Character factorisations and the Littlewood map

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The Theory of Group Characters and Matrix Representations of Groups

Second Edition

Dudley E. Littlewood

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Page 133 of Littlewood:

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$$r[\mu_{q1}+m-1]+q,\quad r[\mu_{q2}+m-2]+q,\quad \ldots,\quad r\mu_{qm}+q,$$

then

$$\{\lambda\}=\theta\{\mu_{01},\mu_{02},\ldots,\mu_{0m}\}'\{\mu_{11},\ldots,\mu_{1m}\}'\ldots\{\mu_{r-1,1},\ldots,\mu_{r-1,m}\}',$$

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The dictionary

$$\lambda \longrightarrow \text{a partition with empty } r\text{-core}$$

$$\mu_{i,j} \longrightarrow \text{the } r\text{-quotient of } \lambda$$

$$\{\lambda\} \longrightarrow \text{the Schur function indexed by } \lambda$$

$$\theta \longrightarrow \text{some sign}$$

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly decreasing sequence

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \cdots$$

such that only finitely $\lambda_i \neq 0$. The number of nonzero λ_i, written $\ell(\lambda)$, is called the length and the sum is $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots$.

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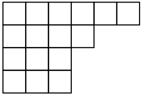
For example $\lambda = (6, 4, 3, 3)$ has $\ell(\lambda) = 4$ and $|\lambda| = 16$. Its Young diagram is given by



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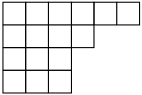
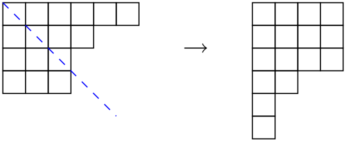


The conjugate partition is obtained by reflecting the Young diagram in the "main diagonal"



so that $(4, 4, 4, 2, 1, 1)$ is the conjugate of $(6, 4, 3, 3)$.

In  Littlewood defines *"S-functions of series"*. These take as input a formal power series

$$f(z) := 1 + \sum_{k \geqslant 1} f_k z^k.$$

Then for any λ the S-function s_λ^f may be defined by the Jacobi–Trudi determinant

$$s_\lambda^f := \det_{1 \leqslant i,j \leqslant \ell(\lambda)} (f_{\lambda_i - i + j}),$$

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If we take

$$f(z) = \prod_{i=1}^{\infty} \frac{1}{1 - z x_i}$$

then the f_k are just the complete homogeneous symmetric functions

$$f_k = h_k(X) := \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_k} x_{i_1} \cdots x_{i_k}.$$

Therefore s_λ^f is just the ordinary Schur function in this case.

Littlewood's theorem is about comparing the ordinary Schur functions with those given by the series

$$f(z^r) = \prod_{i=1}^{\infty} \frac{1}{1 - z^r x_i} = \sum_{k \geqslant 0} h_k z^{kr}.$$

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This is best encoded by an operator $\varphi_r : \Lambda \longrightarrow \Lambda$ where $r \geqslant 2$ which acts on the h_k by

$$\varphi_r h_k = \begin{cases} h_{k/r} & \text{if } r \mid k, \\ 0 & \text{otherwise.} \end{cases}$$

Since any symmetric function can be written as a polynomial in the h_r we can extend this action to any $f \in \Lambda$.

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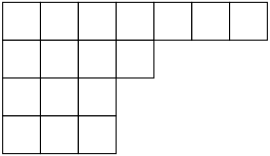
Thus Littlewood computed

$$\varphi_r s_\lambda.$$

Cores and quotients

We say μ is contained in λ if its Young diagram fits inside the Young diagram of λ, written $\mu \subseteq \lambda$.



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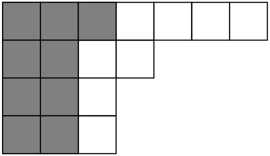
So $(3, 2, 2, 2) \subseteq (7, 4, 3, 3)$, and we form the skew shape by removing μ's Young diagram from λ's.



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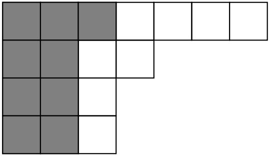
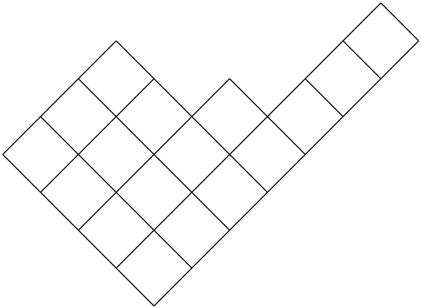


What remains here is a connected skew shape with no 2×2 square. Such a shape is called a ribbon. Since it has 8 cells, it is an 8-ribbon. The height of this ribbon is 3, with definition

$$\text{ht}(\lambda) = \#\text{rows} - 1.$$

$(7, 4, 3, 3)$



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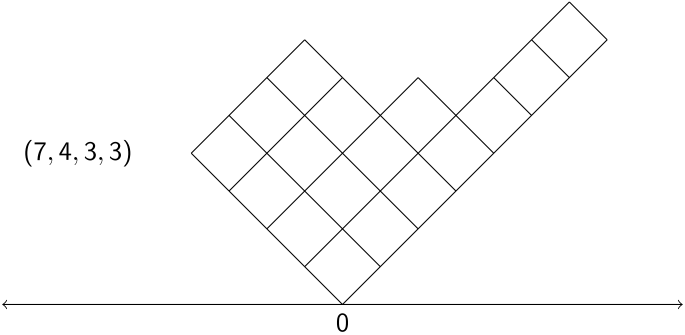

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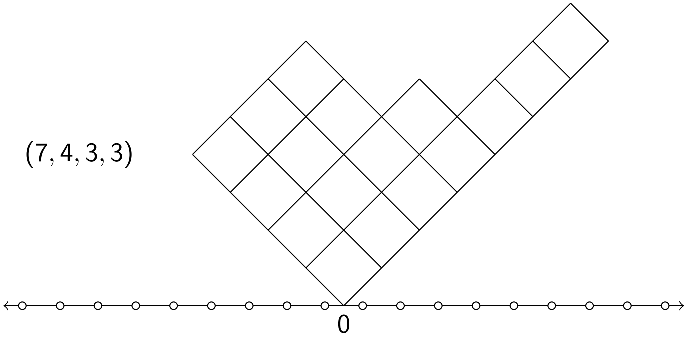

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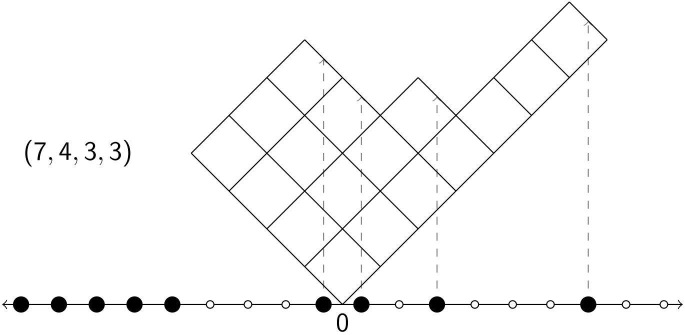
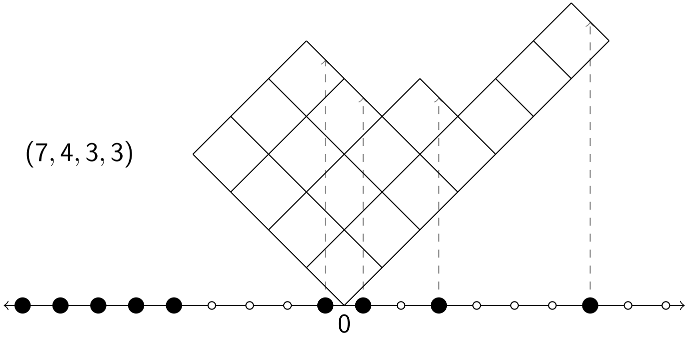


Place “beads” at the positions given by the beta set

$$\beta(\lambda) := \left\{\lambda_i - i + \frac{1}{2} : i \geqslant 1\right\}.$$

For fixed $r \geqslant 2$ moving a bead r places to the left to an empty space is equivalent to removing an r-ribbon from λ such that what remains is a partition. For example with $r = 4$:

$(7, 4, 3, 3)$



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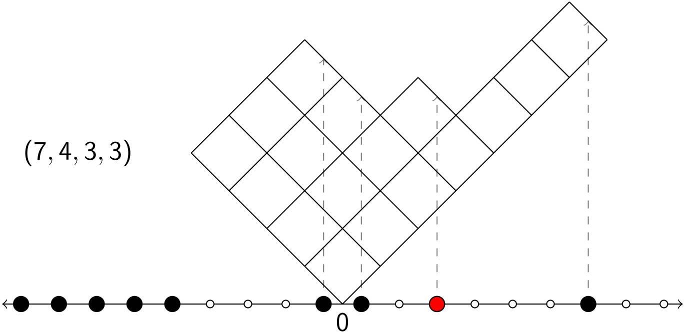

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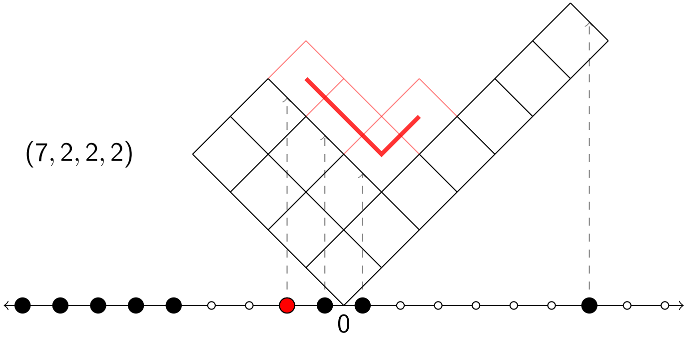

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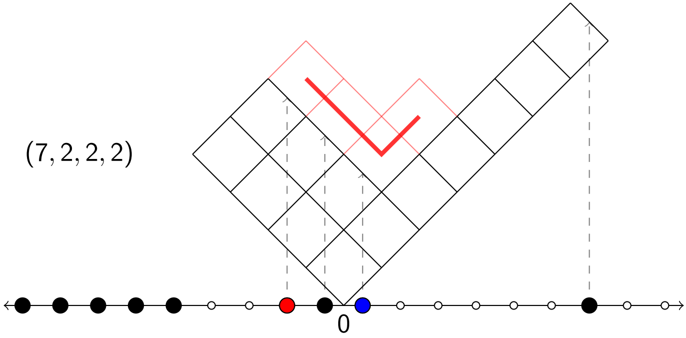

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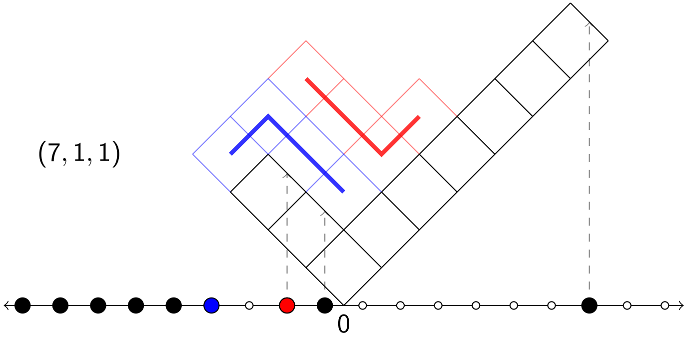

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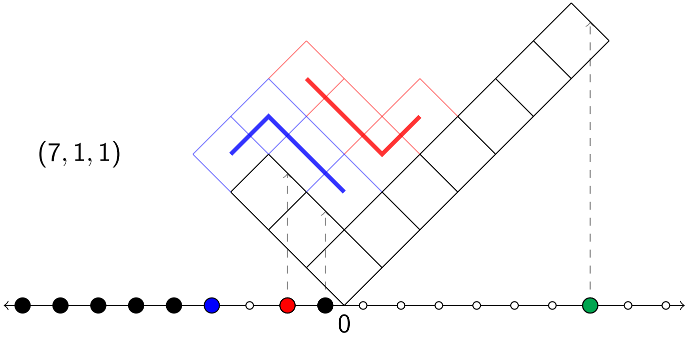

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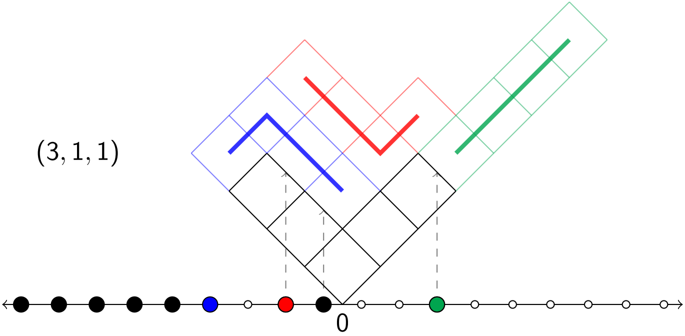

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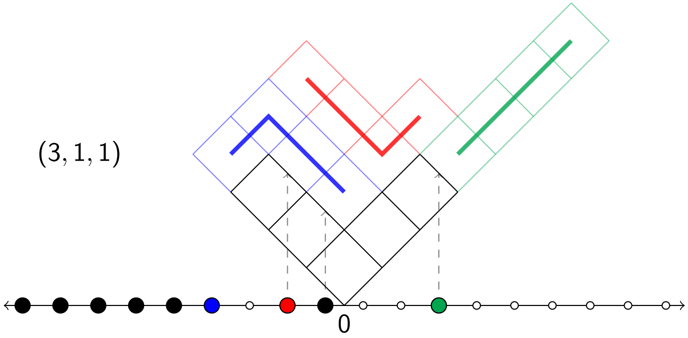

Note that the height of a removed ribbon is equal to the number of beads "jumped over" in the Maya diagram picture!

Theorem (Nakayama 1940): For each integer $r \geqslant 2$ and partition λ successively removing r-ribbons from λ (in any valid order) leaves a unique partition $r\text{-core}(\lambda)$ which has no hook of length r.

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For any partition the skew shape $\lambda/r\text{-core}(\lambda)$ has a decomposition (ribbon tiling)

$$r\text{-core}(\lambda) =: \nu^{(0)} \subseteq \nu^{(1)} \subseteq \cdots \subseteq \nu^{(k-1)} \subseteq \nu^{(k)} := \lambda,$$

where $k = (|\lambda| - |r\text{-core}(\lambda)|)/r$ and $\nu^{(i)}/\nu^{(i-1)}$ is an r-ribbon for $1 \leqslant i \leqslant k$.

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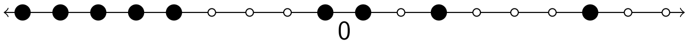
Lemma: For any ribbon tiling of λ/r-core(λ) the sign

$$(-1)^{\sum_{i=1}^{k} \mathrm{ht}(\nu^{(i)}/\nu^{(i-1)})}$$

is the same, and we denote it by $\mathrm{sgn}_r(\lambda/r\text{-core}(\lambda))$.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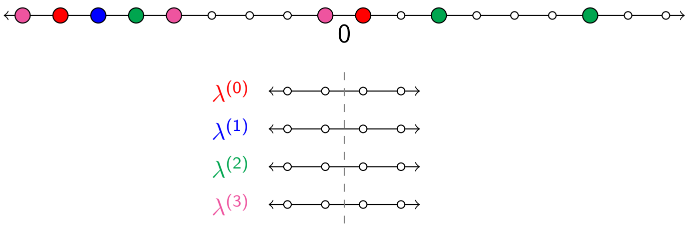
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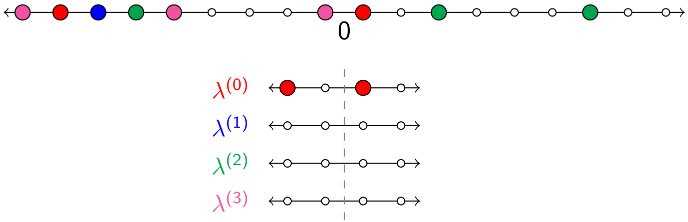

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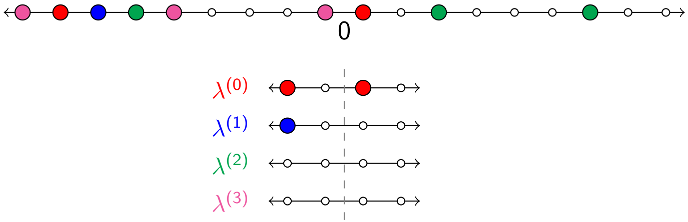
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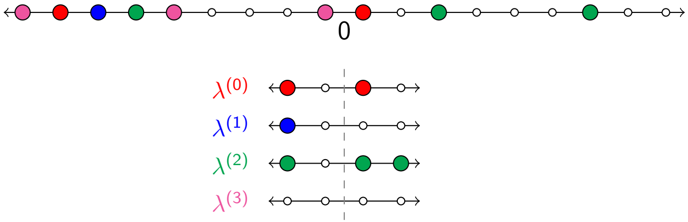
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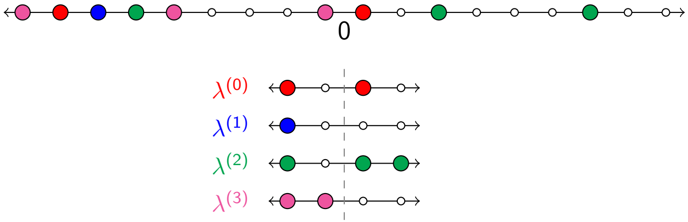
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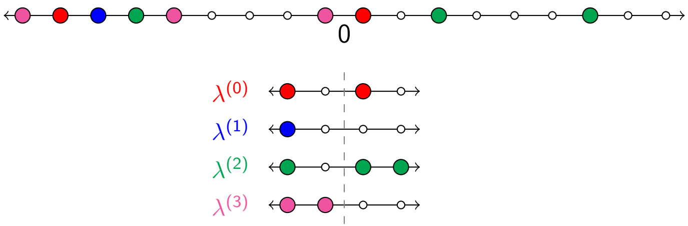

So the quotient of $\lambda = (7,4,3,3)$ is

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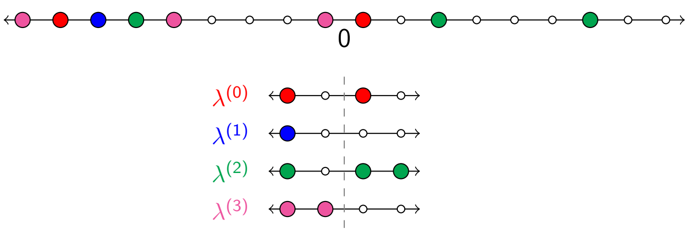

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The core can be obtained from the above picture by simply pushing all beads to the left as far as possible.

Theorem (Littlewood 1951): For each $r \geqslant 2$ the above defines a bijection

$$\begin{aligned} \mathcal{P} &\longrightarrow \mathcal{C}_r \times \mathcal{P}^r \\ \lambda &\longmapsto \big(r\text{-core}(\lambda), (\lambda^{(0)}, \ldots, \lambda^{(r)})\big). \end{aligned}$$

such that

$$|\lambda| = |r\text{-core}(\lambda)| + r\big(|\lambda^{(0)}| + \cdots + |\lambda^{(r)}|\big).$$

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In our example we have that

$$(7,4,3,3) \longmapsto \big((3,1,1), ((1), \varnothing, (1,1), \varnothing)\big),$$

and of course

$$17 = 5 + 4 \cdot 3.$$

Theorem (Littlewood 1940): For any partition λ and integer $r \geqslant 2$ we have that $\varphi_r s_\lambda = 0$ unless r-core$(\lambda) = \varnothing$, in which case

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There is a version for skew Schur functions

$$s_{\lambda/\mu} = \det_{1 \leqslant i,j \leqslant \ell(\lambda)} \big(h_{\lambda_i - \mu_j - i + j}\big).$$

Theorem (Farahat 1958 & Macdonald 1995): We have that $\varphi_r s_{\lambda/\mu} = 0$ unless r-core$(\lambda) = r$-core(μ) and $\mu^{(i)} \subseteq \lambda^{(i)}$ for $0 \leqslant i \leqslant r-1$ (equivalently, λ/μ has a ribbon decomposition), in which case

$$\varphi_r s_{\lambda/\mu} = \mathrm{sgn}_r(\lambda/\mu) \prod_{i=0}^{r-1} s_{\lambda^{(i)}/\mu^{(i)}}.$$

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1. Littlewood's proof uses an equivalent formulation and evaluates the Schur function

$$s_\lambda(x_1, \ldots, \zeta^{r-1}x_1, \ldots, x_n, \ldots, \zeta^{r-1}x_n)$$

where ζ is a primitive r-th root of unity using the ratio of alternants

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j+n-j})}{\det_{1\leqslant i,j\leqslant n}(x_i^{n-j})}.$$

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3. Lascoux, Leclerc and Thibon give more combinatorial proof using ribbon tableaux and the adjoint relation

$$\langle \varphi_r s_\lambda, h_\mu \rangle = \langle s_\lambda, h_\mu \circ p_r \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Hall inner product on Λ and $\circ$ denotes plethysm.

Inspired by a rediscovery of Littlewood's theorem by Prasad, Ayyer and Kumari proved similar factorisation theorems for the characters of the classical groups $\mathrm{Sp}(2n, \mathbb{C})$, $\mathrm{O}(2n, \mathbb{C})$ and $\mathrm{SO}(2n+1, \mathbb{C})$. Their proofs mimic (1) above, "twisting" by a root of unity.

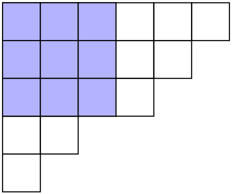
Inspired by a rediscovery of Littlewood's theorem by Prasad, Ayyer and Kumari proved similar factorisation theorems for the characters of the classical groups $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{O}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$. Their proofs mimic (1) above, "twisting" by a root of unity.

We have shown that their formulae admit lifts to the universal characters, defined by Koike and Terada using Weyl's Jacobi–Trudi-type formulae:

$$\begin{aligned} \mathrm{sp}_\lambda &:= \frac{1}{2} \det_{1\leqslant i,j\leqslant \ell(\lambda)} \big(h_{\lambda_i-i+j} + h_{\lambda_i-i-j+2}\big) \\ \mathrm{o}_\lambda &:= \det_{1\leqslant i,j\leqslant \ell(\lambda)} \big(h_{\lambda_i-i+j} - h_{\lambda_i-i-j}\big) \\ \mathrm{so}^{\pm}_\lambda &:= \det_{1\leqslant i,j\leqslant \ell(\lambda)} \big(h_{\lambda_i-i+j} \pm h_{\lambda_i-i-j+1}\big). \end{aligned}$$

These now are symmetric functions, and specialising to $x_1^{\pm},\dots,x_n^{\pm}$ give *actual* characters of the labelled groups.

The Durfee square of a partition is the largest square which fits inside the Young diagram



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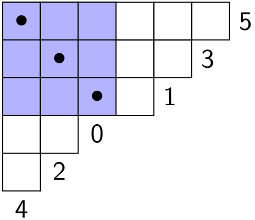

Call the side length $d(\lambda)$. The Frobenius notation for a partition records how many cells are below/to the right of each cell on the main diagonal. For example

$$(6, 5, 4, 2, 1) = (5, 3, 1 \mid 4, 2, 0).$$

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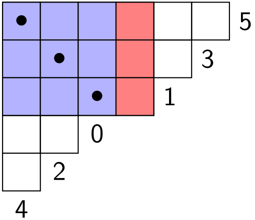

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Ayyer and Kumari call a partition z-asymmetric if it can be written in Frobenius notation as

$$(a_1 + z, \ldots, a_d + z \mid a_1, \ldots, a_d).$$

0-asymmetric partitions are just self-conjugate.

Theorem: We have that $\varphi_r \mathrm{o}_\lambda = 0$ unless r-core(λ) is 1-asymmetric, in which case

$$\varphi_r \mathrm{o}_\lambda = (-1)^{|r\text{-core}(\lambda)|/2} \mathrm{sgn}_r(\lambda / r\text{-core}(\lambda)) \times \mathrm{o}_{\lambda^{(0)}} \prod_{i=1}^{\lfloor (r-1)/2 \rfloor} \mathrm{rs}_{\lambda^{(i)}, \lambda^{(r-i)}} \times \begin{cases} \mathrm{so}^-_{\lambda^{(r/2)}} & r \text{ even}, \\ 1 & r \text{ odd}. \end{cases}$$

The factorisations for so_λ and sp_λ are similar and I spare you the details.

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All however involve the symmetric function

$$\mathrm{rs}_{\lambda,\mu} := \det_{1 \leqslant i,j \leqslant \ell(\lambda)+\ell(\mu)} \begin{pmatrix} \left(h_{\lambda_i - i + j}\right)_{1 \leqslant i,j \leqslant \ell(\lambda)} & \left(h_{\lambda_i - i - j + 1}\right)_{\substack{1 \leqslant i \leqslant \ell(\lambda) \\ 1 \leqslant j \leqslant \ell(\mu)}} \\ \left(h_{\mu_i - i - j + 1}\right)_{\substack{1 \leqslant i \leqslant \ell(\mu) \\ 1 \leqslant j \leqslant \ell(\lambda)}} & \left(h_{\mu_i - i + j}\right)_{1 \leqslant i,j \leqslant \ell(\mu)}. \end{pmatrix}$$

This object arises from the rational representations of $\mathrm{GL}(n, \mathbb{C})$, and is the correct universal character analogue of the Schur function with $2n$ variables

$$s_\nu(x_1, 1/x_1, \dots, x_n, 1/x_n).$$

The structure of the factorisation is best explained through Weyl's formula

$$\mathrm{o}_\lambda = \sum_{\mu \text{ 1-asymmetric}} (-1)^{|\mu|/2} s_{\lambda/\mu},$$

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combined with

Theorem (Garvan, Kim and Stanton 1990): Let λ be 1-symmetric. Then

1. r-core(λ) and $\lambda^{(0)}$ are 1-symmetric, and
2. for $1 \leqslant i \leqslant r-1$,

$$\lambda^{(i)} = (\lambda^{(r-i)})'.$$

If r is even this means that $\lambda^{(r/2)}$ is self-conjugate. For example if $\lambda = (9,7,6,6,6,2,1,1)$ then

$$\big(4\text{-core}(\lambda), (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})\big) = \big((2), ((3,1),(1,1),(1),(2))\big).$$

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The formula for $\varphi_r s_{\lambda/\mu}$ is the other tool required for this proof (apart from figuring out the sign).

Weyl's formula has a generalisation which involves a sum over z-asymmetric partitions due to Bressoud and Wei

$$\chi_\lambda(z) := \det_{1\leqslant i,j\leqslant \ell(\lambda)} \left(h_{\lambda_i-i+j} + (-1)^z h_{\lambda_i-i-j-z+1}\right)$$

$$= \sum_{\mu\in\mathcal{P}_z} (-1)^{(|\mu|+(z-1)d(\mu))/2} s_{\lambda/\mu}.$$

where here $z \geqslant 0$ (but can be made to work for all $z \in \mathbb{Z}$).

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