Card Shuffling, q-Analogues, and Derangements

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"Invariant theory of the free left-regular band and a q-analogue"

+

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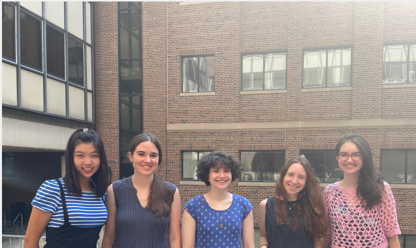


BIG IDEA: Answer questions in probability theory using COMBINATORICS & REPRESENTATION THEORY

OUTLINE:

I. SHUFFLING

* $R2T_n$
* $T2R_n$
* $R2R_n$

Random walks on permutations

II. q- ANALOGUES

* $R2T_n^{(q)}$
* $T2R_n^{(q)}$
* $R2R_n^{(q)}$

Random walks on complete flags of $\mathbb{F}_q^n$

III ANSWERS AND CONJECTURES

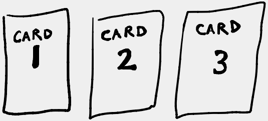
GOAL :

* eigen values
* eigenspaces as S_n & $GL_n(\mathbb{F}_q)$- reps...

I. SHUFFLING

SCENARIO:

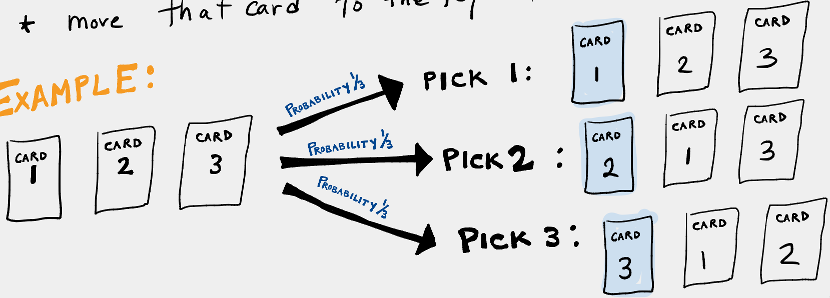
You have a deck of cards:



Shuffle via **RANDOM-TO-TOP**

* pick a card at random
* move that card to the top of the deck

EXAMPLE:



QUESTION: How many times do you need to do this?

SIMPLIFICATION: Let's think of a deck of cards as a word of length n in the set $\{1, 2, \ldots, n\}$.

EXAMPLE:

DECK	WORD
CARD 1, CARD 2, CARD 3	123
CARD 1, CARD 3, CARD 2	132
CARD 2, CARD 1, CARD 3	213
CARD 2, CARD 3, CARD 1	231
CARD 3, CARD 1, CARD 2	312
CARD 3, CARD 2, CARD 1	321

For a word $(w_1, \ldots w_n)$ define (and extend linearly ...)

RANDOM-TO-TOP : bring any card to the front with equal probability

- $R2T_n(w_1 \ldots w_n) = \frac{1}{n}\left(\sum_{i=1}^{n} w_i\, w_1 \ldots \hat{w_i} \ldots w_n\right)$

$$R2T_3(123) = \frac{1}{3}(123 + 213 + 312) + 0\,(132 + 231 + 321).$$

FOR EXPERTS: This is a linear map $\mathbb{Q}[S_n] \longrightarrow \mathbb{Q}[S_n]$

We can similarly define...

TOP-TO-RANDOM : move top card anywhere in deck with equal probability

- $T2R_n(w_1 \dots w_n) := \frac{1}{n}\left(\sum_{i=1}^{n} w_2 \dots w_{i-1}\, w_1 \dots w_n\right)$

$$T2R_3(123) = \frac{1}{3}(123 + 213 + 231)$$
$$+\, 0\,(132 + 312 + 321).$$

RANDOM-TO-RANDOM : move random card to random spot in deck

- $R2R_n := T2R_n \circ R2T_n$

$$R2R_3(123) = \frac{1}{3}(123) + \frac{2}{9}(213) + \frac{2}{9}(132)$$
$$+ \frac{1}{9}(231) + \frac{1}{9}(312) + 0\,(321)$$

These shuffling processes are **MARKOV CHAINS**

initial state: word w $\xrightarrow{\text{SHUFFLE}}$ word u with some probability

IDEA: For K large enough...

initial state: a word $\xrightarrow{\text{SHUFFLE } K \text{ TIMES}}$ any word will appear with equal probability!

HOW DO WE MAKE THIS PRECISE?

Answer: Probability Matrix

* **Rows** and **Columns** are indexed by all "states"

  For us: all words of length n

* j^{th} column and i^{th} row:

  Probability of going from $\boxed{j}$ to $\boxed{i}$

Example 1. Probability Matrix: $R2T_3$

$$R2T_3(123) = \tfrac{1}{3}\cdot(123) + 0\cdot(132) + \tfrac{1}{3}\cdot(213) + 0\cdot(231) + \tfrac{1}{3}(312) + 0\cdot(321)$$

$$R2T_3 := \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{bmatrix} 1/3 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 0 & 1/3 \end{bmatrix} \end{array}$$

EXAMPLE 2.

THE UNIFORM MATRIX:

$$
M = \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\
\begin{bmatrix}
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\
1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6
\end{bmatrix}
\end{array}
$$

INTERPRETATION: No matter where I start, I could end anywhere with equal probability

KEY IDEA:

shuffling **K** times $\longleftrightarrow$ $(R2T_n)^K$, $(T2R_n)^K$, $(R2R_n)^K$

CLAIM:

(Perron-Frobenius Theorem + theory of Markov chains)

* For **K** large enough,

  $(R2T_n)^K$, $(T2R_n)^K$, $(R2R_n)^K$ will converge to **THE UNIFORM MATRIX**

* The **EIGENVALUES** of the transition matrix tell us how large **K** must be ...

GOAL:

(1) Determine EIGENVALUES of

$$\left.\begin{array}{l} R2T_n \\ T2R_n \\ R2R_n \end{array}\right\} \; n! \times n! \quad \text{matrices}$$

(2) Describe their EIGENSPACES

as S_n - representations...

SIMPLIFICATION:

(i) $(R2T_n)^T = T2R_n$, so eigenvalues are the same

$$R2T_3 = \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \end{array}$$

$$T2R_3 = \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \end{array}$$

(ii) Scale operators so matrices have entries in $\mathbb{Z}$

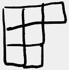
$$\begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \end{array} \longrightarrow \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

REPRESENTATION THEORY OF THE SYMMETRIC GROUP

* BIG IDEA: We can study S_n-reps using combinatorics of tableaux!

* Smallest building blocks: χ^λ where $\lambda \vdash n$

* Write λ as a YOUNG DIAGRAM



$(3,2,2) \vdash 7$

* There is a way to think of them as SYMMETRIC FUNCTIONS

The Frobenius characteristic map

(virtual) representations of $S_n \longrightarrow$ Symmetric functions

$\chi^\lambda \longmapsto s_\lambda$ ← Schur function.

TRIVIAL REP of S_n → $\chi^{\boxed{\;\cdots\;}} \longmapsto h_n$ ← homogeneous

MOTIVATION TO STUDY EIGENSPACE REPS...

1. Can help answer eigenvalue questions

↳ In particular $R2R_n$ (+ other operators studied by Lafrenière)

2. Motivation from monoid theory

↳ Invariant theory of the "free left regular band"

3. Kernel of $R2T_n$, $R2R_n$ known to have a very nice description

↳ THE DERANGEMENT REPRESENTATION...

THE DERANGEMENT REPRESENTATION...

* Standard Young tableau (SYT): label boxes of Young diagram $1, 2, \ldots n$, s.t. rows and columns increase south and east.

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{matrix}$$

✓ SYT

$$\begin{matrix} 1 & 3 & 5 \\ 4 & 2 & \\ 6 & & \end{matrix}$$

✗ SYT

* Given Q a SYT of shape λ,

$\mathrm{Des}(Q) = \{ i \in [n-1] : i+1$ appears south and weakly west of $i \}$

$$\mathrm{Des}\left(\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{matrix} \right) = \{3, 5\} \qquad \mathrm{Des}\left(\begin{matrix} 1 & 3 \\ 2 & 6 \\ 4 & \\ 5 & \end{matrix} \right) = \{1, 3, 4\}$$

THE DERANGEMENT REPRESENTATION...

A **DESARRANGEMENT TABLEAU** is a SYT Q such that the **MINIMUM** of $\{1, \dots n\} \setminus \mathrm{Des}(Q)$ is **EVEN**

* $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$ is **NOT DESARRANGEMENT** because

$$\{1, 2, \dots 6\} \setminus \mathrm{Des}\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\right) = \{\overset{\text{odd}}{1}, 2, 4, 6\}$$

* $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$ is **DESARRANGEMENT** because

$$\{1, 2, \dots 6\} \setminus \mathrm{Des}\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 \\ \cline{1-1} \\ \cline{1-1}\end{array}\right) = \{\overset{\text{even}}{2}, 5, 6\}$$

(Definition due to Désarménian and Wachs)

DEFINITION: (Désarménian and Wachs):

The **DERANGEMENT REPRESENTATION** is the symmetric function $Q_n := \sum s_{\lambda(Q)}$

where the sum is over all **DESARRANGEMENT TABLEAUX** of size n

EXAMPLE: When $n=4$, the **DESARRANGEMENT TABLEAUX** are:

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 \\\cline{1-1} 4 \\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & 4\\\hline\end{array}\qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4\\\hline 2 \\\cline{1-1}\end{array}$$

So $Q_4 = s_{\ydiagram{1,1,1,1}} + s_{\ydiagram{2,1,1}} + s_{\ydiagram{2,2}} + s_{\ydiagram{3,1}}$

PREVIEW: $\mathrm{Ker}(R2T_n)$ has Frobenius image Q_n

Where is the name from?

* The **DERANGEMENT** number is

$$d_n := n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = n! \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{(-1)^n}{n!} \right)$$

* d_n COUNTS...
  - $\#\{\sigma \in S_n : \#\mathrm{Fix}(\sigma) = 0\}$, e.g. derangements of S_n,
  - $\dim(\mathbb{Q}_n)$
  - $\#\{\sigma \in S_n : \sigma \text{ is a Desarrangement}\}$

defined similarly to our def for SYT...

II. q- ANALOGUES

assume $q = p^\ell$ for p a prime...

IDEA: * Introduce a parameter q

* we recover classical objects as $q \rightsquigarrow 1$

CLASSICAL OBJECT	q- ANALOGUE
$k \in \mathbb{Z}_{\geq 0}$	$[k]_q := 1 + q + q^2 + \cdots + q^{k-1}$
$n! = n \cdot (n-1) \cdots 2 \cdot 1$	$[n]!_q := [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q$
set $\{1, 2, \ldots, n\}$	Vector space $\mathbb{F}_q^n$, for $\mathbb{F}_q$ a finite field
$S_n = \mathrm{Aut}\{1, 2, \ldots, n\}$	$GL_n(\mathbb{F}_q) = \mathrm{Aut}(\mathbb{F}_q^n)$

For us: replace words with complete flags

e.g. $V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{F}_q^n$

where V_k = k-dim'l subspace of $\mathbb{F}_q^n$

EXAMPLE: $V = \mathbb{F}_q^n$ for $n=2$, $q=2$.

So $V = \langle e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$.

The complete flags are

$f_1 = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle = V$

$f_2 = \langle e_2 \rangle \subset \langle e_1, e_2 \rangle = V$

$f_3 = \langle e_1 + e_2 \rangle \subset \langle e_1, e_2 \rangle = V$

With Commins and Reiner, we introduced q-shuffling operators.. on the space of complete flags $\mathcal{F}(V)$...

DEFINE:

$\mathbf{R2T}_n^{(q)}$: Pick any line L in $\mathbb{F}_q^n$ and "bring it to the front" with equal probability...

remove repeats

$$\mathbf{R2T}_n^{(q)}(V_1 \subset V_2 \subset \cdots \subset V_n) := \sum_{\substack{\text{lines} \\ L \subset V}} \widehat{(L \subset L+V_1 \subset L+V_2 \subset \cdots \subset L+V)}$$

EXAMPLE: $V = \mathbb{F}_q^n$ for $n=2$, $q=2$.

$$\begin{aligned} \mathbf{R2T}_2^{(2)}(\langle e_1\rangle \subset \langle e_1, e_2\rangle) &= \widehat{(\langle e_1\rangle \subset \cancel{\langle e_1\rangle + \langle e_1\rangle} \subset \langle e_1\rangle + V)} \\ &+ \widehat{(\langle e_2\rangle \subset \langle e_2\rangle + \langle e_1\rangle \subset \cancel{\langle e_2\rangle + V})} \\ &+ \widehat{(\langle e_1+e_2\rangle \subset \langle e_1+e_2\rangle + \langle e_1\rangle \subset \cancel{\langle e_1+e_2\rangle + V})} \end{aligned}$$

We can similarly define...

$$\mathbf{T2R}_n^{(q)} := \left[R2T_n^{(q)}\right]^T$$

$$\mathbf{R2R}_n^{(q)} := T2R_n^{(q)} \circ R2T_n^{(q)}$$

FOR EXPERTS: $\mathcal{F}(V)$ = set of complete flags of $V = \mathbb{F}_q^n$.
(e.g. $GL_n(\mathbb{F}_q)/B$)...

Then these are maps $\mathbb{C}[\mathcal{F}(V)] \longrightarrow \mathbb{C}[\mathcal{F}(V)]$

GOALS:

(1) Determine **EIGENVALUES** of

$$\left.\begin{array}{l} R2T_n^{(q)} \\ T2R_n^{(q)} \\ R2R_n^{(q)} \end{array}\right\} [n]!_q \times [n]!_q \text{ - matrices}$$

(2) Describe their **EIGENSPACES**

as $GL_n(\mathbb{F}_q)$ - representations...

REPRESENTATION THEORY OF THE $GL_n(\mathbb{F}_q)$:

* We are interested in the UNIPOTENT REPRESENTATIONS

↳ Irreducibles indexed by partitions λ of n

$$\chi_q^\lambda \longleftrightarrow \lambda \vdash n.$$

↳ There is q-Frobenius characteristic map

$$\text{UNIPOTENT REPRESENTATIONS} \longrightarrow \text{RING OF SYMMETRIC FUNCTIONS}$$

$$\chi_q^\lambda \longmapsto s_\lambda$$

← Schur function.

TRIVIAL REP of $GL_n(\mathbb{F}_q)$

$$\chi_q^{\boxed{\;\cdots\;}} \longmapsto h_n$$

← homogeneous

UPSHOT: We can still use symmetric functions!

EXAMPLE: The q-derangement representation

* q-Frobenius characteristic image is $Q_n := \sum_{\substack{Q \text{. a} \\ \text{desarrangement} \\ \text{tableau}}} S_{\lambda(Q)}$

* has dimension given by the q-derangement numbers of Désarménian and Wachs:

$$d_n(q) = [n]!_q \sum_{k=0}^{n} \frac{(-1)^k}{[k]!_q}.$$

III Answers and Conjectures

Recall our **Goal:**

(1) Determine EIGENVALUES

(2) Describe EIGENSPACES

of...	"CLASSICAL" OBJECT	q- ANALOGUE
	$R2T_n$	$R2T_n^{(q)}$
	$T2R_n$	$T2R_n^{(q)}$
	$R2R_n$	$R2R_n^{(q)}$

For **$R2T_n$** ....

* Phatarfod (1991): **EIGENVALUES** of $R2T_n$ are $0, 1, 2, \ldots, n$, where j occurs with multiplicity
$$f_j := \#\{\sigma \in S_n : \#\mathrm{Fix}(\sigma) = j\}$$
$$= \binom{n}{j} \cdot d_{n-j}$$

* Uyemura-Reyes (2002): The **j-EIGENSPACE** of $R2T_n$ has Frobenius image
$$h_j \cdot \mathcal{Q}_{n-j}$$

NOTE: This is a rephrasing of his result...

THEOREM (B, Commins, Reiner, 2022):

* The eigenvalues of $R2T_n^{(q)}$ are $[0]_q, [1]_q, \ldots, [n]_q$ where $[j]_q$ occurs with multiplicity $\begin{bmatrix} n \\ j \end{bmatrix}_q \cdot d_{n-j}(q)$

* The $[j]_q$-eigenspace of $R2T_n^{(q)}$ has q-Frobenius image

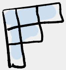
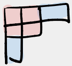
$$h_j \cdot \mathcal{Q}_{n-j}$$

* In fact, we prove these results for $R2T_n$ & $R2T_n^{(q)}$ in parallel!

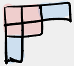
For $R2R_n$ .... this was an open problem for 20 years!

To state the solution, we need a few definitions...

★ SKEW DIAGRAM λ/μ:

$\lambda =$  $\mu =$  $\Rightarrow$  Delete pink, keep blue

★ HORIZONTAL STRIP : skew partition with at most one cell in each column.

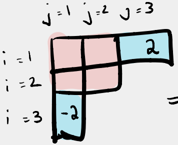


✓ horizontal strip.

X <u>not</u> a horizontal strip

★ $\mathrm{diag}(\lambda/\mu) \in \mathbb{Z}$ is a statistic on λ/μ

$$\mathrm{diag}(\lambda/\mu) = \sum_{\substack{\text{cells } (i,j) \\ \text{in } \lambda/\mu}} (j-i)$$



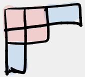
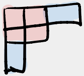
$\Rightarrow$ diag is $-2+2=0$.

THEOREM (Dieker–Saliola, 2018)

All eigenvalues of $R2R_n$ are of the form $\mathrm{eig}(\lambda/\mu)$, where λ/μ is a horizontal strip and

$$\mathrm{eig}(\lambda/\mu) = \mathrm{diag}(\lambda/\mu) + \sum_{j=1}^{|\lambda/\mu|} j + |\mu| \in \mathbb{Z}_{\geq 0}$$

BIG DEAL!

EXAMPLE: $\mathrm{eig}\Big($  $\Big) = \mathrm{diag}\Big($  $\Big) + (1+4) + (2+4)$

$= 0 + 5 + 6 = 11.$

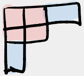
CONJECTURE (Axelrod-Freed, B, Chiang, Commins, Lang, 2023+)

All eigenvalues of $R2R_n^{(q)}$ are of the form $\mathrm{eig}_q(\lambda/\mu)$, where (λ/μ) is a horizontal strip and

$$\mathrm{eig}_q(\lambda/\mu) = \mathrm{diag}_q(\lambda/\mu) + \sum_{j=1}^{|\lambda/\mu|} q^{|\lambda/\mu|-j}[j+|\mu|]_q$$

statistic on λ/μ in $\mathbb{Z}[q]$

THUS $\mathrm{eig}_q(\lambda/\mu) \in \mathbb{Z}_{\geq 0}[q]$

EXAMPLE: $\mathrm{eig}_q\Big($  $\Big) = [0]_q + q^1[5]_q + q^0[6]_q$

$= q \cdot [5]_q + [6]_q$

THEOREM (Dieker-Saliola, 2018)

Every eigenvalue ε of $R2R_n$ has eigenspace with Frobenius image

$$\sum_{\substack{(\lambda/\mu) \\ \mathrm{eig}(\lambda/\mu)=\varepsilon}} d^{\mu} \cdot S_{\lambda}$$

# of desarrangement tableaux of shape μ

CONJECTURE (Axelrod-Freed, B, Chiang, Commins, Lang, 2022+)

Every eigenvalue ε_q of $R2R_n^{(q)}$ has eigenspace with q-Frobenius image

$$\sum_{\substack{(\lambda/\mu) \\ \mathrm{eig}_q(\lambda/\mu)=\varepsilon_q}} d^{\mu} \cdot S_{\lambda}$$

NEXT STEPS...

* Prove our conjectures!

STRATEGY: The Type A Hecke algebra $\mathcal{H}_n(q)$ ...

* There is a way to realize $R2T_n^{(q)}$, $T2R_n^{(q)}$, $R2R_n^{(q)}$ in $\mathcal{H}_n(q)$

* Goal: generalize methods of Dieker-Saliola using $\mathcal{H}_n(q)$ combinatorics & representation theory

THANK YOU!

CONTACT ME!

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How does $\mathcal{H}_n(q)$ act on permutations?

$$\mathcal{H}_n(q) = \left\langle T_1, \dots, T_{n-1} \;:\; \begin{array}{l} T_i^2 = (1-q)T_i + q \\ T_{i+1} T_i T_{i+1} = T_i T_{i+1} T_i \\ T_i T_j T_i = T_j T_i T_j \end{array} \right\rangle$$

Think of $q \in [0,1]$. Then...

$$T_i \cdot (w_1 \dots, w_i, w_{i+1} \dots w_n) = \begin{cases} w_1 \dots, w_{i+1}, w_i, \dots w_n & \text{if } w_i > w_{i+1} \\ q \cdot (w_1 \dots w_{i+1}, w_i, \dots w_n) + (1-q)(w_1 \dots w_i w_{i+1} \dots w_n) & \text{if } w_{i+1} > w_i \end{cases}$$

* Note this is a rescaling from the usual presentation of $\mathcal{H}_n(q)$

$T_i \longmapsto -\tilde{T}_i \longleftarrow$ where $\tilde{T}_i$ are "standard generators"