Fully Complementary Higher Dimensional Partitions

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Partitions

A partition $\lambda = (\lambda_1, \lambda_2, \dots,)$ is a weakly decreasing sequence of non-negative integers with all but finitely many entries equal to 0. We define the size $|\lambda| = \lambda_1 + \lambda_2 + \cdots$.

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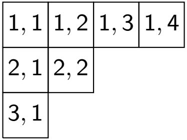

A Young diagram λ is a finite subset of $\mathbb{N}^2_{>0}$ such that $(x_1, x_2) \in \lambda$ implies $(y_1, y_2) \in \lambda$ for $1 \leq y_i \leq x_i$ for $1 \leq i \leq 2$.

Generating functions I

Theorem

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$$\sum_{\lambda} q^{|\lambda|} = \prod_{i \geq 1} \frac{1}{1-q^i}.$$

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A Young diagram λ is contained in an (a,b)-box if $\lambda \subseteq [a] \times [b]$.

Theorem

The generating function of Young diagrams inside an (a,b)-box is

$$\sum_{\lambda} q^{|\lambda|} = \begin{bmatrix} a+b \\ a \end{bmatrix}_q = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{1-q^{i+j}}{1-q^{i+j-1}}.$$

Plane partitions

A plane partition π is an array $(\pi_{i,j})$ of non-negative integers and finite support, which is weakly decreasing along rows and columns.

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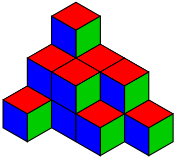

A 2-dimensional Young diagram λ is a finite subset of $\mathbb{N}^3_{>0}$ such that $(x_1, x_2, x_3) \in \lambda$ implies $(y_1, y_2, y_3) \in \lambda$ for $1 \leq y_i \leq x_i$ for $1 \leq i \leq 3$.

Generating functions II

Theorem (MacMahon)

The generating function for plane partitions is

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Theorem (MacMahon)

The generating function for 2-dimensional Young diagrams inside an (a, b, c)-box is

$$\sum_{\lambda} q^{|\lambda|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}.$$

d-dimensional partitions

A d-dimensional partition π is an array $(\pi_{i_1,\ldots,i_d})$ of non-negative integers and finite support, such that

$$\pi_{i_1,\ldots,i_k+1,\ldots,i_d} \geq \pi_{i_1,\ldots,i_d},$$

for all $i_1, \ldots, i_d \in \mathbb{N}_{>0}$ and $1 \leq k \leq d$.

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$$\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \qquad \Leftrightarrow \qquad \{(1,1,1,1),(1,1,2,1)\}$$

A d-dimensional Young diagram λ is a finite subset of of $\mathbb{N}_{>0}^{d+1}$ such that $\mathbf{x} \in \lambda$ implies $\mathbf{y} \in \lambda$ for $1 \leq y_i \leq x_i$ for $1 \leq i \leq d+1$.

Generating functions III

Conjecture (MacMahon)

The generating function of d-dimensional partitions π is

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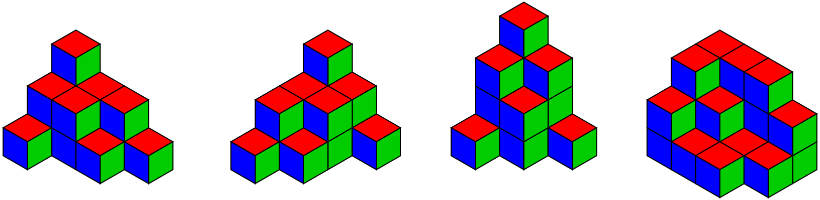
Theorem (Amanov–Yeliussizov, 2023)

The generating function of d-dimensional partitions π with respect to two statistics cor and $|\cdot|_{ch}$ is given by

$$\sum_{\pi} t^{cor(\pi)} q^{|\pi|_{ch}} = \prod_{i\geq 1} (1-tq^i)^{-\binom{i+d-2}{d-1}}.$$

Symmetries of boxed plane partitions



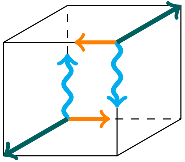
reflection rotation complementation

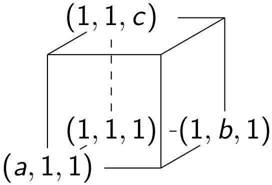
Self-complementary VS fully complementary

A $2d$-Young diagram λ inside an (a, b, c)-box is called self-complementary if it is equal to its complementation.

Self-complementary VS fully complementary

A $2d$-Young diagram λ inside an (a, b, c)-box is called self-complementary if λ, and λ "placed" at the corner (a, b, c) fills the (a, b, c)-box without overlap.





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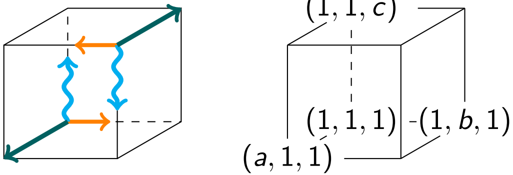

A $2d$-Young diagram λ inside an (a,b,c)-box is called fully complementary if λ, and λ "placed" at the corners $(a,b,1)$, $(a,1,c)$ and $(1,b,c)$ fill the (a,b,c)-box without overlap.

Fully Complementary in higher dimensions

Let $\mathbf{n} = (n_1, \dots, n_{d+1})$ be a sequence of positive integers and $I \subseteq [d+1] = \{1, 2, \dots, d+1\}$. We define for $\mathbf{x} = (x_1, \dots, x_{d+1})$

$$\rho_{I,\mathbf{n}}(\mathbf{x}) := \left(\begin{cases} x_i & i \notin I, \\ 2n_i + 1 - x_i & i \in I, \end{cases} \right)_{1 \leq i \leq d+1} .$$

A d-dimensional Young diagram λ is called fully complementary inside a $(2n_1, \dots, 2n_{d+1})$-box if

- for all even sized $I \neq J \subseteq [d+1]$ holds $\rho_{I,2\mathbf{n}}(\lambda) \cap \rho_{J,2\mathbf{n}}(\lambda) = \emptyset$,
- and $\displaystyle\bigcap_{\substack{I \subseteq [d+1] \\ |I| \text{ even}}} \rho_{I,2\mathbf{n}}(\lambda) = [2n_1] \times \cdots [2n_{d+1}]$.

An example

For $\mathbf{n} = (1,1,1,1)$ the 3-dimensional Young diagram $\lambda = \{(1,1,1,1),(2,1,1,1)\}$ is fully complementary inside the $(2,2,2,2)$-box.

$$\rho_{\{1,2\}}(\lambda) = \{(2,2,1,1),(1,2,1,1)\}, \qquad \rho_{\{2,3\}}(\lambda) = \{(1,2,2,1),(2,2,2,1)\},$$
$$\rho_{\{1,3\}}(\lambda) = \{(2,1,2,1),(1,1,2,1)\}, \qquad \rho_{\{2,4\}}(\lambda) = \{(1,2,1,2),(2,2,1,2)\},$$
$$\rho_{\{1,4\}}(\lambda) = \{(2,1,1,2),(1,1,1,2)\}, \qquad \rho_{\{3,4\}}(\lambda) = \{(1,1,2,2),(2,1,2,2)\},$$
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There are three further Young diagrams which are fully complementary inside $(2,2,2,2)$:

$$\{(1,1,1,1),(1,2,1,1)\}$$
$$\{(1,1,1,1),(1,1,2,1)\}$$
$$\{(1,1,1,1),(1,1,1,2)\}$$

Generating functions IV

We call a d-dimensional partition π fully complementary inside a $(2n_1, \ldots, 2n_{d+1})$-box if its Young diagram is fully complementary in this box.

Theorem (SA, 2023)

Let $\mathbf{x} = (x_1, \ldots, x_{d+1})$, $\mathbf{n} = (n_1, \ldots, n_{d+1}) \in \mathbb{N}_{>0}^{d+1}$ and denote by $\mathrm{FCP}(\mathbf{n})$ the set of fully complementary partitions inside a $(2n_1, \ldots, 2n_{d+1})$-box. Then

$$\sum_{\mathbf{n} \in \mathbb{N}_{>0}^{d+1}} |\,\mathrm{FCP}(\mathbf{n})|\mathbf{x}^{\mathbf{n}} = \frac{\left(d+1-\sum_{i=1}^{d+1} x_i\right) \prod_{i=1}^{d+1} x_i}{\left(1-\sum_{i=1}^{d+1} x_i\right) \prod_{i=1}^{d+1} (1-x_i)}.$$

Stretching maps

For $1 \leq k \leq d$ define the map $\varphi_k : \mathrm{FCP}(\mathbf{n}) \to \mathrm{FCP}(\mathbf{n} + e_k)$

$$\varphi_k(\pi)_{i_1,\ldots,i_d} = \begin{cases} \pi_{i_1,\ldots,i_d} & i_k \leq n_k, \\ n_{d+1} & i_k \in \{n_k+1, n_k+2\} \\ & \text{ and } i_j \leq n_j \text{ for all } 1 \leq j \neq k \leq d, \\ \pi_{i_1,\ldots,i_k-2,\ldots,i_d} & i_k > n_k + 2, \\ 0 & \text{otherwise,} \end{cases}$$

and the map $\varphi_{d+1} : \mathrm{FCP}(\mathbf{n}) \to \mathrm{FCP}(\mathbf{n} + e_{d+1})$ as

$$\varphi_{d+1}(\pi)_{i_1,\ldots,i_d} = \begin{cases} \pi_{i_1,\ldots,i_d} + 2 & i_j \leq n_j \text{ for all } 1 \leq j \leq d, \\ \pi_{i_1,\ldots,i_d} & \text{ otherwise.} \end{cases}$$

A recursive structure

Proposition

Let $\mathbf{n} = (n_1, \ldots, n_{d+1})$ *be a sequence of positive integers. Then* $\mathrm{FCP}(\mathbf{n})$ *is equal to the disjoint union*

$$\mathrm{FCP}(\mathbf{n}) = \dot{\bigcup}_{1 \le k \le d+1} \varphi_k\big(\mathrm{FCP}(\mathbf{n} - e_k)\big).$$

Note, that if exactly one $n_i = 0$, we have to define $\mathrm{FCP}(\mathbf{n})$ to consist of "the empty array" and extend the definitions of the φ_k appropriately.

Symmetry classes of FCPs I

There are many interesting symmetry classes of plane partitions. Is the same true for fully complementary plane partitions?

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symmetric	$\Rightarrow$	quasi symmetric
cyclically symmetric	$\Rightarrow$	?
self-complementary	$\Rightarrow$	self-complementary
transpose-complementary	$\Rightarrow$	quasi transpose-complementary

Symmetry classes of FCPs II

Proposition

1. *Denote by* $\mathrm{QS}(a, c)$ *the set of quasi symmetric fully complementary plane partitions (FCPP) inside an* (a, a, c)*-box, then holds*

$$\sum_{a,c \geq 0} |\,\mathrm{QS}(a, c)| x^a y^c = \frac{x + y - 2x^2 - xy}{(1 - x)(1 - 2x - y)}.$$

2. *The number of self-complementary FCPPs inside an* $(2a, 2b, 2c)$*-box is* $\binom{a+b}{a}$.

3. *The number of quasi transpose-complementary FCPPs inside an* $(2a, 2a, 2c)$*-box is* 2^a.

Quasi transpose complementary PPs

A plane partition π inside an (a, a, c)-box is called quasi transpose complementary if

$$\pi_{i,j} + \pi_{a+1-j,a+1-i} = c,$$

holds for all $1 \leq i, j \leq n$ with $i + j \neq a + 1$.

Theorem

The number of quasi transpose-symmetric plane partitions inside an (a, a, c)-box is equal to the number of symmetric plane partitions inside an (a, a, c)-box.

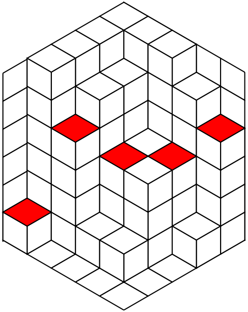
As a side product we stumble upon the relation

$$2^{n-1}\,\mathrm{TCPP}(n, n, 2c) = \mathrm{SPP}(n-1, n-1, 2c+1).$$

I have also some conjectures for other quasi symmetry classes.

Proof idea

6	6	6	5	4
6	5	3	3	1
6	5	3	3	0
6	4	1	1	0
1	0	0	0	0



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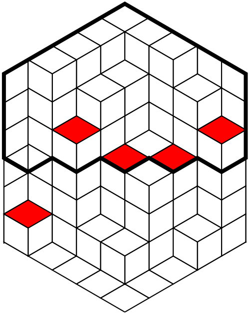

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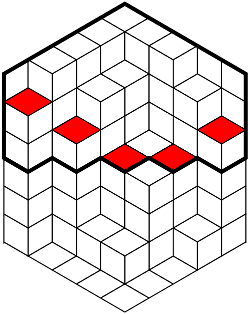

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