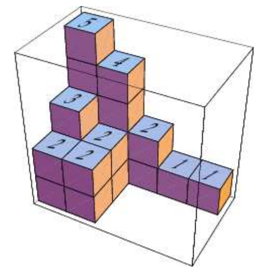
Alternating Sign Arrays and Plane Partitions

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



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Slides: `https://mat.univie.ac.at/~ifischer/SLC90.pdf`

Outline

I. Four types of objects counted by $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$

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V. Schur function expansion, TSPPs and a Cauchy-type identity

VI. A complicated bijection for the ASM-DPP relation

VII. DASASMs and the six-vertex model approach

I. Four types of objects counted by $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$

Alternating Sign Matrices = ASMs

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Square matrix with entries in $\{0, \pm 1\}$ such that in each row and each column

- the non–zero entries appear with alternating signs, and
- the sum of entries is 1.

How many?

n	1	2	3	4
	(1)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$3! + \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	42

ASMs generalize permutation matrices!

The number of $n \times n$ ASMs

Theorem (Zeilberger 1996). The number of $n \times n$ alternating sign matrices is

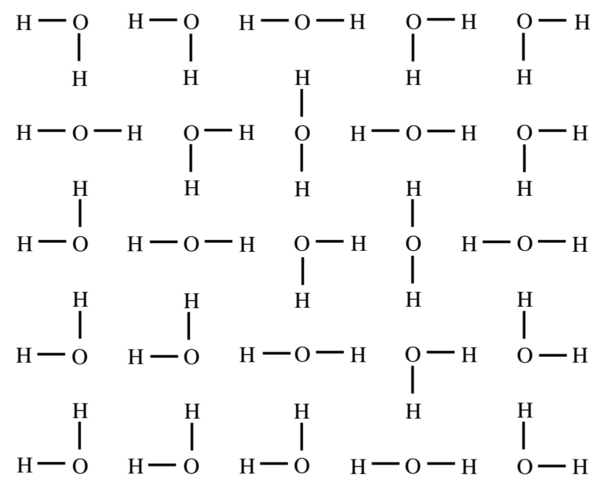
$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

- Conjectured by **Mills, Robbins** and **Rumsey** in the 1980s.
- Zeilberger gave the first proof (of a generalization including an additional parameter) in 1996.
- Recommended reading: **"Dave ROBBINS's ART of GUESSING"**
- Kuperberg gave another proof (of the special case) using methods from **statistical physics** such as the **Yang-Baxter equation.**

ASMs in statistical physics

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

ASM

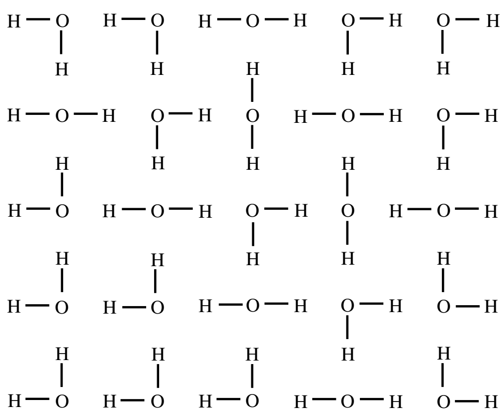


Square ice

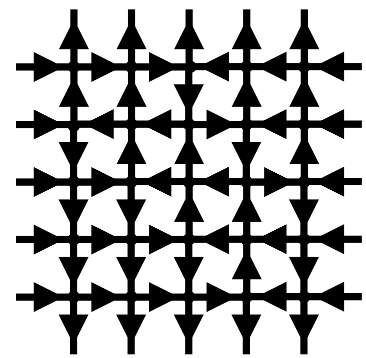
Bijection: 1's correspond to horizontal water molecules and –1's correspond to vertical water molecules.

The connection was first mentioned in 1992 in a paper by Elkies, Kuperberg, Larsen and Propp.

Square ice → 6-vertex configuration



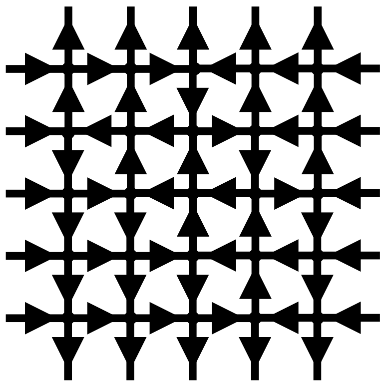
Square ice



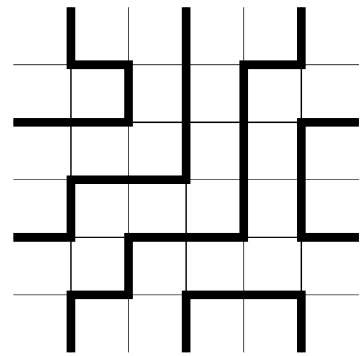
6-vertex configuration with DWBC

Bijection: Hydrogen bonds correspond to inward pointing edges

6-vertex configuration → fully packed loop configurations



6-vertex configuration



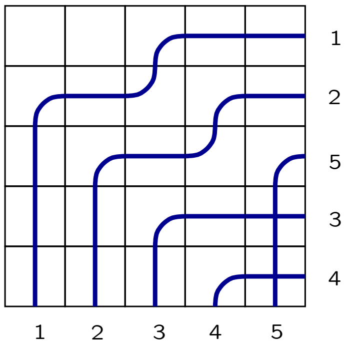
fully packed loop configuration

Bijection: To obtain the fully packed loop configuration (= collection of paths and loops), choose for "odd" vertices the inward pointing edges and for the "even" vertices the outward pointing edges.

The **Razumov-Stroganov (ex-)conjecture** states that the stationary distribution of the $O(1)$ loop model is proportional to the number of fully packed loop configurations with given link patterns. The proof was given by Cantini and Sportiello in 2011.

ASMs and Bumpless Pipe Dreams

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Bijection: 1's correspond to right-turns, while −1's correspond to left-turns.

In 1982, Grothendieck polynomials have been introduced to study the K-theory of the complete flag variety and they can be written as a certain generating function of reduced bumpless pipe dreams (as revealed by Weigandt in 2020).

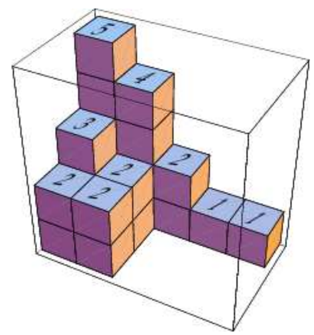
Plane partitions

A plane partition in an $a \times b \times c$ box is a subset

$$PP \subseteq \{1,2,\ldots,a\} \times \{1,2,\ldots,b\} \times \{1,2,\ldots,c\}$$

with

$$(i,j,k) \in PP \Rightarrow (i',j',k') \in PP \quad \forall (i',j',k') \leq (i,j,k).$$



$a = 5, b = 3, c = 5$

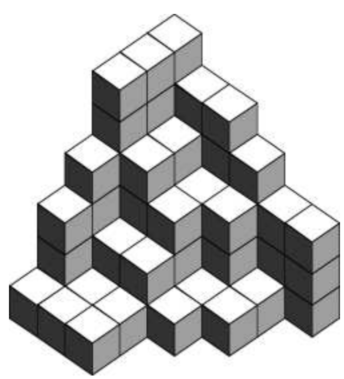
Cyclically symmetric plane partitions = CSPPs

An $n \times n \times n$ plane partition PP is cyclically symmetric if

$$(i,j,k) \in PP \Rightarrow (j,k,i) \in PP.$$

In 1979, George Andrews proved that the number of $n \times n \times n$ cyclically symmetric plane partitons is

$$\prod_{i=0}^{n-1} \frac{(3i+2)(3i)!}{(n+i)!}.$$



A determinant in Andrews' proof

In his proof, Andrews shows that the number of CSPPs of order n is given by the following determinant

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{i+j}{i}\right)$$

and then proves that

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{i+j}{i}\right)=\prod_{i=0}^{n-1}\frac{(3i+2)(3i)!}{(n+i)!}.$$

Then he also considered the following more general determinant:

$$\det_{0\le i,j\le n-1}\left(\delta_{i,j}+\binom{k+i+j}{i}\right):=D_n(k)$$

$D_n(k)$ for small values of n

$$2$$

$$k+5$$

$$(k+4)(k+5)$$

$$\frac{1}{12}(k+4)^2(k+9)(k+11)$$

$$\frac{1}{72}(k+4)^2(k+6)(k+9)(k+11)^2$$

$$\frac{(k+4)^2(k+6)^2(k+11)^2(k+13)(k+15)(k+17)}{8640}$$

$$\frac{(k+4)^2(k+6)^2(k+8)(k+10)(k+11)(k+13)(k+15)^2(k+17)^2}{518400}$$

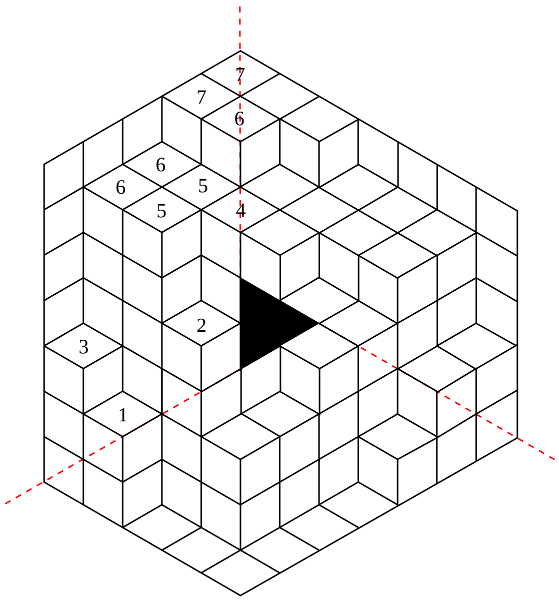
$$\frac{(k+4)^2(k+6)^2(k+8)^2(k+10)^2(k+15)^2(k+17)^3(k+19)(k+21)(k+23)}{870912000}$$

Surprise:

$$D_n(2) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

Combinatorial interpretation for $D_n(2)$

Christian Krattenthaler (2003):

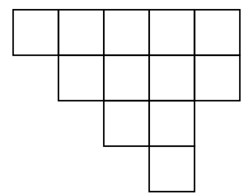


Cyclically symmetric lozenge tilings of a hexagon with side lengths $n+2, n, n+2, n, n+2, n$ with a central hole of size 2.

To obtain the combinatorial interpretation for any k, replace 2 by k!

Column strict shifted plane partitions of a fixed class aka DPPs

• With each strict partition (= partition with distinct parts), we associate a shifted Ferrers diagram. The shifted Ferrers diagram of $(5,4,2,1)$ is



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• A column strict shifted plane partition is a filling of a shifted Ferrers diagram with positive integers such that the rows are weakly decreasing and the columns are strictly decreasing.

Example.

$$\begin{array}{ccccc} 7 & 7 & 6 & 6 & 3 \\ & 6 & 5 & 5 & 1 \\ & & 4 & 2 & \end{array}$$

• A column strict shifted plane partition is of class k if the first part of each row exceeds the length of the row by precisely k. (Mills, Robbins and Rumsey 1987; Andrews 1979) The example is of class 2.

• There is a simple bijection between column strict shifted plane partitions of class k where the length of the top row does not exceed n and cyclically symmetric rhombus tilings of a hexagon with side lengths $n+k, n, n+k, n, n+k, n$ with a central triangular hole of size k.

Totally symmetric self-complementary plane partitions

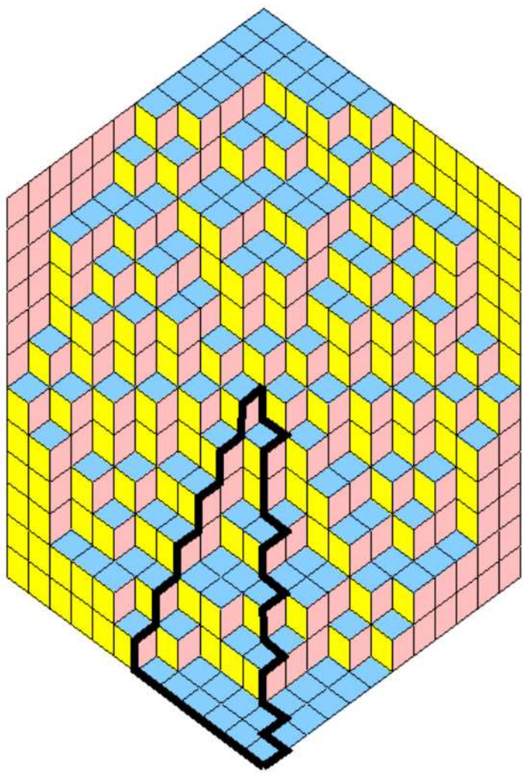


Figure by Di Francesco / Zinn-Justin

- **Totally symmetric:**
$(i,j,k) \in PP \Rightarrow \sigma(i,j,k) \in PP \ \forall \sigma \in \mathcal{S}_3$
(MacMahon 1899, 1915/16)

- **Self-complementary:**
Equal to its complement in the $2n \times 2n \times 2n$ box
(Mills, Robbins and Rumsey 1986)

Theorem (Andrews 1994). The number of TSSCPPs in a $2n\times 2n\times 2n$ box is (also) $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

Alternating sign triangles = ASTs

An AST of order n is a triangular array of 1's, -1's and 0's with n centered rows

$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & & \bullet & \bullet & \bullet & & \\ & & & \bullet & & & \end{array}$$

such that

(1) the non-zero entries alternate in each row and each column,

(2) all row sums are 1, and

(3) the topmost non-zero entry of each column is 1 (if such an entry exists).

Example:

$$\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & -1 & 1 & 0 & 0 & \\ & & 1 & -1 & 1 & & \\ & & & 1 & & & \end{array}$$

ASTs of order 3

$$
\begin{array}{ccccc|ccccc|ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & 1 & 0 & 0 & & & 1 & 0 & 0 & & & 1 & 0 & 0 & & & 0 & 0 & 1 & \\
 & & 1 & & & & & 1 & & & & & 1 & & & & & 1 & &
\end{array}
$$

$$
\begin{array}{ccccc|ccccc|ccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
 & 0 & 0 & 1 & & & 0 & 0 & 1 & & & 1 & -1 & 1 & \\
 & & 1 & & & & & 1 & & & & & 1 & &
\end{array}
$$

Theorem (Ayyer, Behrend, and F., 2020). The number of ASTs with n rows is $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

II. Alternating Sign Triangles and Trapezoids

Number of -1's in ASMs and ASTs

Let A be an ASM or an AST. Then we define

$$\mu(A) = \# \text{ of } -1\text{'s in } A.$$

Obviously

$$|\{A \in \mathrm{ASM}(n) \mid \mu(A) = 0\}| = n! = |\{A \in \mathrm{AST}(n) \mid \mu(A) = 0\}|.$$

Generalization of our theorem: Let m, n be non-negative integers. Then

$$|\{A \in \mathrm{ASM}(n) \mid \mu(A) = m\}| = |\{A \in \mathrm{AST}(n) \mid \mu(A) = m\}|.$$

Inversion numbers

Let $\pi = (\pi_1, \ldots, \pi_n)$ be a permutation and A be the permutation matrix of π, that is π_i is the column of the unique 1 in row i. Then

$$\mathrm{inv}(A) = \sum_{1 \le i' < i \le n, 1 \le j' \le j \le n} a_{i'j} a_{ij'}$$

is the number of inversions in π. We use this to define the inversion number of ASMs.

Let $A = (a_{i,j})_{1 \le i \le n, i \le j \le 2n-i}$ be an AST. We define

$$\mathrm{inv}(A) = \sum_{i' < i, j' \le j} a_{i'j} a_{ij'}.$$

Generalization of the generalization of our theorem: Let m, n, i be non-negative integers. Then

$$|\{A \in \mathsf{ASM}(n) \mid \mu(A) = m, \mathsf{inv}(A) = i\}|$$
$$= |\{A \in \mathsf{AST}(n) \mid \mu(A) = m, \mathsf{inv}(A) = i\}|.$$

The case $n = 3$

ASM	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$
μ	0	0	0	0
inv	0	1	1	2
AST	$\begin{array}{ccccc}1&0&0&0&0\\&1&0&0&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&1&0&0&0\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}1&0&0&0&0\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&0&0&1\\&1&0&0&\\&&1&&\end{array}$

ASM	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$
μ	0	0	1
inv	2	3	1
AST	$\begin{array}{ccccc}0&0&0&1&0\\&1&0&0&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&0&0&1\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&1&0&0\\&1&-1&1&\\&&1&&\end{array}$

Refined ASM-Theorem

Observation: There is a unique 1 in the top row of an ASM.

Theorem (Zeilberger, 1996): The number of $n \times n$ ASMs with a 1 in the top row and column r is

$$\binom{n+r-2}{n-1}\frac{(2n-r-1)!}{(n-r)!}\prod_{j=0}^{n-2}\frac{(3j+1)!}{(n+j)!} = A_{n,r}.$$

Find a statistic on ASTs that has the same distribution as the column of the 1 in the top row of an ASM!

Equivalent statistic on ASTs

In an AST, the elements of a column add up to 0 or 1. We say that a column is a 1-column if they add up to 1.

Let T be an AST with n rows. Define

$$\rho(T) = (\#\text{1-columns in the left half of } T \text{ that have a 0 at the bottom}) \\ + (\#\text{1-columns in the right half of } T \text{ that have a 1 at the bottom}) + 1.$$

This statistic was introduced by Behrend.

Theorem (F. 2019). The number of ASTs T with n rows and $\rho(T) = r$ is equal to $A_{n,r}$.

The case $n = 3$

ASM	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$
μ	0	0	0	0
inv	0	1	1	2
Top 1	1	1	2	2
ρ	1	3	2	2
AST	$\begin{array}{ccccc}1&0&0&0&0\\&1&0&0&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&1&0&0&0\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}1&0&0&0&0\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&0&0&1\\&1&0&0&\\&&1&&\end{array}$

ASM	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$
μ	0	0	1
inv	2	3	1
Top 1	3	3	2
ρ	1	3	2
AST	$\begin{array}{ccccc}0&0&0&1&0\\&1&0&0&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&0&0&1\\&0&0&1&\\&&1&&\end{array}$	$\begin{array}{ccccc}0&0&1&0&0\\&1&-1&1&\\&&1&&\end{array}$

Remark. The statistics inv can also be replaced by inv of the rotated or reflected ASM and/or the reflected AST. Rotation by 90° and reflection replaces inv by $3 - \text{inv}$. However, also after such a replacement it is **not possible to have a bijection that preserves all three statistics.**

Back to Andrews' determinant

$$D_n(k) = \mathsf{det}_{0 \le i,j \le n-1} \left(\delta_{i,j} + \binom{k+i+j}{i} \right)$$

Recall:

• $D_n(2)$ is the number of $n \times n$ ASMs as well as the number of ASTs with n rows.

• $D_n(k)$ is the number of cyclically symmetric lozenge tilings of a hexagon with central triangular hole of size k.

Is there a combinatorial realization of $D_n(k)$ in terms of ASM-like objects ?

Alternating sign trapezoids

For $n \geq 1, l \geq 2^*$, an (n,l)-*alternating sign trapezoid* is an array of 1's, −1's and 0's with n centered rows and l elements in the bottom row, arranged as follows

$$\begin{array}{cccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
 & & \ddots & & & & & \iddots & & \\
 & & & \bullet & \bullet & \bullet & \bullet & & &
\end{array},$$

such that the following conditions are satisfied.

(1) In each row and column, the non-zero entries alternate.

(2) All row sums are 1.

(3) The topmost non-zero entry in each column is 1.

(4) The column sums are 0 for the middle $l-2$ columns.

*Can be extended to $l = 1$.

Example

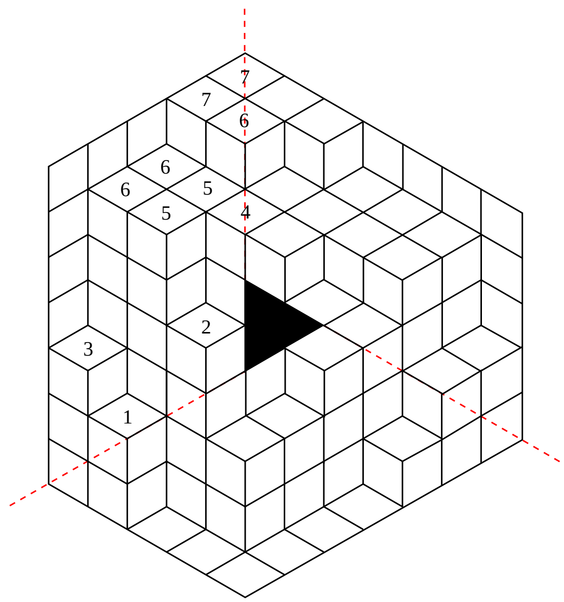
A $(5, 4)$-alternating sign trapezoid.

$$\begin{array}{cccccccccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & & 0 \\
& 0 & & 0 & & 0 & & 0 & & 1 & & 0 & & -1 & & 1 & & 0 & & 0 & \\
& & 0 & & 1 & & 0 & & -1 & & 0 & & 1 & & -1 & & 1 & & & & \\
& & & 0 & & 0 & & 0 & & 1 & & -1 & & 1 & & & & & & & \\
& & & & 1 & & 0 & & -1 & & 1 & & & & & & & & & &
\end{array}$$

ASTs with n rows are equivalent to $(n-1, 3)$-alternating sign trapezoids. (Delete the bottom row of the AST.)

Alternating sign trapezoids and cyclically symmetric rhombus tilings of a holey hexagon

Theorem (Behrend, F. 2018). There is the same number of (n, l)-alternating sign trapezoids as there is of cyclically symmetric rhombus tilings of a hexagon with side lengths $n+l-1, n, n+l-1, n, n+l-1, n$ that has a central triangular hole of size $l-1$.



Product formula

Corollary. The number of (n, l)-alternating sign trapezoids is

$$2^n \prod_{i=0}^{n-1} q_i(l)^{n-i-1},$$

where

$$q_i(l) = \begin{cases} \frac{(l+3i)(2+l+3i)(4+l+3i)}{(l+2i)(2+l+2i)(4+4i)}, & i \text{ even}, \\ \frac{2\left(-\frac{1}{2}+l+\frac{3}{2}i\right)\left(\frac{1}{2}+l+\frac{3}{2}i\right)\left(\frac{3}{2}+l+\frac{3}{2}i\right)}{(l+2i)(2+l+2i)(l+i)}, & i \text{ odd}. \end{cases}$$

Three statistics on alternating sign trapezoids

- A 1-column is a column with sum 1.
- A 10-column is a 1-column whose bottom element is 0.

Simple fact: An (n, l)-alternating sign trapezoid has n 1-columns

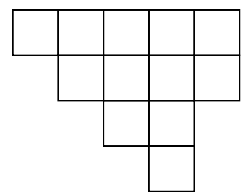
The statistics on (n, l)-alternating sign trapezoids T:

$$\begin{aligned} \mathsf{p}(T) &= \# \text{ of 10-columns among the } n \text{ leftmost columns,} \\ \mathsf{q}(T) &= \# \text{ of 10-columns among the } n \text{ rightmost columns,} \\ \mathsf{r}(T) &= \# \text{ of 1-columns among the } n \text{ leftmost columns.} \end{aligned}$$

In the example above, we have $\mathsf{p}(T) = 1, \mathsf{q}(T) = 0, \mathsf{r}(T) = 2$.

Column strict shifted plane partitions of a fixed class aka DPPs

• With each strict partition (= partition with distinct parts), we associate a shifted Ferrers diagram. The shifted Ferrers diagram of $(5, 4, 2, 1)$ is



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• A column strict shifted plane partition is a filling of a shifted Ferrers diagram with positive integers such that the rows are weakly decreasing and the columns are strictly decreasing.

Example.

$$\begin{array}{ccccc} 7 & 7 & 6 & 6 & 3 \\ & 6 & 5 & 5 & 1 \\ & & 4 & 2 & \end{array}$$

• A column strict shifted plane partition is of class k if the first part of each row exceeds the length of the row by precisely k. (Mills, Robbins and Rumsey 1987; Andrews 1979) The example is of class 2.

• There is a simple bijection between column strict shifted plane partitions of class k where the length of the top row does not exceed n and cyclically symmetric rhombus tilings of a hexagon with side lengths $n+k, n, n+k, n, n+k, n$ with a central triangular hole of size k.

Three statistics on column strict shifted plane partitions

For $d \in \{1, \ldots, k\}$ and a column strict shifted plane partition $C = (c_{i,j})$ of class k, we define

$$\begin{aligned} \mathrm{p}_d(C) &= \# \ (i,j) \text{ s.t. } c_{i,j} = j - i + d, \\ \mathrm{q}(C) &= \# \text{ of 1's}, \\ \mathrm{r}(C) &= \# \text{ of rows}. \end{aligned}$$

In the example above, we have $\mathrm{p}_1(C) = 1, \mathrm{q}(C) = 1, \mathrm{r}(C) = 3$.

Theorem (F. 2019). The number of (n,l)-alternating sign trapezoids T with $\mathrm{p}(T) = p, \mathrm{q}(T) = q, \mathrm{r}(T) = r$ is equal to the number of column strict shifted plane partitions of class $l-1$ with $\mathrm{p}_d(C) = p, \mathrm{q}(C) = q, \mathrm{r}(C) = r$, where the length of the first row does not exceed n.

Hans Höngesberg could add another statistic (number of -1's on the alternating sign trapezoid side). He also provides a statistic-preserving bijection for the case $r = 1$.

The case $n = 2, l = 4$

Alternating sign trapezoids:

$$
\begin{array}{ccccccccccc} 1&&0&&0&&0&&0&&0\\ &&1&&0&&0&&0&&\\ &&&&&(0,0,2)&&&&& \end{array}
\qquad
\begin{array}{ccccccccccc} 1&&0&&0&&0&&0&&0\\ &&0&&0&&0&&1&&\\ &&&&&(0,0,1)&&&&& \end{array}
\qquad
\begin{array}{ccccccccccc} 0&&1&&0&&0&&0&&0\\ &&0&&0&&0&&1&&\\ &&&&&(1,0,1)&&&&& \end{array}
\qquad
\begin{array}{ccccccccccc} 0&&0&&1&&0&&0&&0\\ &&1&&-1&&0&&1&&\\ &&&&&(0,0,1)&&&&& \end{array}
$$

$$
\begin{array}{ccccccccccc} 0&&0&&0&&1&&0&&0\\ &&1&&0&&-1&&1&&\\ &&&&&(0,0,1)&&&&& \end{array}
\qquad
\begin{array}{ccccccccccc} 0&&0&&0&&0&&1&&0\\ &&1&&0&&0&&0&&\\ &&&&&(0,1,1)&&&&& \end{array}
\qquad
\begin{array}{ccccccccccc} 0&&0&&0&&0&&0&&1\\ &&1&&0&&0&&0&&\\ &&&&&(0,0,1)&&&&& \end{array}
\qquad
\begin{array}{ccccccccccc} 0&&0&&0&&0&&0&&1\\ &&0&&0&&0&&1&&\\ &&&&&(0,0,0)&&&&& \end{array}
$$

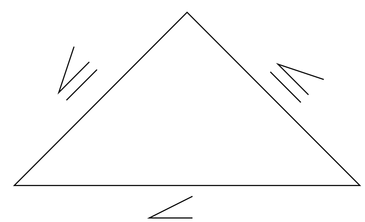
Column strict shifted plane partitions:

	$\varnothing$	4	5 1	5 2	5 3	5 4	5 5	$\begin{smallmatrix}5&5\\&4\end{smallmatrix}$
$d = 1$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(1,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,2)$
$d = 2$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(0,0,1)$	$(1,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,2)$
$d = 3$	$(0,0,0)$	$(0,0,1)$	$(0,1,1)$	$(0,0,1)$	$(0,0,1)$	$(1,0,1)$	$(0,0,1)$	$(0,0,2)$

Proofs

Monotone triangles

Triangular arrays of integers with monotonicity requirements:



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 1 & & 4 & & & \\ & & 1 & & 2 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Monotone triangles with bottom row $1, 2, \ldots, n \Leftrightarrow n \times n$ ASMs

Formula for the number of monotone triangles with prescribed bottom row

Antisymmetrizer:

$$\mathbf{ASym}_{Y_1,\ldots,Y_n} F(Y_1,\ldots,Y_n) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma F(Y_{\sigma(1)},\ldots,Y_{\sigma(n)})$$

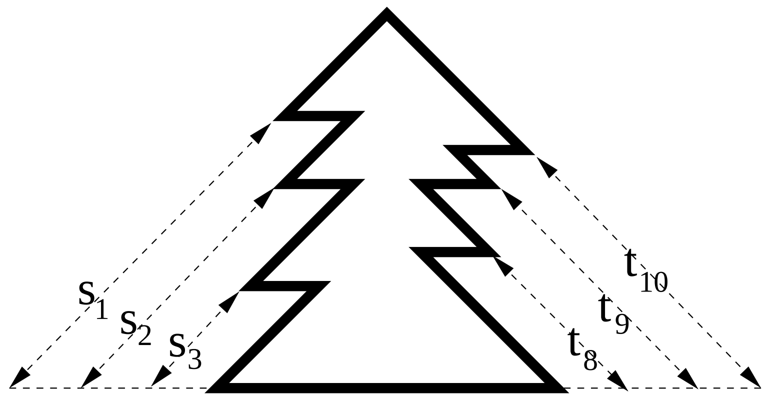
Define

$$M_n(\mathbf{x}) = \mathrm{CT}_{Y_1,\ldots,Y_n} \frac{\mathbf{ASym}_{Y_1,\ldots,Y_n}\left[\prod_{i=1}^{n}(1+Y_i)^{x_i} \prod_{1 \le i < j \le n}(1+Y_j+Y_iY_j)\right]}{\prod_{1 \le i < j \le n}(Y_j - Y_i)},$$

where $\mathrm{CT}_{Y_1,\ldots,Y_n}$ denotes the constant term w.r.t. $Y_1,\ldots,Y_n$.

Theorem (F., 2006). The number of monotone triangles with bottom row $b_1,\ldots,b_n$ is $M_n(b_1,\ldots,b_n)$.

Truncated monotone triangles: (s,t)-trees



- $\mathbf{s} = (s_1, s_2, \ldots, s_l)$ weakly decreasing sequence: prescribes the number of entries deleted at the bottom of the $\nearrow$-diagonals.

- $\mathbf{t} = (t_{n-r+1}, \ldots, t_n)$ weakly increasing sequence: prescribes the number of entries deleted at the bottom of the $\searrow$-diagonals.

The number of (s, t)-trees

Forward difference operator: $\overline{\Delta}_x p(x) = p(x+1) - p(x)$
Backward difference operator: $\underline{\Delta}_x p(x) = p(x) - p(x-1)$

The evaluation of

$$(-\overline{\Delta}_{x_1})^{s_1} \cdots (-\overline{\Delta}_{x_l})^{s_l} \underline{\Delta}_{x_{n-r+1}}^{t_{n-r+1}} \cdots \underline{\Delta}_{x_n}^{t_n} M_n(\mathbf{x})$$

at $\mathbf{x} = (b_1, \ldots, b_n)$ is the number of $(\mathbf{s}, \mathbf{t})$-trees of order n with the following properties:

- The bottom entry of the i-th $\nearrow$-diagonal is b_i for $1 \le i \le n - r$.
- The bottom entry of the i-th $\searrow$-diagonal is b_i for $n - r + 1 \le i \le n$.

From alternating sign triangles to truncated monotone triangles

$$
\begin{array}{ccccccccccccccccccccc}
0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& & & 1 & & 0 & & -1 & & 0 & & 0 & & 0 & & 1 & & 0 & & 0 & \\
& & & & 0 & & 0 & & 0 & & 1 & & 0 & & -1 & & 1 & & & & \\
& & & & & 0 & & 1 & & -1 & & 0 & & 1 & & & & & & & \\
& & & & & & 0 & & 0 & & 1 & & & & & & & & & & \\
& & & & & & & 1 & & & & & & & & & & & & &
\end{array}
$$

1-column = a column with sum 1.

- An AST with n rows has precisely n 1-columns.

- First goal: Constant term formula for the number of ASTs with prescribed positions of the 1-columns.

$$\begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$\Downarrow$$

$$\begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{matrix}$$

−5	−4	−3	−2	−1	0	1	2	3	4	5
0	0	0	1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	1	0	0	0
0	1	0	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	1	0	0
0	1	0	0	1	0	1	1	1	0	0
0	1	0	0	1	1	1	1	1	0	0

$$\Downarrow$$

$$\begin{array}{ccccccccccc}
 & & & & & -2 & & & & & \\
 & & & & -4 & & 2 & & & & \\
 & & & -4 & & 0 & & 3 & & & \\
 & & -4 & & -1 & & 2 & & 3 & & \\
 & -4 & & -1 & & 1 & & 2 & & 3 & \\
-4 & & -1 & & 0 & & 1 & & 2 & & 3
\end{array}$$

• Orange entries are redundant. Delete them in order to obtain an $(\mathbf{s}, \mathbf{t})$-tree.

• Number of deleted entries in a fixed diagonal equals the absolute value of the bottom entry in the truncated diagonal.

The number of ASTs with prescribed positions of the 1-columns

Using the formula for the number of $(\mathbf{s}, \mathbf{t})$-tree, we can deduce the following (after a few pages of calculations).

Theorem (F. 2019). The number of ASTs with n rows that have the 1-columns in positions $j_1, j_2, \ldots, j_{n-1}$, where we count from the left starting with 0 and disregard the central column, is the coefficient of $X_1^{j_1} X_2^{j_2} \ldots X_{n-1}^{j_{n-1}}$ in

$$\prod_{i=1}^{n-1} (1 + X_i) \prod_{1 \le i < j \le n-1} (1 + X_i + X_i X_j)(X_j - X_i).$$

Total number of ASTs

The number of ASTs with n rows is the constant term of

$$\sum_{0\le j_1<j_2<\ldots<j_{n-1}} \prod_{i=1}^{n-1}(1+X_i^{-1})X_i^{j_i} \prod_{1\le i<j\le n-1}(1+X_i^{-1}+X_i^{-1}X_j^{-1})(X_j^{-1}-X_i^{-1})$$

$$= \frac{\prod\limits_{i=1}^{n-1}(1+X_i)X_i^{i-2n+2}\prod\limits_{1\le i<j\le n-1}(1+X_j+X_iX_j)(X_i-X_j)}{\prod\limits_{i=1}^{n-1}\left(1-\prod\limits_{j=i}^{n-1}X_j\right)}.$$

"Trick:" Apply the symmetrizer w.r.t. $X_1,\ldots,X_n$. The constant term is then multiplied by $n!$.

"Magic:" The symmetrizer can actually be computed!

End of the proof

Lemma. Let $n \geq 1$. Then

$$\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(1+X_j+X_iX_j)\prod_{i=1}^{n}X_i^{i-1}\left(1-\prod_{j=i}^{n}X_j\right)^{-1}\right]$$
$$=\prod_{i=1}^{n}(1-X_i)^{-1}\prod_{1\le i<j\le n}\frac{(1+X_i+X_j)(X_j-X_i)}{(1-X_iX_j)}.$$

After some further steps, one can see that the number is the constant term of

$$(-1)^{\binom{n-1}{2}}\sum_{0\le b_1<b_2<\ldots<b_{n-1}}\mathrm{det}_{1\le i,j\le n-1}\left((1+X_j)^i X_j^{i-2n+2+b_j}\right),$$

and this leads directly to an expression that gives the number of **totally symmetric self-complementary plane partitions** in a $2n\times 2n\times 2n$ box (Lindström-Gessel-Viennot). See my TAMS-paper from 2019.

The number of alternating sign trapezoids with prescribed positions of the 1-columns

Theorem (Schreier-Aigner). The number of (n,l)-alternating sign trapezoids with the 1-columns in positions $0 \leq j_1 < j_2 < \ldots < j_n \leq 2n-1$ where we index the columns from left to right starting with 0 and disregard the $l-2$ central columns is the coefficient of $X_1^{j_1} X_2^{j_2} \ldots X_n^{j_n}$ in

$$\prod_{i=1}^{m}(1+X_i) \prod_{i=m+1}^{n} X_i^{-l+3}(1+X_i)^{l-2} \prod_{1 \leq i<j \leq n}(X_j - X_i)(1+X_i+X_iX_j),$$

where m is maximal such that $j_m \leq n-1$.

Only for $l=3$ (ASTs!), there is no dependency on m.

Crucial step in the enumeration of AS-trapezoids

Definition.

$$\mathbf{Subsets}_m\, F(Y_1,\ldots,Y_n) = \sum_{\substack{\sigma\in\mathcal{S}_n \\ \sigma(1)<\sigma(2)<\ldots<\sigma(m),\sigma(m+1)<\sigma(m+2)<\ldots<\sigma(n)}} F(Y_{\sigma(1)},\ldots,Y_{\sigma(n)})$$

The number is the constant term of

$$\sum_{m=0}^{n} \mathbf{Subsets}_m \prod_{i=m+1}^{n} (1+Y_i)^{l-1} \prod_{i,j=1}^{m} \frac{1}{1+Y_i+Y_j} \prod_{i,j=m+1}^{n} \frac{1}{1-Y_iY_j} \times \prod_{i=1}^{m} \prod_{j=m+1}^{n} \frac{1+Y_j+Y_iY_j}{(Y_j-Y_i)(1+Y_i+Y_j)(1-Y_iY_j)}.$$

Using the Cauchy determinant, it follows that

$$\prod_{i=m+1}^{n}(1+Y_i)^{l-1}\prod_{i,j=1}^{m}\frac{1}{1+Y_i+Y_j}\prod_{i,j=m+1}^{n}\frac{1}{1-Y_iY_j}$$
$$\times\prod_{i=1}^{m}\prod_{j=m+1}^{n}\frac{1+Y_j+Y_iY_j}{(Y_j-Y_i)(1+Y_i+Y_j)(1-Y_iY_j)}$$
$$=\frac{\det_{1\le i,j\le n}\left(\begin{cases}\frac{1}{1+Y_i+Y_j}, & i\le m\\ \frac{(1+Y_i)^{l-1}}{1-Y_iY_j}, & i>m\end{cases}\right)}{\prod\limits_{1\le i<j\le n}(Y_j-Y_i)^2}.$$

Applying $\mathbf{Subsets}_m$ and summing over all m then gives

$$\frac{\det_{1\le i,j\le n}\left(\frac{1}{1+Y_i+Y_j}+\frac{(1+Y_i)^{l-1}}{1-Y_iY_j}\right)}{\prod\limits_{1\le i<j\le n}(Y_j-Y_i)^2}.$$

After some further manipulations we obtain Andrews generalization of the determinant for the number of cyclically symmetric plane partitions. See my Adv. Math. paper from 2019.

III. A Littlewood-type identity related to ASMs

The classical (unbounded) Littlewood identity

$$\sum_{\lambda} s_{\lambda}(X_1,\ldots,X_n) = \prod_{i=1}^{n} \frac{1}{1-X_i} \prod_{1\le i<j\le n} \frac{1}{1-X_i X_j},$$

Proof: RSK and exploiting its symmetry.

We rewrite the classical Littlewood identity:

$$s_{(\lambda_1,\ldots,\lambda_n)}(X_1,\ldots,X_n) = \frac{\det_{1\le i,j\le n}\left(X_i^{\lambda_j+n-j}\right)}{\prod_{1\le i<j\le n}(X_i-X_j)} = \frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{i=1}^{n} X_i^{\lambda_i+n-i}\right]}{\prod_{1\le i<j\le n}(X_i-X_j)},$$

with $\mathbf{ASym}_{X_1,\ldots,X_n} f(X_1,\ldots,X_n) = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}\sigma \cdot f(X_{\sigma(1)},\ldots,X_{\sigma(n)})$

Change of variables: $\lambda_1 \ge \cdots \ge \lambda_n \ge 0 \Rightarrow \underbrace{\lambda_1+n-1}_{k_n} > \underbrace{\lambda_2+n-2}_{k_{n-1}} > \ldots > \underbrace{\lambda_n}_{k_1} \ge 0$

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\sum_{0\le k_1<k_2<\ldots<k_n} X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)} = \prod_{i=1}^{n} \frac{1}{1-X_i} \prod_{1\le i<j\le n} \frac{1}{1-X_i X_j}$$

Littlewood-type identity related to ASMs

In two of my papers from 2019:

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{\mathbf{1\le i<j\le n}}(\mathbf{1}+\mathbf{X_j}+\mathbf{X_i X_j})\sum_{0\le k_1<k_2<\ldots<k_n} X_1^{k_1}X_2^{k_2}\cdots X_n^{k_n}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$$
$$=\prod_{i=1}^{n}\frac{1}{1-X_i}\prod_{1\le i<j\le n}\frac{\mathbf{1}+\mathbf{X_i}+\mathbf{X_j}}{1-X_iX_j}$$

Since then Hans Höngesberg and I realized that we can introduce two additional parameters:

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i<j\le n}(\mathbf{Q}+(\mathbf{Q}+\mathbf{r})\mathbf{X_i}+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n}\prod_{i=1}^{n}\left(\frac{X_i(\mathbf{1}+\mathbf{X_i})}{\mathbf{Q}+\mathbf{X_i}}\right)^{k_i}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$$
$$=\prod_{i=1}^{n}\frac{\mathbf{Q}+X_i}{\mathbf{Q}-X_i^2}\frac{\prod_{1\le i<j\le n}\mathbf{Q}(1+X_i)(1+X_j)+rX_iX_j}{\prod_{1\le i<j\le n}(\mathbf{Q}-X_iX_j)}$$

Set $\mathbf{Q}=\mathbf{1}$ and $\mathbf{r}=-\mathbf{1}$ to obtain the previous identity.

I would love to see a combinatorial proof of this identity! `arXiv:2301.00175`

Combinatorial interpretation of the LHS

Gelfand-Tsetlin patterns

A Gelfand-Tsetlin pattern is a triangular array of integers of the form

$$\begin{array}{ccccccc} & & & a_{1,1} & & & \\ & & a_{2,1} & & a_{2,2} & & \\ & \dots & & \dots & & \dots & \\ a_{n,1} & & \dots & & \dots & & a_{n,n} \end{array}$$

with weak increase in $\nearrow$- and $\searrow$-direction.

The weight of a Gelfand-Tsetlin pattern is $\prod_{i=1}^{n} X_i^{\sum_j a_{i,j} - \sum_j a_{i-1,j}}$ and $s_\lambda(X_1, \dots, X_n)$ is the sum of weights of all Gelfand-Tsetlin patterns with bottom row $(0, \dots, 0, \lambda_l, \dots, \lambda_1)$.

Example:

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & 3 & & 5 & & & \\ & & 2 & & 4 & & 6 & & \\ & 1 & & 3 & & 5 & & 7 & \\ 1 & & 1 & & 4 & & 7 & & 8 \end{array}$$

Arrowed Gelfand-Tsetlin patterns

An arrowed Gelfand-Tsetlin pattern is a Gelfand-Tsetlin pattern where each entry is decorated with an element from $\{\nwarrow, \nearrow, \nwarrow\!\!\!\!\!\nearrow, \emptyset\}$ such that for the **little triangles** in the pattern

$$\begin{array}{ccc} & y & \\ x & & z \end{array}$$

we have the following:

- If $x = y$ and $\mathrm{decor}(x) \in \{\nearrow, \nwarrow\!\!\!\!\!\nearrow\}$, then $z = y = x$ and $\mathrm{decor}(z) \in \{\nwarrow, \nwarrow\!\!\!\!\!\nearrow\}$, and
- if $y = z$ and $\mathrm{decor}(z) \in \{\nwarrow, \nwarrow\!\!\!\!\!\nearrow\}$, then $x = y = z$ and $\mathrm{decor}(x) \in \{\nearrow, \nwarrow\!\!\!\!\!\nearrow\}$.

Both instances contribute -1 to the sign.

Summary: Arrows between diagonal neighbors indicate that the entries are different, except when we have two such occurrences appearing in a little triangle. In this case, we have a contribution of -1 to the sign.

Example:

$$\begin{array}{ccccccccc}
 & & & & {}^{\nwarrow}3^{\nearrow} & & & & \\
 & & & 3 & & {}^{\nwarrow}5 & & & \\
 & & 2^{\nearrow} & & {}^{\nwarrow}4 & & 6 & & \\
 & 1^{\nearrow} & & 3^{\nearrow} & & 4^{\nearrow} & & {}^{\nwarrow}7^{\nearrow} & \\
{}^{\nwarrow}1 & & 3^{\nearrow} & & {}^{\nwarrow}3^{\nearrow} & & 7 & & {}^{\nwarrow}8
\end{array}$$

Generating function

We associate the following **weight** to a given arrowed Gelfand-Tsetlin pattern $A = (a_{i,j})_{1 \le j \le i \le n}$:

$$\mathrm{W}(A) = \mathrm{sgn}(A) \cdot t^{\# \emptyset} u^{\# \nearrow} v^{\# \nwarrow} w^{\# \nwarrow\!\!\!\!\nearrow} \prod_{i=1}^{n} X_i^{\sum_{j=1}^{i} a_{i,j} - \sum_{j=1}^{i-1} a_{i-1,j} + \# \nearrow \text{in row } i - \# \nwarrow \text{in row } i}$$

The weight of our example is

$$-t^3 u^5 v^3 w^3 X_1^3 X_2^4 X_3^4 X_4^6 X_5^6.$$

Compare to the Schur function weight for Gelfand-Tsetlin patterns!

Theorem (F. and Schreier-Aigner). The generating function of arrowed Gelfand-Tsetlin patterns with bottom row $k_1, \ldots, k_n$ is

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1 \le i \le j \le n} (v + w X_i + t X_j + u X_i X_j) \prod_{i=1}^{n} X_i^{k_i - 1}\right]}{\prod_{1 \le i < j \le n} (X_j - X_i)}.$$

This is a generalization of **Hall-Littlewood polynomials** (set $t = 1$, $u = v = 0$ and $w = -t$).

Application to our LHS

Our Littlewood-type identity, slightly rewritten:

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1\le i\le j\le n}(1+wX_i+X_j+X_iX_j)\sum_{0\le k_1<k_2<\ldots<k_n} X_1^{k_1-1}X_2^{k_2-1}\ldots X_n^{k_n-1}\right]}{\prod_{1\le i<j\le n}(X_j-X_i)}$$
$$=\prod_{i=1}^{n}\frac{X_i^{-1}+(1+w)+X_i}{1-X_i}\prod_{1\le i<j\le n}\frac{1+X_i+X_j+wX_iX_j}{1-X_iX_j}$$

The left-hand side is the generating function of all arrowed Gelfand-Tsetlin patterns with strictly increasing bottom row of non-negative integers when setting $t=u=v=1$.

Combinatorial interpretation of the RHS of the Littlewood-type identity: decorated two-line arrays (straightforward).

Bounded classical Littlewood identity

Bounded? $\sum_{0 \le k_1 < k_2 < \ldots < k_n} \to \sum_{0 \le k_1 < k_2 < \ldots < k_n \le m}$

$$\sum_{\lambda \subseteq (m^n)} s_\lambda(X_1, \ldots, X_n) = \frac{\det_{1 \le i,j \le n}\left(X_i^{j-1} - X_i^{m+2n-j}\right)}{\prod_{i=1}^{n}(1 - X_i) \prod_{1 \le i < j \le n}(X_j - X_i)(1 - X_i X_j)}$$

Macdonald in his book.

Bounded Littlewood identity related to ASMs

$$\frac{1}{\prod_{1 \le i < j \le n} (X_j - X_i)} \mathbf{ASym}_{X_1,\dots,X_n} \Bigg[\prod_{1 \le i < j \le n} (Q + (Q+r) X_i + X_j + X_i X_j)$$

$$\times \sum_{0 \le k_1 < k_2 < \dots < k_n \le \mathbf{m}} \left(\frac{X_1(1+X_1)}{Q+X_1} \right)^{k_1} \left(\frac{X_2(1+X_2)}{Q+X_2} \right)^{k_2} \cdots \left(\frac{X_n(1+X_n)}{Q+X_n} \right)^{k_n} \Bigg]$$

$$= \frac{\det_{1 \le i,j \le n} \left(a_{j,m,n}(Q,r;X_i) \right)}{\prod_{1 \le i \le j \le n} (Q - X_i X_j) \prod_{1 \le i < j \le n} (X_j - X_i)}$$

with

$$a_{j,m,n}(Q,r;X) = (1+QX^{-1}) X^j (1+X)^{j-1} (Q + rX + QX)^{n-j}$$
$$- X^{2n} Q^{-n} \left(\frac{(1+X)X}{Q+X} \right)^m (1+X) \left(QX^{-1} \right)^j (1+QX^{-1})^{j-1} (Q + rQX^{-1} + Q^2 X^{-1})^{n-j}.$$

• The proof has more than 7 pages, but it is elementary.

• Moritz Gangl has a nice application of this relating AS-pentagons to Magog-pentagons. Note that there are some old conjectures on the relation between Gog trapezoids and Magog trapezoids that are still open.

The case $Q = 1$

$$\frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1 \le i \le j \le n}(1 + wX_i + X_j + X_iX_j)\sum_{0 \le k_1 < k_2 < \ldots < k_n \le m} X_1^{k_1-1}X_2^{k_2-1}\ldots X_n^{k_n-1}\right]}{\prod\limits_{1 \le i < j \le n}(X_j - X_i)}$$

$$= \prod_{i=1}^{n}(X_i^{-1} + 1 + w + X_i)$$

$$\times \frac{\det_{1 \le i,j \le n}\left(X_i^{j-1}(1+X_i)^{j-1}(1+wX_i)^{n-j} - X_i^{m+2n-j}(1+X_i^{-1})^{j-1}(1+wX_i^{-1})^{n-j}\right)}{\prod\limits_{i=1}^{n}(1-X_i)\prod\limits_{1 \le i < j \le n}(1-X_iX_j)(X_j - X_i)}.$$

LHS: Generating function of arrowed Gelfand-Tsetlin patterns with strictly increasing bottom that are bounded by m.

What about the RHS ? $\longrightarrow$ I have a combinatorial interpretation for the RHS as well.

$$\mathbf{1, 4, 60, 3328, 678912 \ldots}$$

RHS of the new Littlewood-type identity for $Q = 1$:

$$\frac{\det_{1 \le i,j \le n} \left(X_i^{j-1} (1 + X_i)^{j-1} (1 + w X_i)^{n-j} - X_i^{m+2n-j} (1 + X_i^{-1})^{j-1} (1 + w X_i^{-1})^{n-j} \right)}{\prod_{i=1}^{n} (1 - X_i) \prod_{1 \le i < j \le n} (1 - X_i X_j)(X_j - X_i)}$$

Setting all $X_i = 1, w = -1$ and $m = n - 1$, we obtain

$$1, 4, 60, 3328, 678912, \ldots = 2^{n(n-1)/2} \prod_{j=0}^{n-1} \frac{(4j+2)!}{(n+2j+1)!}.$$

- This is a consequence of our Theorem 1 below.

- In fact, these theorems involve the **additional parameter** m, and the special case $m = n - 1$ is an unpublished conjecture of Florian Schreier-Aigner from 2018.

- Note that $m = n - 1$ just means that we consider arrowed Gelfand-Tsetlin patterns with bottom row $(0, 1, \ldots, n - 1)$.

These numbers also appear in recent work of Di Francesco related to the **twenty vertex model** and **domino tilings.**

He showed that both families of objects are counted by

$$\det_{0\le i,j\le n-1}\left(2^i\binom{i+2j+1}{2j+1}-\binom{i-1}{2j+1}\right)$$

and conjectured the following theorem, which was subsequently proven by Koutschan.

Theorem.

$$\det_{0\le i,j\le n-1}\left(2^i\binom{i+2j+1}{2j+1}-\binom{i-1}{2j+1}\right)=2^{n(n-1)/2}\prod_{j=0}^{n-1}\frac{(4j+2)!}{(n+2j+1)!}.$$

Explicit product formulas in case $X_i = 1$ and $w = -1$

Theorem 1 (F. and Schreier-Aigner, 2023). For $(X_1, \ldots, X_n) = (1, \ldots, 1)$ and $w = -1$, we have that

$$\frac{\det_{1 \le i,j \le n}\left(X_i^{j-1}(1+X_i)^{j-1}(1+wX_i)^{n-j} - X_i^{m+2n-j}(1+X_i^{-1})^{j-1}(1+wX_i^{-1})^{n-j}\right)}{\prod_{i=1}^{n}(1-X_i) \prod_{1 \le i < j \le n}(1-X_iX_j)(X_j-X_i)}$$

is

$$2^n \prod_{i=1}^{n} \frac{(m-n+3i+1)_{i-1}(m-n+i+1)_i}{\left(\frac{m-n+i+2}{2}\right)_{i-1}(i)_i}.$$

We have signless versions for the objects (decorated Gelfand-Tsetlin patterns) in the theorem.

Open problem: Find a bijection between Di Francesco's twenty vertex configurations and our objects for $m = n - 1$.

See our paper `arXiv:2302.04164` for details.

Explicit product formulas in case $X_i = 1$ and $w = 0$

Theorem 2 (F. and Schreier-Aigner, 2023). For $(X_1, \ldots, X_n) = (1, \ldots, 1)$ and $w = 0$, we have that

$$\frac{\det_{1 \le i,j \le n}\left(X_i^{j-1}(1+X_i)^{j-1}(1+wX_i)^{n-j} - X_i^{m+2n-j}(1+X_i^{-1})^{j-1}(1+wX_i^{-1})^{n-j}\right)}{\prod_{i=1}^{n}(1-X_i)\prod_{1\le i<j\le n}(1-X_iX_j)(X_j-X_i)}$$

is

$$3^{\binom{n+1}{2}}\prod_{i=1}^{n}\frac{(2n+m+2-3i)_i}{(i)_i}.$$

Also here we have signless versions for the objects in the theorem.

IV. Introducing $n+3$ parameters in the ASM-DPP relation

Goal: Take the (multivariate) generating function of arrowed monotone triangles and find corresponding plane partitions related to DPPs with the same generating function.

Arrowed monotone triangles → monotone triangles

Claim: When setting $u = v = 1, w = -1$ and $(X_1, \dots, X_n) = (1, \dots, 1)$ in the generating function of arrowed monotone triangles, we obtain the number of monotone triangles with bottom row $(k_1, \dots, k_n)$.

Why?

• Fix a **monotone triangle** and consider all arrowed monotone triangles that can be obtained by decorating the entries of that monotone triangle.

• Namely, an entry in the monotone triangle that is **equal to its $\nwarrow$-neighbor can only be decorated by $\nearrow$**, while an entry that is **equal to its $\nearrow$-neighbor can only be decorated by $\nwarrow$.**

• Let l **be the number of entries of the first type** and r **be the number of entries of the second type.** All other entries can be decorated by any element in $\{\nwarrow, \nearrow, \nwarrow\!\!\!\!\nearrow\}$ and we let f be their number.

• Setting $(x_1, \dots, x_n) = (1, \dots, 1)$ in the generating function, we see that the contribution of the fixed monotone triangle is

$$u^l v^r (u + v + w)^f$$

and this reduces to 1 when setting $u = v = 1$ and $w = -1$.

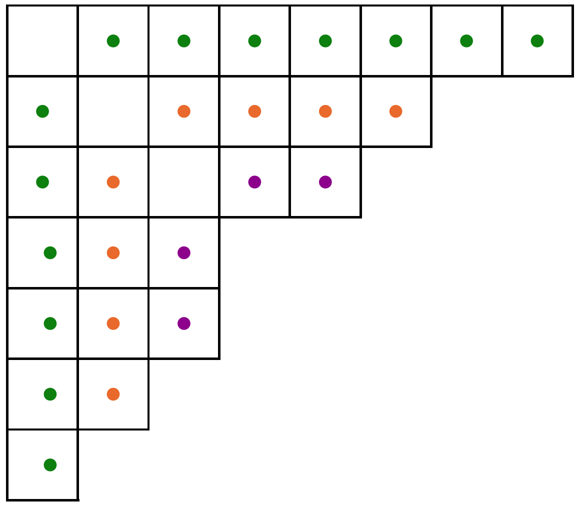
Arrowed monotone triangles with bottom row $(1, 2)$

In the following, we use $^{\nwarrow}e^{\nearrow}$ instead of $\overset{\nwarrow\!\nearrow}{e}$ in our arrowed monotone triangles.

AMT	W	SBCSPP	AMT	W	SBCSPP
$^{\nwarrow}1$ $^{\nwarrow}1 \quad ^{\nwarrow}2$	v^3	$\varnothing$	$^{\nwarrow}2$ $^{\nwarrow}1^{\nearrow} \quad 2^{\nearrow}$	$uvwX_1X_2^2$	2 2 1
$^{\nwarrow}1^{\nearrow}$ $^{\nwarrow}1 \quad ^{\nwarrow}2$	v^2wX_1	1	$^{\nwarrow}1^{\nearrow}$ $^{\nwarrow}1 \quad 2^{\nearrow}$	$uvwX_1X_2^2$	2,1 1
$1^{\nearrow}$ $^{\nwarrow}1 \quad ^{\nwarrow}2$	$uv^2X_1^2$	1 1	$2^{\nearrow}$ $^{\nwarrow}1 \quad 2^{\nearrow}$	$u^2vX_1^3X_2$	2 1 1 1
$^{\nwarrow}1$ $^{\nwarrow}1 \quad ^{\nwarrow}2^{\nearrow}$	v^2wX_2	2	$^{\nwarrow}2^{\nearrow}$ $^{\nwarrow}1^{\nearrow} \quad 2^{\nearrow}$	$uw^2X_1^2X_2^2$	2 2 1 1
$^{\nwarrow}2$ $^{\nwarrow}1 \quad 2^{\nearrow}$	$uv^2X_1X_2$	2 1	$1^{\nearrow}$ $^{\nwarrow}1 \quad 2^{\nearrow}$	$u^2vX_1^2X_2^2$	2 2 1 1
$^{\nwarrow}1^{\nearrow}$ $^{\nwarrow}1 \quad ^{\nwarrow}2^{\nearrow}$	$vw^2X_1X_2$	2,1	$2^{\nearrow}$ $^{\nwarrow}1^{\nearrow} \quad 2^{\nearrow}$	$u^2wX_1^3X_2^2$	2 2 1 1 1
$^{\nwarrow}1$ $^{\nwarrow}1 \quad 2^{\nearrow}$	$uv^2X_2^2$	2 2	$^{\nwarrow}2$ $1^{\nearrow} \quad 2^{\nearrow}$	$u^2vX_1X_2^3$	2 2 2 1
$^{\nwarrow}2^{\nearrow}$ $^{\nwarrow}1 \quad 2^{\nearrow}$	$uvwX_1^2X_2$	2 1 1	$^{\nwarrow}2^{\nearrow}$ $1^{\nearrow} \quad 2^{\nearrow}$	$u^2wX_1^2X_2^3$	2 2 2 1 1
$1^{\nearrow}$ $^{\nwarrow}1 \quad ^{\nwarrow}2^{\nearrow}$	$uvwX_1^2X_2$	2,1 1	$2^{\nearrow}$ $1^{\nearrow} \quad 2^{\nearrow}$	$u^3X_1^3X_2^3$	2 2 2 1 1 1

What is a SBCSPP?

Example:



This is the Ferrers diagram of the partition $(8,6,5,3,3,2,1)$, where an integer partition is simply a weakly decreasing sequence of non-negative integers. In Frobenius notation, we write $(7,4,2|6,4,2)$.

Balanced shape: Let $\lambda = (a_1,\ldots,a_l|b_1,\ldots,b_l)$ be a partition in Frobenius notation, i.e., a_i is the number of cells right of the diagonal cell (i,i) in the same row, while b_i is the number of cells below (i,i) in the same column. We say that λ is balanced if, for all i, either $a_i = b_i$ or $a_i = b_i + 1$. The weight is

$$\mathrm{W}(\lambda) = w^{l+\sum_{i=1}^{l}(b_i-a_i)}.$$

Set-valued balanced column strict plane partitions

A **set-valued balanced column strict plane partition (SBCSPP)** D of shape λ and order n is a filling of a balanced shape with non-empty subsets of $\{1, 2, \ldots, n\}$ such that strictly above the diagonal the subsets are singletons, and

1. rows decrease weakly in the sense that the maxima of the sets form a decreasing sequence if read from left to right, and

2. columns decrease strictly in the sense that for two adjacent cells in a column, all elements in the top cell are strictly greater than all elements in the bottom cell.

The weight of D is as follows

$$\mathrm{W}(D) = \mathrm{W}(\lambda) \cdot u^{\#\text{of cells strictly above the main diagonal}} \cdot v^{\binom{n+1}{2} - \#\text{of entries on and below the main diagonal}} \cdot w^{\#\text{of entries} - \#\text{of cells}} \cdot \prod_{i=1}^{n} X_i^{\#\text{ of } i \text{ in D}}.$$

The exponents of the $u, v, w, X_1, \ldots, X_n$ are the $n+3$ statistics from the title.

Example:

8	8	8	7	7	6	4	1
7	7	7	6	5	5		
6	6	5	4	4	4		
5	4	3	$3,2$	3	2		
3	2	$2,1$	1				
2	1						
1							

Letting $n = 9$, the weight is

$$u^{16}v^{26}w^{3}X_1^5X_2^5X_3^4X_4^5X_5^4X_6^4X_7^5X_8^3.$$

The case $n = 2$

AMT	W	SBCSPP	AMT	W	SBCSPP
$\begin{smallmatrix} & {}^{\nwarrow}1 & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2 \end{smallmatrix}$	v^3	$\varnothing$	$\begin{smallmatrix} & {}^{\nwarrow}2 & \\ {}^{\nwarrow}1^{\nearrow} & & 2^{\nearrow} \end{smallmatrix}$	$uvwX_1X_2^2$	$\begin{smallmatrix} 2 & 2 \\ 1 & \end{smallmatrix}$
$\begin{smallmatrix} & {}^{\nwarrow}1^{\nearrow} & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2 \end{smallmatrix}$	v^2wX_1	$\begin{smallmatrix} 1 \end{smallmatrix}$	$\begin{smallmatrix} & {}^{\nwarrow}1^{\nearrow} & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{smallmatrix}$	$uvwX_1X_2^2$	$\begin{smallmatrix} 2,1 & 1 \end{smallmatrix}$
$\begin{smallmatrix} & 1^{\nearrow} & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2 \end{smallmatrix}$	$uv^2X_1^2$	$\begin{smallmatrix} 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} & 2^{\nearrow} & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{smallmatrix}$	$u^2vX_1^3X_2$	$\begin{smallmatrix} 2 & 1 & 1 \\ 1 & & \end{smallmatrix}$
$\begin{smallmatrix} & {}^{\nwarrow}1 & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2^{\nearrow} \end{smallmatrix}$	v^2wX_2	$\begin{smallmatrix} 2 \end{smallmatrix}$	$\begin{smallmatrix} & {}^{\nwarrow}2^{\nearrow} & \\ {}^{\nwarrow}1^{\nearrow} & & 2^{\nearrow} \end{smallmatrix}$	$uw^2X_1^2X_2^2$	$\begin{smallmatrix} 2 & 2 \\ 1 & 1 \end{smallmatrix}$
$\begin{smallmatrix} & {}^{\nwarrow}2 & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{smallmatrix}$	$uv^2X_1X_2$	$\begin{smallmatrix} 2 & 1 \end{smallmatrix}$	$\begin{smallmatrix} & 1^{\nearrow} & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{smallmatrix}$	$u^2vX_1^2X_2^2$	$\begin{smallmatrix} 2 & 2 & 1 \\ 1 & & \end{smallmatrix}$
$\begin{smallmatrix} & {}^{\nwarrow}1^{\nearrow} & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2^{\nearrow} \end{smallmatrix}$	$vw^2X_1X_2$	$\begin{smallmatrix} 2,1 \end{smallmatrix}$	$\begin{smallmatrix} & 2^{\nearrow} & \\ {}^{\nwarrow}1^{\nearrow} & & 2^{\nearrow} \end{smallmatrix}$	$u^2wX_1^3X_2^2$	$\begin{smallmatrix} 2 & 2 & 1 \\ 1 & 1 & \end{smallmatrix}$
$\begin{smallmatrix} & {}^{\nwarrow}1 & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{smallmatrix}$	$uv^2X_2^2$	$\begin{smallmatrix} 2 & 2 \end{smallmatrix}$	$\begin{smallmatrix} & {}^{\nwarrow}2 & \\ 1^{\nearrow} & & 2^{\nearrow} \end{smallmatrix}$	$u^2vX_1X_2^3$	$\begin{smallmatrix} 2 & 2 & 2 \\ 1 & & \end{smallmatrix}$
$\begin{smallmatrix} & {}^{\nwarrow}2^{\nearrow} & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{smallmatrix}$	$uvwX_1^2X_2$	$\begin{smallmatrix} 2 & 1 \\ 1 & \end{smallmatrix}$	$\begin{smallmatrix} & {}^{\nwarrow}2^{\nearrow} & \\ 1^{\nearrow} & & 2^{\nearrow} \end{smallmatrix}$	$u^2wX_1^2X_2^3$	$\begin{smallmatrix} 2 & 2 & 2 \\ 1 & 1 & \end{smallmatrix}$
$\begin{smallmatrix} & 1^{\nearrow} & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2^{\nearrow} \end{smallmatrix}$	$uvwX_1^2X_2$	$\begin{smallmatrix} 2,1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} & 2^{\nearrow} & \\ 1^{\nearrow} & & 2^{\nearrow} \end{smallmatrix}$	$u^3X_1^3X_2^3$	$\begin{smallmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{smallmatrix}$

Theorem (F. and Schreier-Aigner, 2021). The generating function of arrowed monotone triangles with bottom row $1, 2, \ldots, n$ is equal to the generating function of set-valued balanced column-strict plane partitions with parts in $\{1, 2, \ldots, n\}$.

But why should we care?

It is an $n+3$-parameter generalization of the ASM-DPP relation!

SBCSPPs $\to$ DPPs

Claim: When setting $u = v = 1, w = -1$ and $(X_1, \ldots, X_n) = (1, \ldots, 1)$ in the generating function of SBCSPPs of order n, we obtain the number of DPPs of order n.

What do we need to do?

• For a given SBCSPP of shape $(a_1, \ldots, a_l | b_1, \ldots, b_l)$, the weight reduces to

$$(-1)^{l+\sum_{i=1}^{l}(b_i - a_i) + (\#\text{ of entries}) - (\#\text{ of cells})} = (-1)^{\#\text{ of entries}}$$

• We define **two sign-reversing involutions** to "cancel" certain subsets.

• The remaining set will be a set of **positive** SBCSPP's that is in **easy bijective correspondence with the set of DPPs.**

The first sign-reversing involution

• A **principal SBCSPP** has singletons in each cell. We can associate a principal SBCSPP to each SBCSPP by just keeping the **maximum** in each cell.

9	8	8	7	7	5	3	2
7	6	5	4	3	3		
6	5	4	3	2			
4	4	3	1				
3	3						
2	1						
1							

$9_{,8}$	8	8	7	7	5	3	2
7	6	5	4	3			
$6_{,5}$	5	4	3	2			
4	4	$3_{,2,1}$	1				
3	$3_{,2}$						
2	1						
1							

• If for a fixed principal SBCSPP with **more than one** SBCSPP associated with it, there is the following sign-reversing involution: Fix the topmost and leftmost cell c that can contain **more than one entry**, and let e be the minimal possible entry for this cell (i.e., $e-1$ is in the cell below). If c contains e remove it, otherwise add it.

In the example: $c=(1,1)$ and $e=8$.

The second sign-reversing involution

• Principal SBCSPPs that have no other SBCSPP associated with it are characterized as follows: for each diagonal entry d, the entries below in the same column are $d-1, d-2, \ldots, 1$.

7	7	7	7	7	5	3	2
6	5	5	4	3	3		
5	4	2	2	2			
4	3	1	1				
3	2						
2	1						
1							

7	7	7	7	7	5	1	1
6	5	5	4	3	3		
5	4	2	2	2			
4	3	1	1				
3	2						
2	1						
1							

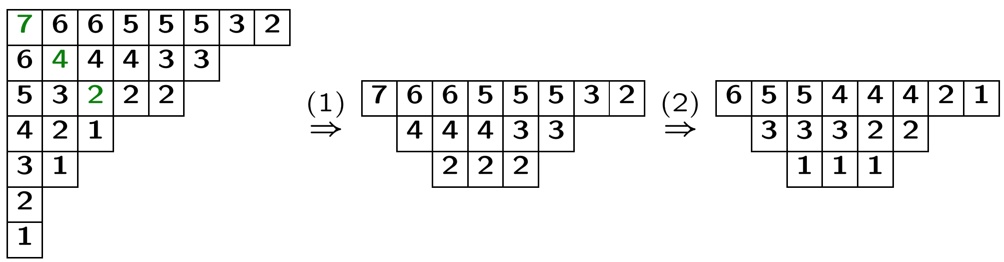
• We define a sign-reversing involution on the subset of the remaining SBCSPPs for which at least one of the following is satisfied: the SBCSPP contains a 1 **strictly above the diagonal** or $a_i \neq b_i + 1$ for an i.

• If $a_i \neq b_i + 1$ for an i, choose the minimal such i. If there is no 1 in row $1, \ldots, i-1$, add a 1 at the end of row i. Otherwise remove the topmost and rightmost 1.

In our examples, we have $a_2 \neq b_2 + 1$. In the left example, we add a 1 to the second row, while in the other example, we delete the last 1 from the first row.

Analyzing the positive remainder

What remains are SBCSPPs such that (1) all cells contain a single element, (2) $a_i = b_i + 1$, (3) weakly below a diagonal entry we have consecutive integers ending with 1, and (4) there are no 1's above the diagonal. All such SBCSPPs have weight 1.



(1) **Remove all cells strictly below the main diagonal** and obtain a column strict shifted plane partition (CSSPP). From $a_i = b_i + 1$, it follows that the **first part of each row is one less than the length.**

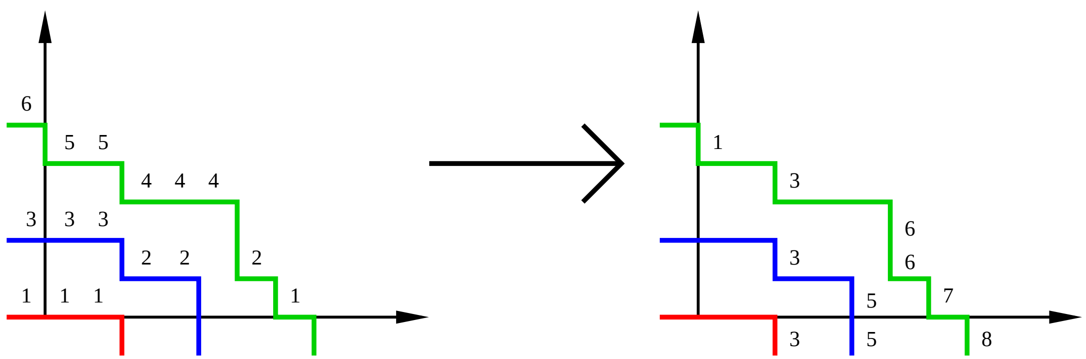
(2) Since there is no 1 in the plane partition, we may **subtract** 1 from each entry and obtain a column strict shifted plane partition such that the **first part of each row is two less than the length of its row.**

(3) By **conjugating the partition in each row**, such CSSPPs with parts no greater than $n-1$ are in easy bijective correspondence with CSSPPs with parts no greater than $n+1$ such that **the first part of each row exceeds its length by precisely** 2**.**

(4) To obtain the corresponding DPP, we **subtract** 1 **from each entry and remove all** 0**s.**

$$\begin{array}{cccccccc} 6 & 5 & 5 & 4 & 4 & 4 & 2 & 1 \\ & 3 & 3 & 3 & 2 & 2 & & \\ & & 1 & 1 & 1 & & & \end{array} \quad \overset{(3)}{\Rightarrow} \quad \begin{array}{cccccc} 8 & 7 & 6 & 6 & 3 & 1 \\ & 5 & 5 & 3 & & \\ & & 3 & & & \end{array} \quad \overset{(4)}{\Rightarrow} \quad \begin{array}{ccccc} 7 & 6 & 5 & 5 & 2 \\ & 4 & 4 & 2 & \\ & & 2 & & \end{array}$$

Lattice paths help in understanding that (3) leads to another CSSPP.



Would a weight-preserving bijection between arrowed monotone triangles and SBCSPPs imply an ASM-DPP bijection?

"Natural" approach: Consider the SBCSPPs that are left after the two sign-reversing involutions (they are equinumerous with DPPs), take the corresponding arrowed monotone triangles in the weight-preserving bijection and delete the arrows to obtain monotone triangles and thus ASMs.

Case $n = 2$.

AMT	W	SBCSPP	AMT	W	SBCSPP
$\begin{array}{ccc} & {}^{\nwarrow}1 & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2 \end{array}$	v^3	$\varnothing$	$\begin{array}{ccc} & {}^{\nwarrow}2 & \\ {}^{\nwarrow}1^{\nearrow} & & 2^{\nearrow} \end{array}$	$uvwX_1X_2^2$	$\begin{array}{lll} 2 & 2 \\ 1 \end{array}$
$\begin{array}{ccc} & {}^{\nwarrow}1^{\nearrow} & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2 \end{array}$	v^2wX_1	$\begin{array}{l} 1 \end{array}$	$\begin{array}{ccc} & {}^{\nwarrow}1^{\nearrow} & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{array}$	$uvwX_1X_2^2$	$\begin{array}{ll} 2,1 & 1 \end{array}$
$\begin{array}{ccc} & 1^{\nearrow} & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2 \end{array}$	$uv^2X_1^2$	$\begin{array}{ll} 1 & 1 \end{array}$	$\begin{array}{ccc} & 2^{\nearrow} & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{array}$	$u^2vX_1^3X_2$	$\begin{array}{lll} 2 & 1 & 1 \\ 1 \end{array}$
$\begin{array}{ccc} & {}^{\nwarrow}1 & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2^{\nearrow} \end{array}$	v^2wX_2	$\begin{array}{l} 2 \end{array}$	$\begin{array}{ccc} & {}^{\nwarrow}2^{\nearrow} & \\ {}^{\nwarrow}1^{\nearrow} & & 2^{\nearrow} \end{array}$	$uw^2X_1^2X_2^2$	$\begin{array}{ll} 2 & 2 \\ 1 & 1 \end{array}$
$\begin{array}{ccc} & {}^{\nwarrow}2 & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{array}$	$uv^2X_1X_2$	$\begin{array}{ll} 2 & 1 \end{array}$	$\begin{array}{ccc} & 1^{\nearrow} & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{array}$	$u^2vX_1^2X_2^2$	$\begin{array}{lll} 2 & 2 & 1 \\ 1 \end{array}$
$\begin{array}{ccc} & {}^{\nwarrow}1^{\nearrow} & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2^{\nearrow} \end{array}$	$vw^2X_1X_2$	$\begin{array}{l} 2,1 \end{array}$	$\begin{array}{ccc} & 2^{\nearrow} & \\ {}^{\nwarrow}1^{\nearrow} & & 2^{\nearrow} \end{array}$	$u^2wX_1^3X_2^2$	$\begin{array}{lll} 2 & 2 & 1 \\ 1 & 1 \end{array}$
$\begin{array}{ccc} & {}^{\nwarrow}1 & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{array}$	$uv^2X_2^2$	$\begin{array}{ll} 2 & 2 \end{array}$	$\begin{array}{ccc} & {}^{\nwarrow}2 & \\ 1^{\nearrow} & & 2^{\nearrow} \end{array}$	$u^2vX_1X_2^3$	$\begin{array}{lll} 2 & 2 & 2 \\ 1 \end{array}$
$\begin{array}{ccc} & {}^{\nwarrow}2^{\nearrow} & \\ {}^{\nwarrow}1 & & 2^{\nearrow} \end{array}$	$uvwX_1^2X_2$	$\begin{array}{ll} 2 & 1 \\ 1 \end{array}$	$\begin{array}{ccc} & {}^{\nwarrow}2^{\nearrow} & \\ 1^{\nearrow} & & 2^{\nearrow} \end{array}$	$u^2wX_1^2X_2^3$	$\begin{array}{lll} 2 & 2 & 2 \\ 1 & 1 \end{array}$
$\begin{array}{ccc} & 1^{\nearrow} & \\ {}^{\nwarrow}1 & & {}^{\nwarrow}2^{\nearrow} \end{array}$	$uvwX_1^2X_2$	$\begin{array}{ll} 2,1 & 1 \end{array}$	$\begin{array}{ccc} & 2^{\nearrow} & \\ 1^{\nearrow} & & 2^{\nearrow} \end{array}$	$u^3X_1^3X_2^3$	$\begin{array}{lll} 2 & 2 & 2 \\ 1 & 1 & 1 \end{array}$

The two SBCSPPs that remain after applying the two sign-reversing involutions are $\varnothing$ and $\begin{array}{lll} 2 & 2 & 2 \\ 1 \end{array}$. If we ignore the arrows in the corresponding arrowed monotone triangles, we obtain the bijection to monotone triangles.

Already for $n = 3$, this can't work!

For $n = 3$, the following 7 SBCSPP are left after applying the two sign-reversing involutions (we also provide the weights):

$$
(\varnothing, v^6),
\left(\begin{array}{|c|c|c|}\hline 2&2&2\\ \hline 1 \\ \cline{1-1}\end{array}, u^2v^4X_1X_2^3\right),
\left(\begin{array}{|c|c|c|c|}\hline 3&2&2&2\\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}, u^3v^3X_1X_2^4X_3\right),
\left(\begin{array}{|c|c|c|c|}\hline 3&3&2&2\\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}, u^3v^3X_1X_2^3X_3^2\right),
$$

$$
\left(\begin{array}{|c|c|c|c|}\hline 3&3&3&2\\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}, u^3v^3X_1X_2^2X_3^3\right),
\left(\begin{array}{|c|c|c|c|}\hline 3&3&3&3\\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}, u^3v^3X_1X_2X_3^4\right),
\left(\begin{array}{|c|c|c|c|}\hline 3&3&3&3\\ \hline 2&2&2&2\\ \hline 1&1 \\ \cline{1-2}\end{array}, u^5vX_1^2X_2^4X_3^4\right).
$$

Crucial observation 1: Only for one of these SBCSPPs, the exponent of X_1 in the weight is greater than 1.

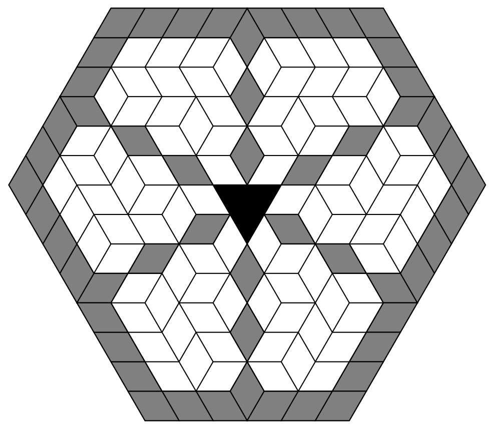
1 ↖1 *2* ↖1 ↖2 *3*	$v^3X_1X_2X_3(uX_1+vX_1^{-1}+w)(uX_2+vX_2^{-1}+w)(uX_3+vX_3^{-1}+w)$
1 ↖1 *3* ↖1 *2* 3↗	$uv^2X_1X_2^2X_3^2(uX_1+vX_1^{-1}+w)(uX_2+vX_2^{-1}+w)(uX_3+vX_3^{-1}+w)$
2 *1* 2↗ ↖1 ↖2 *3*	$uv^2X_1^2X_2^2X_3(uX_1+vX_1^{-1}+w)(uX_2+vX_2^{-1}+w)(uX_3+vX_3^{-1}+w)$
2 *1* *3* ↖1 *2* 3↗	$uvX_1^2X_2^2X_3^2(uX_1+vX_1^{-1}+w)(uX_2+vX_2^{-1}+w)^2(uX_3+vX_3^{-1}+w)$
2 ↖2 *3* *1* 2↗ 3↗	$u^2vX_1^2X_2^2X_3^3(uX_1+vX_1^{-1}+w)(uX_2+vX_2^{-1}+w)(uX_3+vX_3^{-1}+w)$
3 *1* 3↗ ↖1 *2* 3↗	$u^2vX_1^3X_2^2X_3^2(uX_1+vX_1^{-1}+w)(uX_2+vX_2^{-1}+w)(uX_3+vX_3^{-1}+w)$
3 *2* 3↗ *1* 2↗ 3↗	$u^3X_1^3X_2^3X_3^3(uX_1+vX_1^{-1}+w)(uX_2+vX_2^{-1}+w)(uX_3+vX_3^{-1}+w)$

Crucial observation 2: For two (out of the 7) monotone triangles, the exponents of X_1 in the weight of the associated arrowed monotone triangles is at least 2 (namely for those that have a 3 at the top).

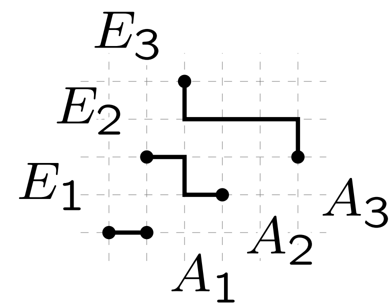
Therefore, “forgetting arrows” can’t work!

Vertical symmetric ASMs and lozenge tilings

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \Leftrightarrow$$



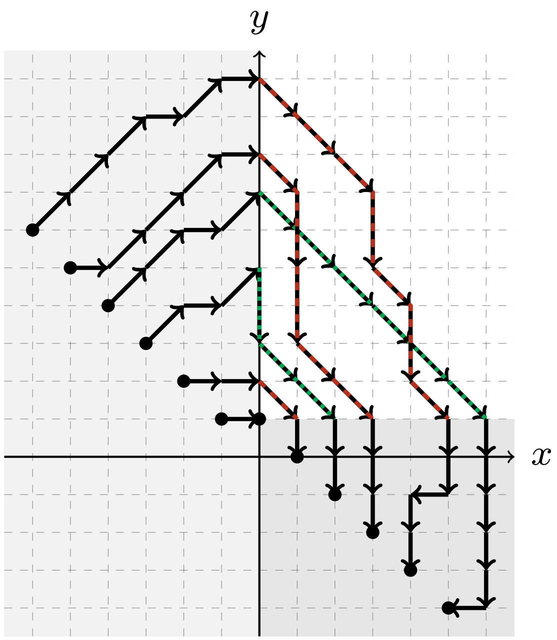
$\longleftrightarrow$



Perfect analogy: There is the same number of vertically symmetric ASMs as there is of cyclically symmetric lozenge tilings with a hole of size 2 that exhibit also a **vertical symmetry.**

Now: lozenge tilings correspond to families of non-intersecting lattice paths.

Theorem (F. and Höngesberg, 2022). For $n \geq 1$, the generating function of arrowed monotone triangles with bottom row $0, 2, \ldots, 2n-2$ is equal to the signed generating function of certain n lattice paths.



When $X_i = 1, u = v = 1$ and $w = -1$, we have constructed a sign-reversing involution and a bijection that takes us from the families of lattice paths from the theorem to the families of non-intersecting lattice paths that correspond to lozenge tilings.

V. Schur function expansion, TSPPs and a Cauchy-type identity

Formula for the generating function

The generating function of arrowed monotone triangles with bottom row $1, 2, \ldots, n$ and of SBCSPPs of order n is

$$\prod_{i=1}^{n} X_i^n \frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1 \le p \le q \le n} \left(uX_q + vX_p^{-1} + w\right)\right]}{\prod_{1 \le i < j \le n}(X_j - X_i)} = \prod_{i=1}^{n} X_i^n \frac{\det_{1 \le i,j \le n}\left((uX_i + w)^j - (-vX_i^{-1})^j\right)}{\prod_{1 \le i < j \le n}(X_j - X_i)}.$$

This is obviously a symmetric function in $X_1, X_2, \ldots, X_n$.

Schur polynomial expansion of $\prod_{i=1}^{n} X_i^{n-1} \frac{\mathbf{ASym}_{X_1,\ldots,X_n}\left[\prod_{1 \le p < q \le n} \left(uX_q + vX_p^{-1} + w\right)\right]}{\prod_{1 \le i < j \le n}(X_j - X_i)}$**?**

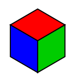
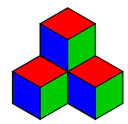
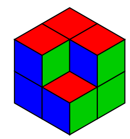
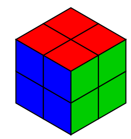
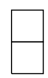
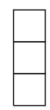
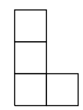
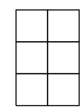
For $n = 3$:

$$v^3 + uv^2 s_{(1,1)}(x_1, x_2, x_3) + uvw s_{(1,1,1)}(x_1, x_2, x_3) + u^2 v s_{(2,1,1)}(x_1, x_2, x_3) + u^3 s_{(2,2,2)}(x_1, x_2, x_3)$$

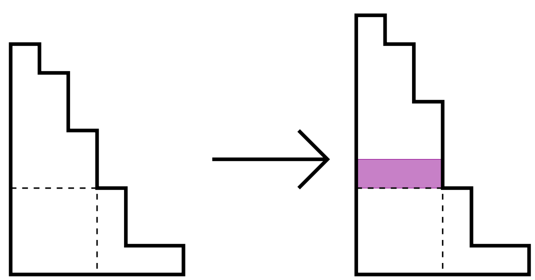
(BTW, this is the generating function of a natural variation of arrowed monotone triangles, which we call **down-arrowed monotone triangles**.)

The case $n = 3$

$$v^3 + uv^2 s_{(1,1)}(x_1, x_2, x_3) + uvw s_{(1,1,1)}(x_1, x_2, x_3) + u^2 v s_{(2,1,1)}(x_1, x_2, x_3) + u^3 s_{(2,2,2)}(x_1, x_2, x_3)$$

T:	$\varnothing$				
$\pi(T)$:	$\varnothing$				
$\omega_{\pi(T)}(u, v))$:	v^3	uv^2	uvw	$u^2 v$	u^3

Here we see all totally symmetric plane partitions in a $2 \times 2 \times 2$ box, its slightly modified "profile" along the diagonal $y = x$ together with a certain weight.

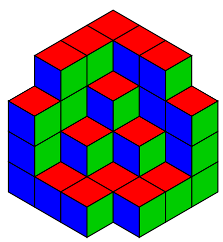


Thin partitions

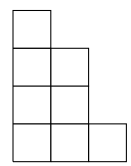
Definition. A partition in Frobenius notation $(a_1,\dots,a_l|b_1,\dots,b_l)$ is said to be thin (a.k.a. modified balanced) if $a_i < b_i$.

Thin partitions whose parts do not exceed $n-1$ are counted by the n-th Catalan number. That is why we see $C_3 = 5$ Schur polynomials in the expansion above.

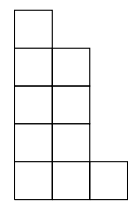
We consider **totally symmetric plane partitions.**



T



diag(T)



$\pi(T)$

Suppose diag$(T) = (a_1,\dots,a_l|b_1,\dots,b_l)$ is the diagonal profil of the totally symmetry plane partition T in Frobenius notation, then it is not terribly hard to see that $\pi(T) = (a_1,\dots,a_l|b_1+1,\dots,b_l+1)$ is a thin partition.

The weight of a thin partition $\lambda = (a_1,\dots,a_l|b_1,\dots,b_l)$ of order n is defined to be

$$\omega_\lambda(u,v) = u^{\sum_{i=1}^l (a_i+1)} v^{\binom{n}{2}-\sum_{i=1}^l b_i} w^{\sum_{i=1}^l (b_i-a_i)}.$$

Theorem (F. Aigner, I. Fischer, M. Konvalinka, P. Nadeau, V. Tewari, FPSAC 2020). The generating function of down-arrowed monotone triangles of order n has the following Schur polynomial expansion.

$$\sum_{T \in \mathsf{TSPP}_{n-1}} \omega_{\pi(T)}(u,v) \cdot s_{\pi(T)}(X_1, \ldots, X_n)$$

Conjectured by **F. Aigner** and **F. Bergeron.**

The vertical symmetric case

Theorem (F. and Höngesberg, 2022) For $n \geq 1$, the generating function of arrowed monotone triangles with bottom row $0, 2, \ldots, 2n-2$ is equal to the generating function of pairs (P, Q) of plane partitions of the same shape with n rows (allowing also rows of length zero) such that

- P is a columnstrict plane partitions such that the entries of P in the i-th row from the bottom are no greater than $2i$,
- Q is a rowstrict plane partition such that the entries of Q in the i-th row from the bottom are no greater than i,

and the weight of such a pair is given by the following monomial.

$$w^{\binom{n+1}{2} - \#\text{of entries in } Q} \prod_{i=1}^{n} X_i^{n-1} (u X_i)^{\#\text{of } 2i-1 \text{ in P}} (v X_i^{-1})^{\#\text{of } 2i \text{ in P}}$$

• The P's are in bijective correspondence with symplectic tableaux and the Q's are in bijective correspondence with TSSCPPs. In this form it was independently conjectured by **F. Schreier-Aigner.**

• Bijective proof? Cauchy-type identity $\longrightarrow$ RSK?

The case $n=2$

$(\varnothing,\varnothing)$: $w^3X_1X_2$ $(\ytableaushort{1},\ytableaushort{1})$: $w^2uX_1^2X_2$ $(\ytableaushort{2},\ytableaushort{1})$: w^2vX_2 $(\ytableaushort{3},\ytableaushort{1})$: $w^2uX_1X_2^2$ $(\ytableaushort{4},\ytableaushort{1})$: w^2vX_1 $(\ytableaushort{1},\ytableaushort{2})$: $w^2uX_1^2X_2$ $(\ytableaushort{2},\ytableaushort{2})$: w^2vX_2 $(\ytableaushort{3},\ytableaushort{2})$: $w^2uX_1X_2^2$ $(\ytableaushort{4},\ytableaushort{2})$: w^2vX_1 $(\ytableaushort{11},\ytableaushort{21})$: $wu^2X_1^3X_2$

$(\ytableaushort{21},\ytableaushort{21})$: $wuvX_1X_2$ $(\ytableaushort{31},\ytableaushort{21})$: $wu^2X_1^2X_2^2$ $(\ytableaushort{41},\ytableaushort{21})$: $wuvX_1^2$ $(\ytableaushort{22},\ytableaushort{21})$: $wv^2X_1^{-1}X_2$ $(\ytableaushort{32},\ytableaushort{21})$: $wuvX_2^2$ $(\ytableaushort{42},\ytableaushort{21})$: wv^2 $(\ytableaushort{33},\ytableaushort{21})$: $wu^2X_1X_2^3$

$(\ytableaushort{43},\ytableaushort{21})$: $wuvX_1X_2$ $(\ytableaushort{44},\ytableaushort{21})$: $wv^2X_1X_2^{-1}$ $\left(\ytableaushort{2,1},\ytableaushort{1,1}\right)$: $wuvX_1X_2$ $\left(\ytableaushort{3,1},\ytableaushort{1,1}\right)$: $wu^2X_1^2X_2^2$ $\left(\ytableaushort{4,1},\ytableaushort{1,1}\right)$: $wuvX_1^2$ $\left(\ytableaushort{3,2},\ytableaushort{1,1}\right)$: $wuvX_2^2$ $\left(\ytableaushort{4,2},\ytableaushort{1,1}\right)$: wv^2 $\left(\ytableaushort{2,1},\ytableaushort{2,1}\right)$: $wuvX_1X_2$ $\left(\ytableaushort{3,1},\ytableaushort{2,1}\right)$: $wu^2X_1^2X_2^2$

$\left(\ytableaushort{4,1},\ytableaushort{2,1}\right)$: $wuvX_1^2$ $\left(\ytableaushort{3,2},\ytableaushort{2,1}\right)$: $wuvX_2^2$ $\left(\ytableaushort{4,2},\ytableaushort{2,1}\right)$: wv^2 $\left(\ytableaushort{21,1},\ytableaushort{21,1}\right)$: $u^2vX_1^2X_2$ $\left(\ytableaushort{22,1},\ytableaushort{21,1}\right)$: uv^2X_2 $\left(\ytableaushort{31,1},\ytableaushort{21,1}\right)$: $u^3X_1^3X_2^2$ $\left(\ytableaushort{32,1},\ytableaushort{21,1}\right)$: $u^2vX_1X_2^2$ $\left(\ytableaushort{33,1},\ytableaushort{21,1}\right)$: $u^3X_1^2X_2^3$

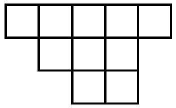
$\left(\ytableaushort{31,2},\ytableaushort{21,1}\right)$: $u^2vX_1X_2^2$ $\left(\ytableaushort{32,2},\ytableaushort{21,1}\right)$: $uv^2X_1^{-1}X_2^2$ $\left(\ytableaushort{33,2},\ytableaushort{21,1}\right)$: $u^2vX_2^3$ $\left(\ytableaushort{41,1},\ytableaushort{21,1}\right)$: $u^2vX_1^3$ $\left(\ytableaushort{42,1},\ytableaushort{21,1}\right)$: uv^2X_1 $\left(\ytableaushort{43,1},\ytableaushort{21,1}\right)$: $u^2vX_1^2X_2$ $\left(\ytableaushort{44,1},\ytableaushort{21,1}\right)$: $uv^2X_1^2X_2^{-1}$

$\left(\ytableaushort{41,2},\ytableaushort{21,1}\right)$: uv^2X_1 $\left(\ytableaushort{42,2},\ytableaushort{21,1}\right)$: $v^3X_1^{-1}$ $\left(\ytableaushort{43,2},\ytableaushort{21,1}\right)$: uv^2X_2 $\left(\ytableaushort{44,2},\ytableaushort{21,1}\right)$: $v^3X_2^{-1}$

VI. A complicated bijection for the ASM-DPP relation

Descending Plane Partitions = DPPs

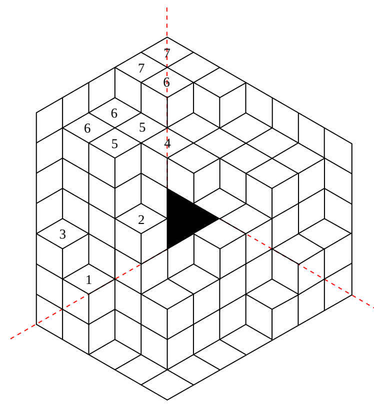
• A strict partition is a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ with distinct parts, i.e., $\lambda_1 > \lambda_2 > \ldots > \lambda_l > 0$. The shifted Young diagram of shape $(5,3,2)$ is as follows.



• A column strict shifted plane partition is a filling of a shifted Young diagram with positive integers such that rows decrease weakly and columns decrease strictly.

$$\begin{array}{|c|c|c|c|c|} \hline 6 & 6 & 5 & 5 & 2 \\ \hline \multicolumn{1}{c|}{} & 5 & 4 & 4 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & 1 & \multicolumn{1}{c}{} \\ \cline{3-4} \end{array}$$

• A DPP is such a column strict PP where the first part in each row is greater than the length of its row and less than or equal to the length of the previous row. Ugly condition?



• DPPs with parts no greater than 3: $\varnothing$, 2 , 3 , $3\ \ 1$, $3\ \ 2$, $3\ \ 3$, $\begin{array}{cc} 3 & 3 \\ & 2 \end{array}$

• The number of DPPs with parts no greater than n is also $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$ (Andrews).

Bijection 1 (Bijective Proof of the Product Formula)

ASM_n = set of $n \times n$ ASMs

$\mathsf{ASM}_{n,i}$ = set of $n \times n$ ASMs $(a_{p,q})_{1 \le p,q \le n}$ with $a_{1,i} = 1$

B_n = set of $(2n-1)$-subsets of $[3n-2] = \{1, 2, \ldots, 3n-2\}$; $|\mathsf{B}_n| = \binom{3n-2}{2n-1}$

$\mathsf{B}_{n,i}$ = set of elements of B_n whose median is $n+i-1$; $|\mathsf{B}_{n,i}| = \binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}$

DPP_n = set of DPPs with parts no greater than n

We have constructed a bijection between the following sets:

$$\mathsf{DPP}_{n-1} \times \mathsf{B}_{n,1} \times \mathsf{ASM}_{n,i} \longrightarrow \mathsf{DPP}_{n-1} \times \mathsf{ASM}_{n-1} \times \mathsf{B}_{n,i}$$

Then we also have a bijection

$$\mathsf{DPP}_{n-1} \times \mathsf{B}_{n,1} \times \mathsf{ASM}_n \longrightarrow \mathsf{DPP}_{n-1} \times \mathsf{ASM}_{n-1} \times \mathsf{B}_n \,.$$

Iterating this, we obtain a bijection

$$\mathsf{DPP}_0 \times \cdots \times \mathsf{DPP}_{n-1} \times \mathsf{B}_{1,1} \times \cdots \times \mathsf{B}_{n,1} \times \mathsf{ASM}_n \longrightarrow \mathsf{DPP}_0 \times \cdots \times \mathsf{DPP}_{n-1} \times \mathsf{B}_1 \times \cdots \times \mathsf{B}_n \,.$$

Example: $\mathrm{DPP}_2 \times \mathrm{B}_{3,1} \times \mathrm{ASM}_{3,2} \longrightarrow \mathrm{DPP}_2 \times \mathrm{ASM}_2 \times \mathrm{B}_{3,2}$ **for** $x = 0$

$$
\begin{array}{lcl@{\qquad}lcl@{\qquad}lcl}
\left(\varnothing, 12345, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23457) &
\left(\varnothing, 12345, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23456) &
\left(\varnothing, 12345, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23456) \\
\left(\varnothing, 12346, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13457) &
\left(\varnothing, 12346, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13456) &
\left(\varnothing, 12346, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13456) \\
\left(\varnothing, 12347, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12457) &
\left(\varnothing, 12347, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12456) &
\left(\varnothing, 12347, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12456) \\
\left(\varnothing, 12356, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13456) &
\left(\varnothing, 12356, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12456) &
\left(\varnothing, 12356, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12456) \\
\left(\varnothing, 12357, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13457) &
\left(\varnothing, 12357, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12457) &
\left(\varnothing, 12357, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12457) \\
\left(\varnothing, 12367, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13467) &
\left(\varnothing, 12367, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12467) &
\left(\varnothing, 12367, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12467) \\
\left(2, 12345, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23467) &
\left(2, 12345, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23467) &
\left(2, 12345, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23457) \\
\left(2, 12346, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 13467) &
\left(2, 12346, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13467) &
\left(2, 12346, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13457) \\
\left(2, 12347, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 12467) &
\left(2, 12347, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12467) &
\left(2, 12347, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (\varnothing, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 12457) \\
\left(2, 12356, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23456) &
\left(2, 12356, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23456) &
\left(2, 12356, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13456) \\
\left(2, 12357, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23457) &
\left(2, 12357, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23457) &
\left(2, 12357, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13457) \\
\left(2, 12367, \begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}1&0\\0&1\end{smallmatrix}, 23467) &
\left(2, 12367, \begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 23467) &
\left(2, 12367, \begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix}\right) & \leftrightarrow & (2, \begin{smallmatrix}0&1\\1&0\end{smallmatrix}, 13467)
\end{array}
$$

The python code is available at `https://www.fmf.uni-lj.si/~konvalinka/asmcode.html`.

Bijection 2 (ASMs and DPPs)

$\mathrm{DPP}_{n,i}$ = subset of DPP_n with DPPs that have $i-1$ occurrences of n.

We have constructed a bijection between the following sets:

$$\mathrm{DPP}_{n-1} \times \mathrm{ASM}_{n,i} \longrightarrow \mathrm{ASM}_{n-1} \times \mathrm{DPP}_{n,i}$$

- Once such a bijection is constructed, it follows that

$$|\mathrm{DPP}_{n-1}| \cdot |\mathrm{ASM}_{n,i}| = |\mathrm{ASM}_{n-1}| \cdot |\mathrm{DPP}_{n,i}|.$$

- By induction, we can assume $|\mathrm{DPP}_{n-1}| = |\mathrm{ASM}_{n-1}|$ and so $|\mathrm{ASM}_{n,i}| = |\mathrm{DPP}_{n,i}|$.
- Summing this over all i implies $|\mathrm{DPP}_n| = |\mathrm{ASM}_n|$.

Example $\mathrm{DPP}_3 \times \mathrm{ASM}_{4,2} \longrightarrow \mathrm{ASM}_3 \times \mathrm{DPP}_{4,2}$ for $x = 0$

$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ 4\ 2\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix},\ 4\ 1\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix},\ 4\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ 4\ 2\right)$
$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ 4\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ 4\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ 4\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ 4\ 1\ 1\right)$
$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ 4\ 1\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ 4\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ 4\ 2\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ 4\right)$
$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix},\ 4\ 1\right)$	$\left(\varnothing,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ 4\ 1\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ 4\ 3\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix},\ 4\ 3\right)$
$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix},\ 4\ 2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ 4\ 3\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix},\ 4\ 3\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix},\ 4\ 3\right)$
$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix},\ 4\ 2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix},\ 4\ 2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ 4\ 3\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ 4\ 3\right)$
$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ 4\ 2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ 4\ 2\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ 4\ 1\right)$	$\left(2,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ 4\ 1\right)$
$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ 4\ 2\ 1\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ 4\ 2\ 2\right)$
$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$
$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix},\ \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$
$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(3\ 3,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix},\ \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix},\ \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}4&3\\&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}4&3&1\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}3&3\\&2\end{smallmatrix},\begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ \begin{smallmatrix}4&3&2\\&2&\end{smallmatrix}\right)$
$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ 4\ 3\ 3\right)$	$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix},\ 4\ 3\ 3\right)$	$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix},\ 4\right)$	$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ 4\ 3\ 3\right)$
$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&-1&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&-1&1\\0&1&0\end{smallmatrix},\ 4\ 3\ 3\right)$	$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&1&0&0\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix},\ 4\ 3\ 3\right)$	$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&0&-1&1\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&0&1\\0&1&0\end{smallmatrix},\ 4\ 2\ 2\right)$	$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix},\ 4\ 1\ 1\right)$
$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ 4\ 3\ 3\right)$	$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ 4\ 3\ 3\right)$	$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix},\ 4\ 1\ 1\right)$	$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix},\ \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$
$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix},\ 4\ 1\right)$	$\left(3\ 2,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix},\ \begin{smallmatrix}4&3&3\\&2&\end{smallmatrix}\right)$	$\left(3\ 1,\begin{smallmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix},\ 4\ 3\ 2\right)$	$\left(3\ 1,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&1&0&0\\0&0&0&1\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix},\ 4\ 3\ 2\right)$
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$\left(3\ 1,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ 4\ 2\ 2\right)$	$\left(3\ 1,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ 4\ 2\ 2\right)$	$\left(3\ 1,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\1&0&0\\0&0&1\end{smallmatrix},\ 4\right)$	$\left(3\ 1,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\1&0&0\\0&1&0\end{smallmatrix},\ 4\right)$
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$\left(3,\begin{smallmatrix}0&1&0&0\\1&-1&1&0\\0&0&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ 4\ 3\ 1\right)$	$\left(3,\begin{smallmatrix}0&1&0&0\\1&-1&0&1\\0&0&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ 4\ 3\ 1\right)$	$\left(3,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\1&-1&0&1\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ 4\ 2\ 1\right)$	$\left(3,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\1&-1&1&0\\0&1&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ 4\ 2\ 1\right)$
$\left(3,\begin{smallmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&1&0\\0&0&1\\1&0&0\end{smallmatrix},\ 4\ 1\ 1\right)$	$\left(3,\begin{smallmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{smallmatrix}\right)\leftrightarrow\left(\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix},\ 4\ 1\ 1\right)$		

A short introduction to signed sets

A signed set is a pair of disjoint finite sets: $\underline{S} = (S^+, S^-)$ with $S^+ \cap S^- = \varnothing$.

- The size of a signed set $\underline{S}$ is $|\underline{S}| = |S^+| - |S^-|$.
- The opposite signed set of $\underline{S}$ is $-\underline{S} = (S^-, S^+)$.
- The Cartesian product of signed sets $\underline{S}$ and $\underline{T}$ is

$$\underline{S} \times \underline{T} = (S^+ \times T^+ \cup S^- \times T^-, S^+ \times T^- \cup S^- \times T^+).$$

- The disjoint union of signed sets $\underline{S}$ and $\underline{T}$ is

$$\underline{S} \sqcup \underline{T} = (\underline{S} \times (\{0\}, \varnothing)) \cup (\underline{T} \times (\{1\}, \varnothing)).$$

- The disjoint union of a family of signed sets $\underline{S}_t$ indexed with a signed set $\underline{T}$ is

$$\bigsqcup_{t \in \underline{T}} \underline{S}_t = \bigcup_{t \in \underline{T}} (\underline{S}_t \times \underline{\{t\}}).$$

Our approach

- We translate some of my non-bijective proofs into combinatorics!

- Note that $|\underline{S} \sqcup \underline{T}| = |\underline{S}| + |\underline{T}|$, $|-\underline{S}| = -|\underline{S}|$, and $|\underline{S} \times \underline{T}| = |\underline{S}| \cdot |\underline{T}|$, and so we can deal with all arithmetic operations accept for division. (The latter explains the “redundant” factors in our bijections.)

- In the original proofs, signs are unavoidable and this makes it necessary to work with signed sets.

- Is there a non-bijective proof that avoids signs? Is there a bijective proof that avoids signed sets (and can this proof be translated into a computation)?

Crucial example: Signed intervals

For $a, b \in \mathbb{Z}$, we set

$$\underline{[a,b]} = \begin{cases} ([a,b], \varnothing) & \text{if } a \le b \\ (\varnothing, [b+1, a-1]) & \text{if } a > b \end{cases},$$

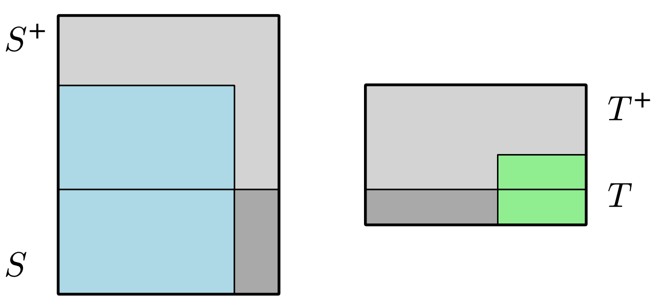
where $[a,b]$ stands for an interval in $\mathbb{Z}$ in the usual sense.

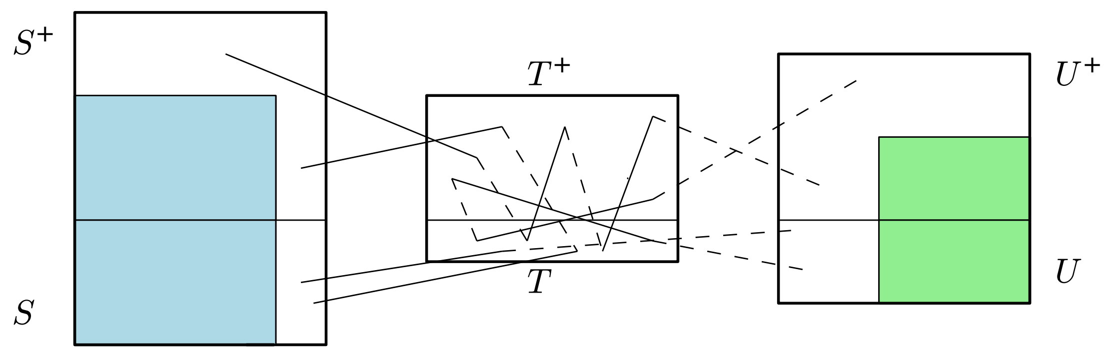
The signed sets in our constructions are typically signed boxes (= Cartesian products of signed intervals) and disjoint unions of signed boxes.

Sijections

The role of bijections for signed sets is played by “signed bijections”, which we call sijections.

A sijection φ from $\underline{S}$ to $\underline{T}$, $\varphi\colon \underline{S} \Rightarrow \underline{T}$, is an involution on the set $(S^+ \cup S^-) \sqcup (T^+ \cup T^-)$ with $\varphi(S^+ \sqcup T^-) = S^- \sqcup T^+$.





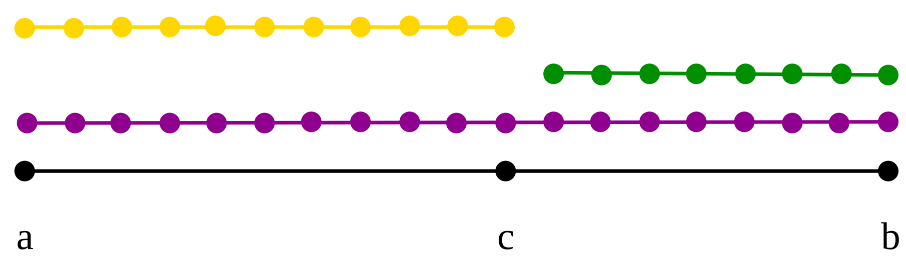
This implies: $|\underline{S}| = |S^+| - |S^-| = |T^+| - |T^-| = |\underline{T}|$

The fundamental sijection

Given $a, b, c \in \mathbb{Z}$, construct a sijection

$$\alpha = \alpha_{a,b,c}\colon \underline{[a,c]} \Rightarrow \underline{[a,b]} \sqcup \underline{[b+1,c]}.$$

Construction: For $a \le b \le c$ and $c < b < a$, there is nothing to prove. For, say, $a \le c < b$, we have that $\underline{[b+1,c]} = -\underline{[c+1,b]}$ is "contained" in $\underline{[a,b]}$, but due to its opposite sign this subset "cancels" and what remains is $\underline{[a,c]}$.



The cases $b < a \le c$, $b \le c < a$, and $c < a \le b$ are analogous.

Cartesian product and disjoint union of sijections

- $\underline{S}_1 \times \cdots \times \underline{S}_k \Rightarrow \underline{T}_1 \times \cdots \times \underline{T}_k$: Suppose we have sijections $\varphi_i : \underline{S}_i \Rightarrow \underline{T}_i$, $i = 1, \ldots, k$. Then define $\varphi = \varphi_1 \times \cdots \times \varphi_k$ by

$$\varphi(s_1, \ldots, s_k) = \begin{cases} (\varphi_1(s_1), \ldots, \varphi_k(s_k)) & \text{if } \varphi_i(s_i) \in \underline{T}_i \text{ for } i = 1, \ldots, k \\ (s_1, \ldots, s_{j-1}, \varphi_j(s_j), s_{j+1}, \ldots, s_k) & \text{if } \varphi_j(s_j) \in \underline{S}_j, \varphi_i(s_i) \in \underline{T}_i \text{ for } i < j \end{cases}$$

if $(s_1, \ldots, s_k) \in \underline{S}_1 \times \cdots \times \underline{S}_k$ and

$$\varphi(t_1, \ldots, t_k) = \begin{cases} (\varphi_1(t_1), \ldots, \varphi_k(t_k)) & \text{if } \varphi_i(t_i) \in \underline{S}_i \text{ for } i = 1, \ldots, k \\ (t_1, \ldots, t_{j-1}, \varphi_j(t_j), t_{j+1}, \ldots, t_k) & \text{if } \varphi_j(t_j) \in \underline{T}_j, \varphi_i(t_i) \in \underline{S}_i \text{ for } i < j \end{cases}$$

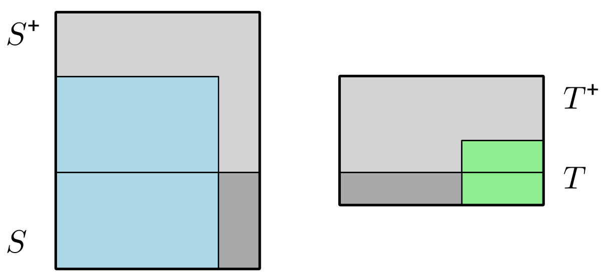
if $(t_1, \ldots, t_k) \in \underline{T}_1 \times \cdots \times \underline{T}_k$.

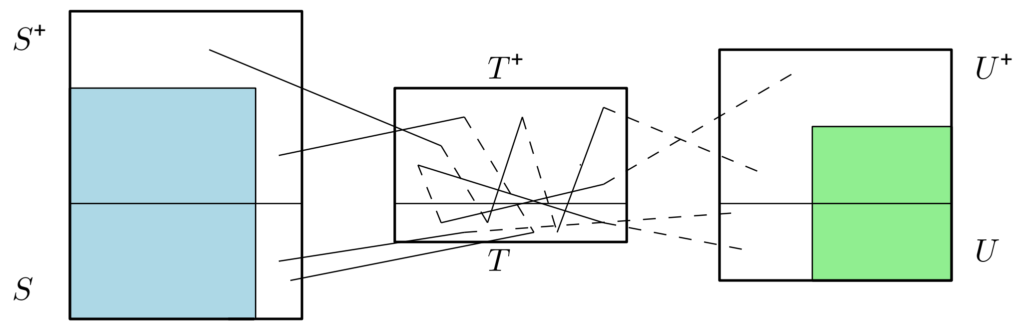
- $\sqcup_{t \in \underline{T}} \underline{S}_t \Rightarrow \sqcup_{t \in \underline{\widetilde{T}}} \underline{S}_t$: Suppose we have signed sets $\underline{T}, \underline{\widetilde{T}}$ and a sijection $\psi : \underline{T} \Rightarrow \underline{\widetilde{T}}$. Furthermore, suppose that for every $t \in \underline{T} \sqcup \underline{\widetilde{T}}$, we have a signed set $\underline{S}_t$ and a sijection $\varphi_t : \underline{S}_t \Rightarrow \underline{S}_{\psi(t)}$ satisfying $\varphi_{\psi(t)} = \varphi_t^{-1}$. Then define $\varphi = \sqcup_{t \in \underline{T} \sqcup \underline{\widetilde{T}}} \varphi_t$ by

$$\varphi(s_t, t) = \begin{cases} (\varphi_t(s_t), t) & \text{if } s_t \in \underline{S}_t, \varphi_t(s_t) \in \underline{S}_t \\ (\varphi_t(s_t), \psi(t)) & \text{if } s_t \in \underline{S}_t, \varphi_t(s_t) \in \underline{S}_{\psi(t)} \end{cases}.$$

Composition of sijections

Suppose $\underline{S}, \underline{T}, \underline{U}$ are signed sets and $\varphi : \underline{S} \Rightarrow \underline{T}$, $\psi : \underline{T} \Rightarrow \underline{U}$, then we can construct a sijection $\psi \circ \varphi : \underline{S} \to \underline{U}$ by alternating applications of φ (solid lines) and ψ (dashed lines) as sketched next.





The special case $S^- = U^- = \varnothing$ is the **Garsia-Milne involution principle.**

VII. DASASMs and the six-vertex model approach

Symmetry classes of ASMs

- **Vertically symmetric ASMs:** $a_{i,j} = a_{i,n+1-j}$
  n odd: Kuperberg (2002)

- **Half-turn symmetric ASMs:** $a_{i,j} = a_{n+1-i,n+1-j}$
  n even: Kuperberg (2002)
  n odd: Razumov/Stroganov (2005)

- **Diagonally symmetric ASMs:** $a_{i,j} = a_{j,i}$
  no formula ?

- **Quarter-turn symmetric ASMs:** $a_{i,j} = a_{j,n+1-i}$
  n even: Kuperberg (2002)
  n odd: Razumov/Stroganov (2005)

Symmetry classes of ASMs (Part 2)

- **Horizontally and vertically symmetric ASMs:** $a_{i,j} = a_{i,n+1-j} = a_{n+1-i,j}$
  n odd: Okada (2004)

- **Diagonally and antidiagonally symmetric ASMs:** $a_{i,j} = a_{j,i} = a_{n+1-j,n+1-i}$
  n odd: **Conjecture** by Robbins (1980s)

- **All symmetries:** $a_{i,j} = a_{j,i} = a_{i,n+1-j}$
  no formula ?

Half of the cases were dealt with in a famous Annals paper by Kuperberg (2002):

Symmetry classes of alternating sign matrices under one roof

Diagonally and antidiagonally symmetric ASMs=DASASMs

Example:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

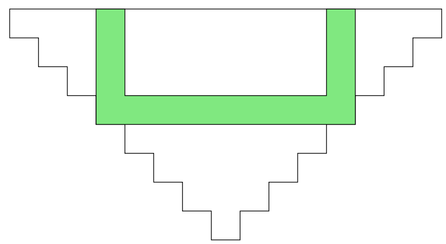
$\mathrm{d}(n)$ = number of $n \times n$ DASASMs

Conjecture (Robbins, 1980s): $\mathrm{d}(2n+1) = \prod_{i=1}^{n} \frac{\binom{3i}{i}}{\binom{2i-1}{i}}$

Sequence starts as follows: $1, 3, 15, 126, 1782, 42471, 1706562 \ldots$

DASASM-triangles

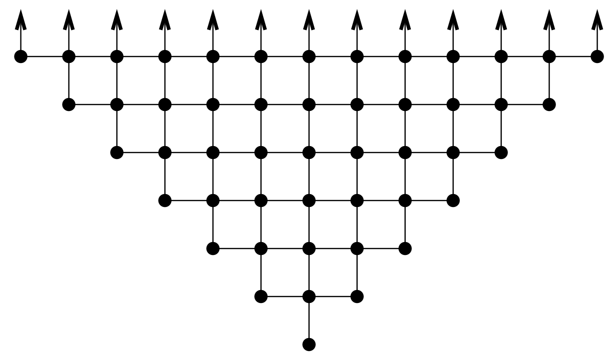
- DASASM $\Rightarrow$ fundamental triangle (DASASM-triangle)



$$\begin{array}{ccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & & \\
 & & & 0 & 1 & -1 & 0 & 1 & 0 & -1 & & & \\
 & & & & -1 & 0 & 1 & -1 & 0 & & & & \\
 & & & & & 1 & 0 & 0 & & & & & \\
 & & & & & & -1 & & & & & &
\end{array}$$

Translation into six-vertex model:

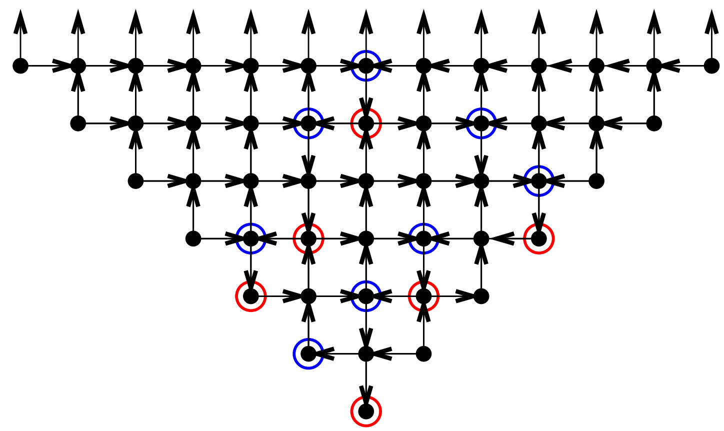
- DASASM-triangle $\Rightarrow$ orientations of triangular graph



Orient edges such that
- all degree 4 vertices are "balanced", and
- all top edges are oriented upward.

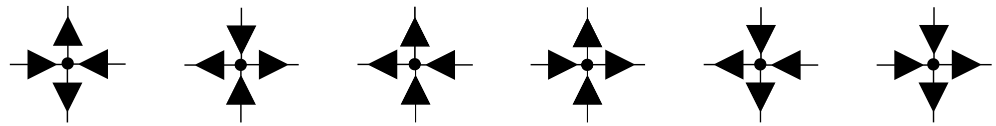
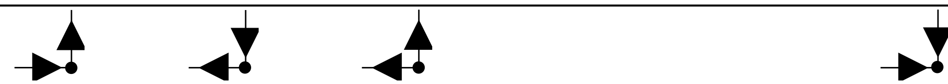
1-1 correspondence with fundamental domains of DASASMs

Example



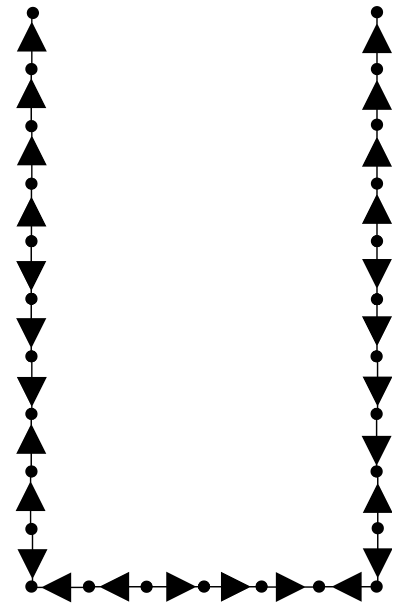
0	0	0	0	0	0	1	0	0	0	0	0	0
	0	0	0	0	1	−1	0	1	0	0	0	
		0	0	0	0	0	0	0	1	0		
			0	1	−1	0	1	0	−1			
				−1	0	1	−1	0				
					1	0	0					
						−1						

Dictionary

degree 4 vertices 	1	-1	0	0	0	0
left boundary 	1	-1		0	0	
right boundary 	1	-1	0			0
bottom vertex 	1	-1				

Why does this work?

$$
\begin{array}{ccccccc}
0 & & & & & & 0 \\
0 & & & & & & 0 \\
0 & & & & & & 0 \\
1 & & & & & & 1 \\
0 & & & & & & 0 \\
0 & & & & & & 0 \\
-1 & & & & & & 0 \\
0 & & & & & & -1 \\
1 & & & & & & 1 \\
0 & 0 & -1 & 0 & 0 & 1 & -1
\end{array}
$$



- Along straight lines, change orientation iff you encounter ± 1.
- As for turns, change orientation iff you encounter 0.

Weighted enumeration

- Principle: sometimes it is easier to prove a generalization!

- Assign to each vertex v a weight $\mathsf{W}(v)$.

- Weight $\mathsf{W}(C)$ of a configuration (=orientation of the triangular graph $\mathcal{T}_n$):

$$\mathsf{W}(C) = \prod_{v \in C} \mathsf{W}(v)$$

- Generating function (partition function):

$$Z_n = \sum_{C \text{ admissible orientation of order } n \text{ triangular graph}} \mathsf{W}(C)$$

- Specialization of the parameters will give the number of configurations, i.e. the number of $(2n+1) \times (2n+1)$ DASASMs.

Very strange vertex weights

The weight of a vertex depends on the orientations of the surrounding edges and the label of the vertex.

Notation: $x^{-1} = \bar{x}$ and $\sigma(x) = x - \bar{x}$; u is the label and q is a global parameter.

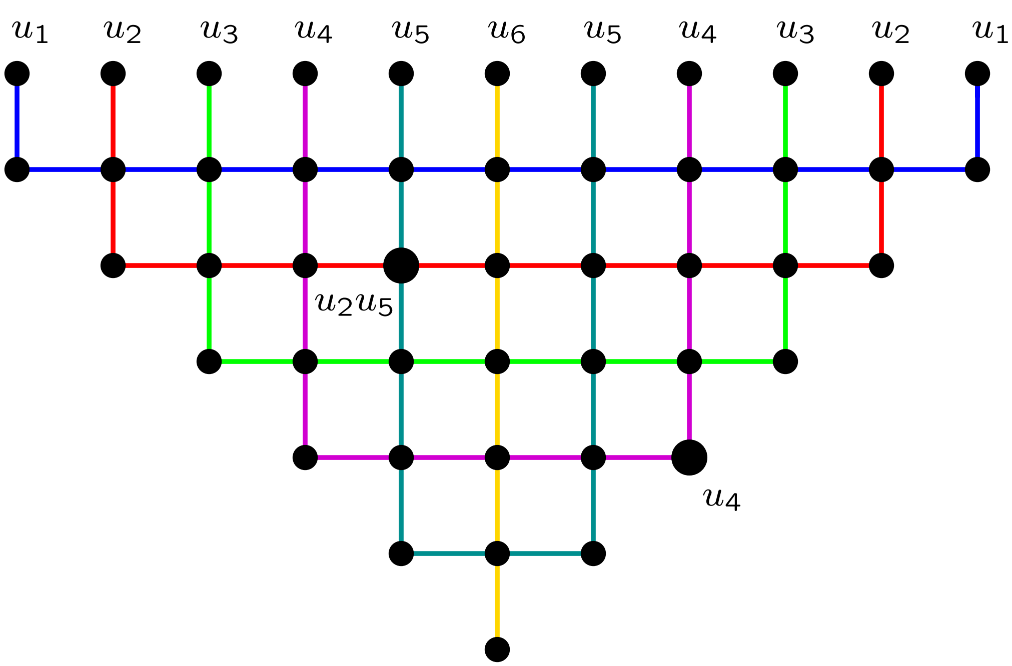
Bulk vertices	Left boundary	Right boundary
$W(\;, u) = W(\;, u) = 1$	$W(\;, u) = W(\;, u) = 1$	$W(\;, u) = W(\;, u) = 1$
$W(\;, u) = W(\;, u) = \frac{\sigma(q^2 u)}{\sigma(q^4)}$	$W(\;, u) = W(\;, u) = \frac{\sigma(qu)}{\sigma(q)}$	
$W(\;, u) = W(\;, u) = \frac{\sigma(q^2 \bar{u})}{\sigma(q^4)}$		$W(\;, u) = W(\;, u) = \frac{\sigma(q\bar{u})}{\sigma(q)}$

All degree 1 vertices have weight 1.

If $u = 1$ and $q = e^{i\pi/6}$, all weights are 1!

Label of a vertex

Each colored path is assigned a parameter u_i as follows.



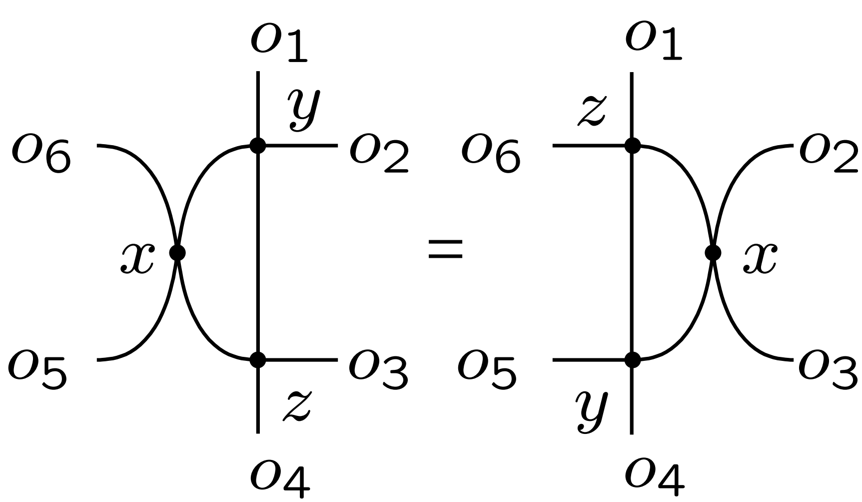
• A degree 4 vertex is contained in two colored paths u_i and u_j $\Rightarrow$ label u_iu_j

• All boundary vertices have a unique path u_i $\Rightarrow$ label u_i

Generating function: $Z_n(u_1, \ldots, u_n; u_{n+1})$.

Yang-Baxter equation

Theorem. If $xyz = q^2$ and $o_1, o_2, \dots, o_6 \in \{\text{in}, \text{out}\}$, then

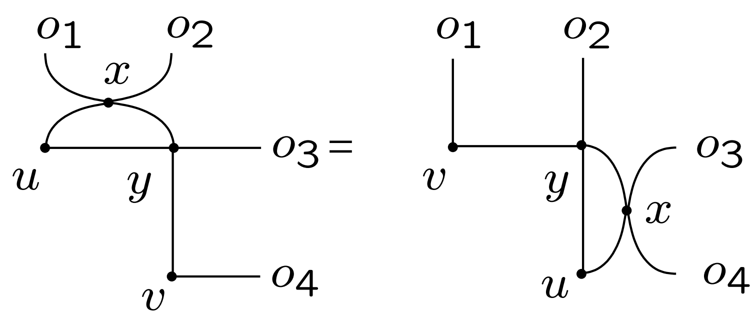


.

A diagram stands for the generating function of all orientations of the graph such that the external edges have the prescribed orientations $o_1, o_2, \dots, o_6$, degree 4 vertices are balanced, and the vertex weights are as given in the table, where the letter close to a vertex indicates its label (rotate until the label is in the SW corner).

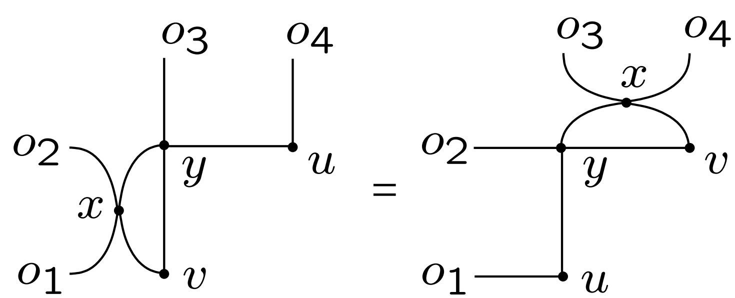
Left and right reflection equation

Theorem (Reflection equations). Suppose $o_1, o_2, o_3, o_4 \in \{\text{in}, \text{out}\}$. If $x = q^2\bar{u}v$ and $y = uv$, then



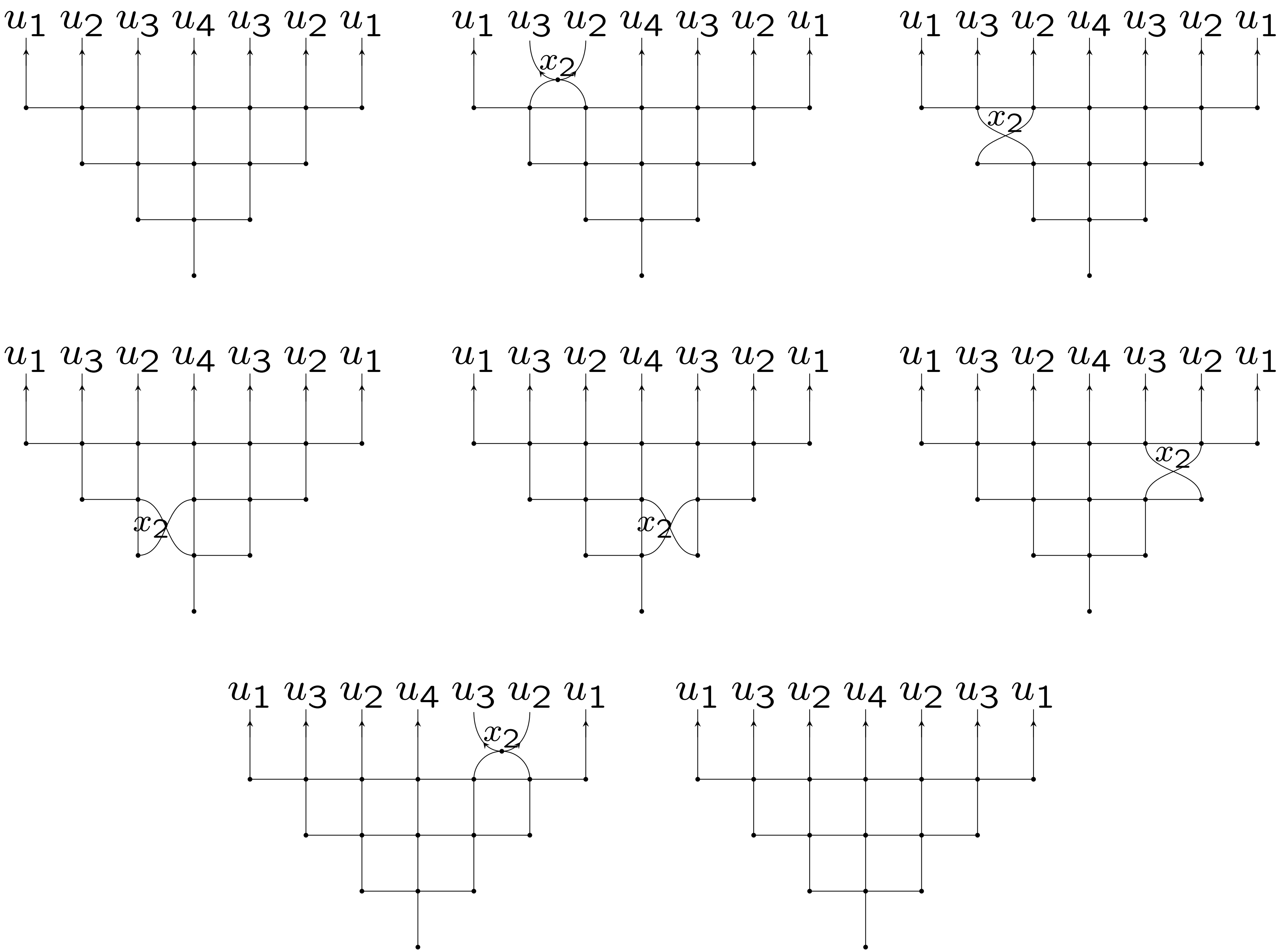
,

and if $x = q^2\bar{u}v$ and $y = \bar{u}\bar{v}$, then



.

$\Rightarrow$ Symmetry of $Z_n(u_1, \ldots, u_n; u_{n+1})$ in $u_1, \ldots, u_n$.



$u_1\ u_2\ u_3\ u_4\ u_3\ u_2\ u_1$
$u_1\ u_3\ u_2\ u_4\ u_3\ u_2\ u_1$
x_2
$u_1\ u_3\ u_2\ u_4\ u_3\ u_2\ u_1$
x_2
$u_1\ u_3\ u_2\ u_4\ u_3\ u_2\ u_1$
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$u_1\ u_3\ u_2\ u_4\ u_3\ u_2\ u_1$
x_2
$u_1\ u_3\ u_2\ u_4\ u_2\ u_3\ u_1$

Another important property

Lemma.

$$
\begin{aligned}
&Z_n(u_1,\dots,u_n;q^2\bar u_1)\\
&\quad = \frac{1}{2}\Big(\big(\mathrm{W}(\;,u_1)+\mathrm{W}(\;,u_1)+\mathrm{W}(\;,u_1)+\mathrm{W}(\;,u_1)\big)Z_{n-1}(u_2,\dots,u_n;u_1)\\
&\qquad +(-1)^{n+1}\big(\mathrm{W}(\;,u_1)+\mathrm{W}(\;,u_1)-\mathrm{W}(\;,u_1)-\mathrm{W}(\;,u_1)\big)Z_{n-1}(u_2,\dots,u_n;-u_1)\Big)\,\Omega,
\end{aligned}
$$

where

$$\Omega = \mathrm{W}(\;,u_1)\prod_{i=2}^{n}\mathrm{W}(\;,u_1u_i)\,\mathrm{W}(\;,q^2\bar u_1u_i).$$

$Z_n(u_1,\dots,u_n;u_{n+1})$ at $u_{n+1}=1$

Theorem (BFK 2015).

$$Z_n(u_1,\dots,u_n;1) = \frac{\sigma(q^2)^n}{\sigma(q)^{2n}\sigma(q^4)^{n^2}} \prod_{i=1}^{n} \sigma(qu_i)\sigma(q\bar{u}_i)\sigma(q^2u_i)\sigma(q^2\bar{u}_i)$$

$$\times \prod_{1\le i<j\le n}\left(\frac{\sigma(q^2u_iu_j)\sigma(q^2\bar{u}_i\bar{u}_j)}{\sigma(u_i\bar{u}_j)}\right)^2 \mathrm{det}_{1\le i,j\le n}\left(\frac{q^2+\bar{q}^2+u_i^2+\bar{u}_j^2}{\sigma(q^2u_iu_j)\,\sigma(q^2\bar{u}_i\bar{u}_j)}\right).$$

Yet another problem: If we set $(u_1,\dots,u_n)=(1,\dots,1)$, then we obtain $\frac{0}{0}$.

Schur function expression for $Z_n(u_1,\dots,u_n;1)$ at $q=e^{i\pi/6}$

Theorem (BFK 2015).

$$Z_n(u_1,\dots,u_n;1)\big|_{q=e^{i\pi/6}} = 3^{-\binom{n}{2}} \times s_{(n,n-1,n-1,n-2,n-2,\dots,1,1)}(u_1^2,\bar{u}_1^2,\dots,u_n^2,\bar{u}_n^2,1)$$

Now we may use the formula

$$s_\lambda(1,\dots,1) = \prod_{1\le i<j\le k} \frac{\lambda_i-\lambda_j+j-i}{j-i}$$

to conclude the proof of the DASASM (ex-)conjecture.

Thanks for the interest!