A generalized RSK correspondence via (type A) quiver representations

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This is a part of my ph.D. work (in progress) supervised by Hugh Thomas.

Underlying Motivations

- Garver, Patrias, and Thomas [2019] introduced an invariant of a quiver representation, its generic Jordan form data, and showed that, in some subcategories of modules, the module could be recovered from this invariant.

- The map from the obvious invariant that counts the number of indecomposable summands of each isomorphism class to the generic Jordan form data is a variant of the Robinson-Schensted-Knuth correspondence.

- A recent paper [D. 23] gives a combinatorial characterization of all the subcategories of module of a type A algebra where any module can be recovered from its generic Jordan form data.

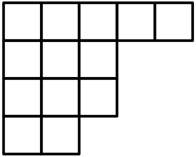
- This talk will focus on a combinatorial translation, and consequences of this result, which is still work in progress. No knowledge about quiver representations is needed !

My slides :

Plan

1. Robinson–Schensted–Knuth correspondence
2. Coxeter elements and Auslander–Reiten quiver
3. Generalized RSK via Coxeter elements of S_n

- An *integer partition* $\lambda = (5, 3, 3, 2)$



- A *filling* of λ :

1	2	1	0	3
2	1	1		
2	1	3		
3	2			

- A *reverse plane partition* of shape λ :

1	3	5	5	7
1	5	5		
4	6	9		
4	10			

Robinson–Schensted–Knuth correspondence

- The RSK correspondence is a bijection from nonnegative integer matrices to pairs of semi-standard Young tableaux. Let us recall the common way we describe this correspondence [Fulton '97, Stanley '99,...] via an example.

- Consider the following matrix.

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

- We can extract from A a two-line array

$$w_A = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 3 & 2 & 2 & 3 & 1 & 2 \end{pmatrix}$$

where the column $\binom{i}{j}$ appears $a_{i,j}$ times and the columns are in lexicographical order.

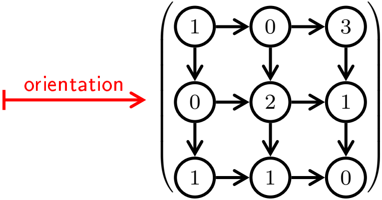
- We construct the pair (P, Q) of semi-standard Young tableaux as it follows.
- First we construct P via a sequence of Schensted row insertions following the second row of w_A.
- Then we construct Q by recording the box added at each row insertion, by inserting the corresponding entry in the first row of w_A.
- We can precisely state that the RSK correspondence gives a bijection from $n \times m$ nonnegative integer matrices to pairs of semi-standard Young tableaux of the same shape with at most n parts and entries in $\{1, \dots, m\}$.

$(1,1)$	**1**	**1**
$(1,3)$	1 **3**	1 **1**
$(1,3)$	1 3 **3**	1 1 **1**
$(1,3)$	1 3 3 **3**	1 1 1 **1**
$(2,2)$	1 **2** 3 3 **3**	1 1 1 1 **2**
$(2,2)$	1 2 **2** 3 3 **3**	1 1 1 1 2 **2**
$(2,3)$	1 2 2 3 **3** 3 3	1 1 1 1 **2** 2 2
$(3,1)$	1 **1** 2 3 3 **2** 3 **3**	1 1 1 1 2 2 2 **3**
$(3,2)$	1 1 2 **2** 3 2 3 **3** 3 P	1 1 1 1 2 2 2 **3** 3 Q

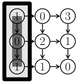
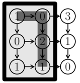
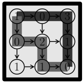
First generalization of RSK

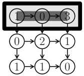
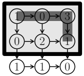
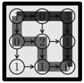
- Another way to describe this correspondence [Gansner '81] gives a description in terms of paths on a graph.
- We first orient our matrix A from left to right and from top to bottom.

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

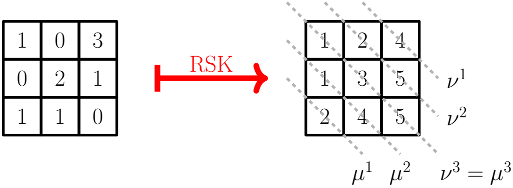


- We calculate two increasing sequences of integer partitions $(\lambda^j)_{1 \leqslant j \leqslant n}$ and $(\mu^j)_{1 \leqslant j \leqslant n}$ (for the inclusion) by the following :
  - $(1j)$ Consider the j first lines (resp. columns) of the matrix ;
  - $(2j)$ We obtain λ_1^j (resp. μ_1^j) as the maximal weight (sum of all the entries) of all the paths starting at the source and ending at the sink of this subquiver.
  - $(3j)$ Recursively, we get λ_i^j (resp. μ_i^j) for $i \geqslant 2$, by subtracting the maximal weight (sum of all the entries without multiplicities) of all the collections of $i-1$ paths starting at the source and ending at the sink of this subquiver, from the maximal weight we get by taking collections of i paths.
- We construct P (resp. Q) by labelling j the boxes of $\mu^j \setminus \mu^{j-1}$ (resp. $\lambda^j \setminus \lambda^{j-1}$).

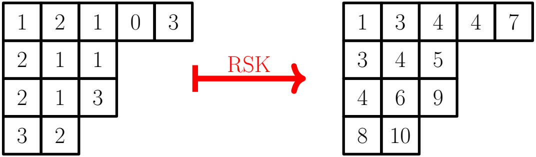
	(2)	1 1
	$(4,1)$	1 1 2 2 2
	$(5,3,1)$	1 1 2 2 3 2 3 3 3

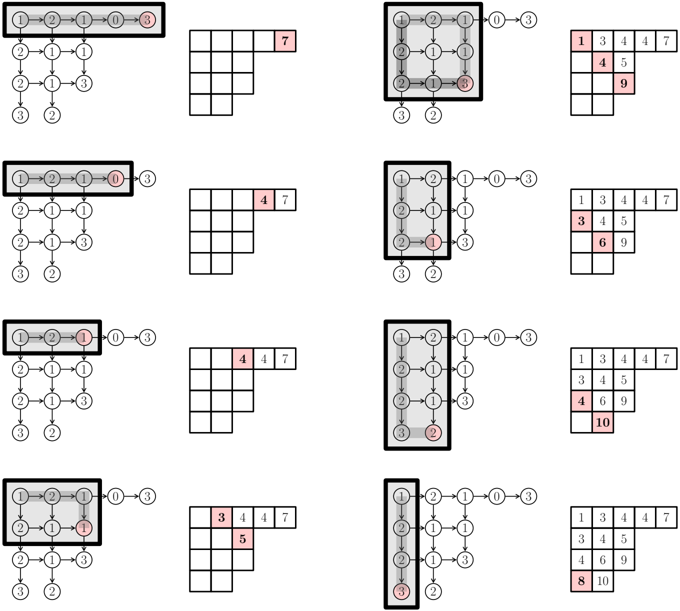
	(4)	1 1 1 1
	$(5,2)$	1 1 1 1 2 2 2
	$(5,3,1)$	1 1 1 1 2 2 2 3 3

- These sequences of integer partitions are the *Greene–Kleitman invariants* of the matrix A.
- Thanks to this invariants, we can generalize the RSK correspondence to fillings of Ferrers diagrams of any integer partition.



- Let us explain how goes this generalized version through the following calculation.





Coxeter elements

Throughout this talk we will work with the Weyl group $W = S_n$ where

$$S_n = \langle s_1, \ldots, s_{n-1} \mid s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = 1 \text{ pour } |i-j| > 1 \rangle.$$

Here s_i corresponds to the adjacent transposition $(i, i+1)$.

Definition 1

A *Coxeter element* $c \in S_n$ is an element obtained by a product of the form $s_{i_1} \cdots s_{i_{n-1}}$ where all the adjacent transpositions appear exactly once.

Example 2

- The cycle $c = (1,2,3,4,5,6,7,8,9)$ is a Coxeter element of S_9 : $c = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8$.
- The cycle $c = (1,3,4,7,9,8,6,5,2)$ is another Coxeter element of S_9 : $c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7$.

Proposition 3

An element $w \in S_n$ is Coxeter if and only if w is a long cycle which could be written

$$w = (w_1, w_2, \ldots, w_m, w_{m+1}, \ldots, w_n)$$

with $1 = w_1 < w_2 < \ldots < w_m = n$ and $n = w_m > w_{m+1} > \ldots > w_n > w_1 = 1$.

Realisation of a Coxeter element

Definition 4

Let c be a Coxeter element of S_n. The *realisation of* c is the quiver(=oriented graph) Q_c of A_{n-1} type defined by :

- The vertices of Q_c are the adjacent transpositions s_i for $1 \leqslant i \leqslant n-1$;
- The arrows Q_c are only between s_i and s_{i+1}, for all $i \in \{1, \dots, n-2\}$;
- We have $s_i \longrightarrow s_{i+1}$ in Q_c if s_i comes before s_{i+1} in a reduced expression of c, otherwise we have $s_i \longleftarrow s_{i+1}$.

Example 5

- The realisation of $c = (1,2,3,4,5,6,7,8,9)$ is :

$$(12) \to (23) \to (34) \to (45) \to (56) \to (67) \to (78) \to (89) \ .$$

- The realisation of $c = (1,3,4,7,9,8,6,5,2)$ is :

$$(12) \leftarrow (23) \to (34) \to (45) \leftarrow (56) \leftarrow (67) \to (78) \leftarrow (89) \ .$$

The Auslander-Reiten quiver

Definition 6

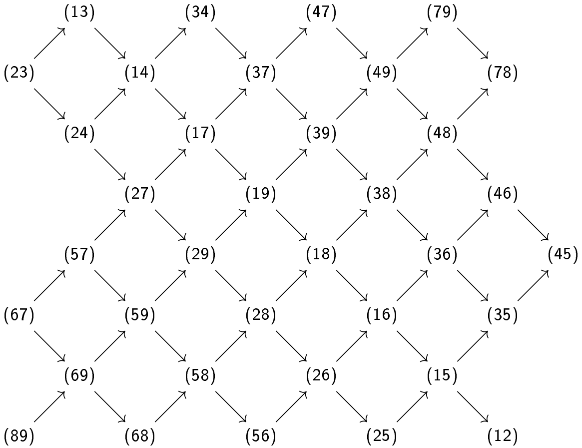
Let c be a Coxeter element of S_n. The *Auslander-Reiten quiver* $\mathrm{AR}(c)$ of c is the quiver defined by :

- The vertices of $\mathrm{AR}(c)$ are the transposition (i, j) with $i < j$ of S_n ;
- The arrows $\mathrm{AR}(c)$ are given for $i < j$ by the following :
  - $(i, j) \longrightarrow (i, c(j))$ if $i < c(j)$;
  - $(i, j) \longrightarrow (c(i), j)$ if $c(i) < j$.

Remark :

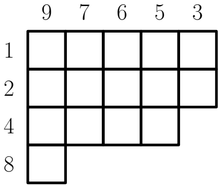
- This definition is actually a combinatorial translation of the algebraic Auslander–Reiten quiver of Q_c : The vertices are the indecomposable modules of the path algebra $\mathbb{K}Q_c$ (up to isomorphism) and the arrows are the irreducible morphisms between those indecomposables modules.
- The Auslander-Reiten quiver gives us a poset structure over transpositions of S_n (see the next slide).

We draw AR(c) for $c = (1,3,4,7,9,8,6,5,2) = (23)(12)(34)(67)(56)(45)(89)(78)$ below.



Interval bipartitions

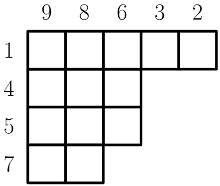
- An *(finite integer) interval* is a subset $[\![i,j]\!] = \{i, i+1, \dots, j\}$ with $i, j \in \mathbb{N}^*$ and $i \leqslant j$.
- An *interval bipartition* is a pair $(\mathbf{B}, \mathbf{E})$ of subsets of $\mathbb{N}$ such that $\{\mathbf{B}, \mathbf{E}\}$ gives a set partition of an interval in $\mathbb{N}^*$. Such a pair $(\mathbf{B}, \mathbf{E})$ is said to be *effective* if the following assertions hold :
  - for all $b \in \mathbf{B}$, there exists $e \in \mathbf{E}$, such that $b < e$;
  - for all $e \in \mathbf{E}$, there exists $b \in \mathbf{B}$, such that $b < e$.
- For any interval bipartition $(\mathbf{B}, \mathbf{E})$, by writing $\mathbf{B} = \{b_1 < \dots < b_p\}$, we define the integer partition $\lambda(\mathbf{B}, \mathbf{E}) = (\lambda_1, \dots, \lambda_p)$ with $\lambda_i = \#\{e \in \mathbf{E} \mid b_i < e\}$.
  Here is an example for $\mathbf{B} = \{1, 2, 4, 8\}$ and $\mathbf{E} = \{3, 5, 6, 7, 9\}$.



Proposition 7

For any nonzero integer partition λ, there exists an interval bipartition $(\mathbf{B}, \mathbf{E})$ such that $\lambda = \lambda(\mathbf{B}, \mathbf{E})$. Moreover, there exists a unique effective one such that $1 \in \mathbf{B}$.

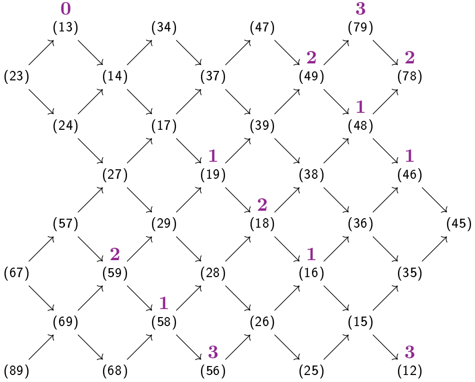
- If we take $\lambda = (5, 3, 3, 2)$ then the unique effective interval bipartition $(\mathbf{B}, \mathbf{E})$ such that $\lambda = \lambda(\mathbf{B}, \mathbf{E})$ and $1 \in \mathbf{B}$ is given by $\mathbf{B} = \{1, 4, 5, 7\}$ and $\mathbf{E} = \{2, 3, 6, 8, 9\}$.



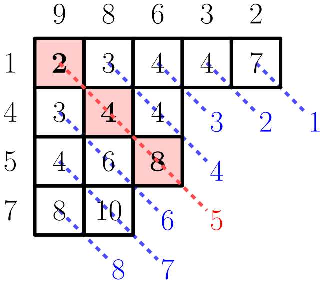
A new generalization of the RSK correspondence

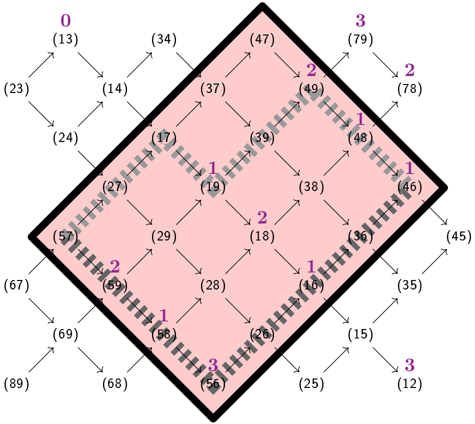
- Consider a filling of shape λ. Let $(\mathbf{B}, \mathbf{E})$ be the unique effective interval bipartition such that $\lambda = \lambda(\mathbf{B}, \mathbf{E})$ and $1 \in \mathbf{B}$.
- Consider a Coxeter element c of S_n with $n = \max(\mathbf{E})$, and its Auslander–Reiten quiver. Here we take again $c = (1, 3, 4, 7, 9, 8, 6, 5, 2)$. We fill this quiver thanks to the value in the filling of λ with respect to $(\mathbf{B}, \mathbf{E})$, as in the example below.

	9	8	6	3	2
1	1	2	1	0	3
4	2	1	1		
5	2	1	3		
7	3	2			



- We label by 1 to $n-1$ the diagonals of λ from right to left, as below.
- To get the entries in the diagonal labelled k, we take the full subquiver given by the vertices (i, j) such that $i \leqslant k < j$, and we proceed to the same "path" calculation as we saw for the previous generalized RSK in this subquiver. See below an example with $k = 5$.





- We note $\mathrm{GK}_c(\lambda)$ the result.

Theorem 8 [D. +23]

Let $n \geqslant 2$, and $c \in S_n$ a Coxeter element. Let λ be an integer partition such that the hook length of the box $(1,1)$ is $n-1$. Then GK_c gives a bijection from fillings of an integer partition λ to reverse plane partitions of shape λ

- By slightly extending a result from [Garver–Patrias–Thomas '19], we can recover the RSK correspondence by taking a Coxeter element adapted to the integer partition : explicitly, if $\lambda = \lambda(\mathbf{B}, \mathbf{E})$, then we have to take c such that $(i, i+1)$ is initial in c $(\ell((i,i+1)c) < \ell(c))$ if and only if $i \in \mathbf{B}$ and $i+1 \in \mathbf{E}$. By duality, we can replace initial by final $(\ell(c(i,i+1)) < \ell(c))$.

- Some open questions :
  (1) **Is there a way to generalize this map by working on integer skew-partitions ?**
  I believe they are linked to another algebraic project I am working on, which could give us a precise combinatorial bijection.

  (2) **Can we hope to replace S_n by any Weyl group ?**
  We can first think about type B/type C Weyl group. I think we could do something which leads us to consider symmetric fillings and symmetric reverse plane partitions. Then, we can hope that we can work with the type D and the type E Weyl groups. In this settings, we can hope to work as we worked with S_n, starting from a representation-theoretic point of view, and translating it into combinatorics.
  I am also working on another kind of representation theoretic generalization of this correspondence, and I hope to get other combinatorial consequences from it.

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Vielen Dank für eure Aufmerksamkeit !

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