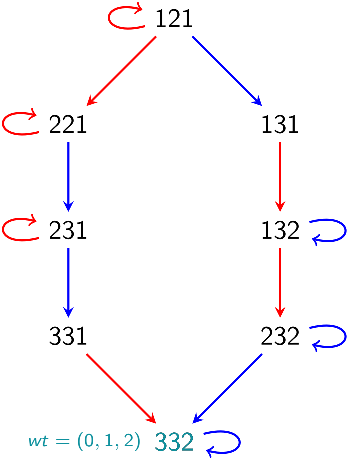

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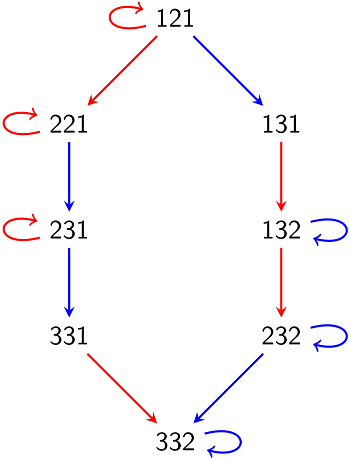

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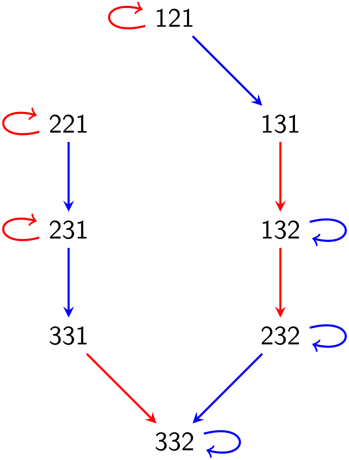

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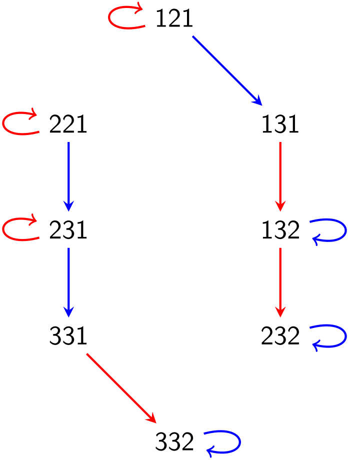

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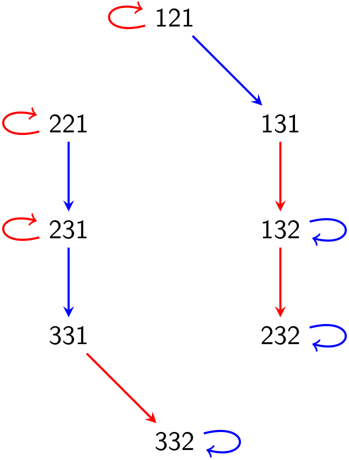

This also illustrates $s_{2,1} = F_{2,1} + F_{1,2}$.

Some references

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Thank you!